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k-Fibonacci numbers and k-Lucas numbers and associated bipartite graphs*

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Abstract: In [6], [8] and [10], the authors studied the generalized Fibonacci numbers. Also, in [7], the author found a class of bipartite graphs whose number of 1-factors is the nth k-Lucas numbers. In this paper, we give a new relationship between $g_n^{(k)}$ and $l_n^{(k)}$ and the number of 1-factors of a bipartite graph.

Key words: k-Fibonacci number, k-Lucas number, permanent, 1-factors

1. Introduction

The Fibonacci sequence has been discussed in so many articles and books. In particular, the Fibonacci number are very important in combinatorial analysis.

The well-known Fibonacci sequence $\{F_n\}$ is defined as follows:

$$F_1 = F_2 = 1 \quad \text{and, for} \quad n > 2, \quad F_n = F_{n-1} + F_{n-2}.$$

We call F_n the nth Fibonacci number. The Fibonacci sequence is

$$(F_0 := 0),\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \ldots.$$

Now, we consider the generalization of the Fibonacci sequence, which is called as the k-Fibonacci sequence for positive integer $k \geq 2$. The *k-Fibonacci sequence* $\{g_n^{(k)}\}$ is defined as follows:

$$g_1^{(k)} = \ldots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \ldots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the nth k-Fibonacci number. By the definition of the k-Fibonacci sequence, we have that $g_j^{(k)} = 2^{j-k}$ for $j = k, k+1, \ldots, 2k-1$.

For example, if $k = 5$, then the 5-Fibonacci sequence is

$$0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 4,\ 8,\ 16,\ 31,\ 61,\ 120,\ 236,\ 464,\ 912,\ 1793,\ 3535,\ 6930, \ldots.$$

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In particular, if $k=2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence $\{F_n\}$.

In [2], the author defined a new subclass of analytic biunivalent functions associated with the Fibonacci numbers. Moreover, the author surveyed the bounds of the coefficients for functions in this class. In [1], the authors introduced a family of companion sequences for some generalized Fibonacci sequence: the r-Fibonacci sequence. They evaluated the generating functions and gave some applications, and they exhibited convolution relations that generalized some known identities such as Cassini's. Afterwards, they calculated the sums of their terms using matrix methods. In [6], [8] and [10], the authors studied the generalized Fibonacci numbers. In [9], the authors considered the factorizations and eigenvalues of k-Fibonacci and symmetric k-Fibonacci matrices. In [12], the authors generalized result on connection permanents of special tridiagonal matrices with Fibonacci numbers, as they shown that more general sequences of tridiagonal matrices is related to the sequence of Fibonacci numbers.

Now, we introduce k-Lucas sequences.

For the Fibonacci number F_n, let L_n be the nth Lucas number, that is, for $n \geq 1$, $L_n = F_{n-1} + F_{n+1}$ where $F_0 = 0$. The Lucas sequence $\{L_n\}$ is

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots.$$

Let us define the generalized Lucas sequence in a similar way to the definition of the generalization of Fibonacci sequence. For the generalized Fibonacci number $\{g_n^{(k)}\}$, let $g_0^{(k)} = 0$, and let the k-*Lucas sequence* $\{l_n^{(k)}\}$ is defined by

$$l_n^{(k)} = g_{n-1}^{(k)} + g_{n+k-1}^{(k)}.$$

We call $l_n^{(k)}$ the nth k-Lucas number. From the definition, we have, for $j = 1, 2, \ldots, k-1$,

$$l_j^{(k)} = 2^{j-1}, \tag{1.1}$$

and $l_k^{(k)} = 1 + 2^{k-1}$.

For example, if $k=5$, then the 5-Lucas sequence is

$$1,\ 2,\ 4,\ 8,\ 17,\ 32,\ 63,\ 124,\ 244,\ 480,\ 943,\ 1854,\ 3645,\ 7166, \ldots.$$

In particular, if $k=2$, then $\{l_n^{(2)}\}$ is the Lucas sequence $\{L_n\}$. One can find a generalization of the Lucas number in [13].

Since the Fibonacci numbers are connected by the fundamental recursion $F_n = F_{n-1} + F_{n-2}$, it follows immediately that the Lucas numbers are likewise related by $L_n = L_{n-1} + L_{n-2}$ for $n > 2$. Hence, we have, for $n > k$,

$$l_n^{(k)} = l_{n-1}^{(k)} + l_{n-2}^{(k)} + \ldots + l_{n-k}^{(k)}.$$

The permanent of an n-square matrix $A = [a_{ij}]$ is defined by

$$\text{per}\, A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ in the symmetric group $\mathfrak{S}_n$. For any square matrix A and any permutation matrices P and Q, $\text{per}\, A = \text{per}\, PAQ$.

For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \ldots, n\}$. For an $n \times n$ matrix A and for $\alpha, \beta \in Q_{k,n}$, let $A[\alpha|\beta]$ denote the matrix lying in rows α and columns β and let $A(\alpha|\beta)$ denote the matrix complementary to $A[\alpha|\beta]$ in A. That is, the submatrix obtained from A by deleting rows α and column β. In particular, the $(n-k) \times n$ submatrix obtained from A by deleting rows α is denoted by $A(\alpha|-)$. Similarly, $A(-|\beta)$ denotes the $n \times (n-k)$ submatrix obtained from A by deleting columns β.

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 and a vertex in V_2. A 1-factor (or perfect matching) of bipartite graph G with $2n$ vertices is a spanning subgraph of G in which every vertex has degree 1. The enumeration or actual construction of 1-factors of a bipartite graph has many applications, for example, in [5], in maximal flow problems, and in [4], in assignment and scheduling problems. Let $A(G)$ be the adjacency matrix of the bipartite graph G, and let $\mu(G)$ denote the number of 1-factors of G. Then, in [11], we have the very well known fact : $\mu(G) = \sqrt{\operatorname{per} A(G)}$. Let G be a bipartite graph with $2n$ vertices, and suppose that no edge joins two vertices in $\{v_1, \ldots, v_n\}$ nor two vertices in $\{v_{n+1}, \ldots, v_{2n}\}$. Then the adjacency matrix of G has the form

$$A(G) = \begin{bmatrix} 0 & B(G) \\ B(G)^T & 0 \end{bmatrix},$$

where $B(G)$ is an $n \times n$ submatrix of $A(G)$. The matrix $B(G)$ is called the subadjacent matrix of the bipartite graph G. Then, in [11], the number of 1-factors of bipartite graph G equals the permanent of its subadjacency matrix. That is,

$$\mu(G) = \sqrt{\operatorname{per} A(G)} = \operatorname{per} B(G).$$

In [7], the author found a class of bipartite graphs whose number of 1-factors is the nth k-Lucas numbers.

Let G and G' denote two general graphs of order n, and let the adjacency matrices of these graphs be denoted by A and A', respectively. Then G and G' are isomorphic if and only if A is transformable into A' by simultaneous permutations of the lines of A. Thus, G and G' are isomorphic if and only if there exists a permutation matrix P of order n such that $P^T A P = A'$.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix with row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is contractible on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. Every contraction used in this paper will be on the first column using the first and second rows. We say that A can be contracted to a matrix B if either $B = A$ or there exist matrices $A_0, A_1, \ldots, A_t$ $(t \geq 1)$ such that $A_0 = A$, $A_t = B$ and A_r is a contraction of A_{r-1} for $r = 1, \ldots, t$. One can find the following fact in [3]: let A be a nonnegative integral matrix of order $n > 1$ and let B be a contraction of A. Then

$$\operatorname{per} A = \operatorname{per} B.$$

2. k-Fibonacci numbers

In this section, we consider the classes of bipartite graphs whose number of 1-factors are k-Fibonacci numbers.

To begin with, we have some definitions for matrices.

A matrix is said to be a $(0,1)$ matrix if each of its entries is either 0 or 1.

Let $n \times n$ $(0,1)$ matrix $\mathfrak{A}_n^{(k)} = [a_{ij}^{(k)}]$ is defined as follows: for $i < j$,

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

for $i \geq j$,

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } j = 1 \text{ or } k \not| (i-j+1), \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $k = 3$ and $n = 8$, then we have the 8×8 $(0,1)$ matrix $\mathfrak{A}_8^{(3)}$ as follows:

$$\mathfrak{A}_8^{(3)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Let $T_n = [t_{ij}]$ be the $n \times n$ tridiagonal $(0,1)$ matrix defined by $t_{ij} = 0$ if and only if $|j-i| \leq 1$ and let $U_n = [u_{ij}]$ be the $n \times n$ $(0,1)$ matrix defined by $u_{ij} = 1$ if and only if $2 \leq j - i \leq k-1$. In [10], the authors had a matrix for which the permanent of the matrix is the $(n+k-1)$st k-Fibonacci number.

Theorem 2.1 *[10] Let $\mathfrak{F}_n^{(k)} = T_n + U_n$. Then*

$$\operatorname{per} \mathfrak{F}_n^{(k)} = g_{n+k-1}^{(k)}.$$

Now, let us compare the matrix $\mathfrak{A}_n^{(k)}$ and $\mathfrak{F}_n^{(k)}$. The here and now, for example, let us compare the number of 1's in $\mathfrak{A}_8^{(3)}$ and the number of 1's in $\mathfrak{F}_8^{(3)}$. Then we know that the number of 1's in $\mathfrak{A}_8^{(3)}$ is 36 but the number of 1's in $\mathfrak{F}_8^{(3)}$ is 28. Thus, in general, we have the following result.

Lemma 2.2 *For the $n \times n$ matrices $\mathfrak{A}_n^{(k)}$ and $\mathfrak{F}_n^{(k)}$, there do not exist permutation matrices P and Q such that*

$$\mathfrak{F}_n^{(k)} = P \mathfrak{A}_n^{(k)} Q$$

Lemma 2.3 *For $k \geq 2$,*

$$\operatorname{per} \mathfrak{A}_k^{(k)} = 2^{k-1}$$

Proof For the $n \times n$ $(0,1)$ matrix $\mathfrak{A}_n^{(k)} = [a_{ij}^{(k)}]$, if $n = k$, then $a_{ij} = 1$ if and only if $-1 \leq j - i$. Since the matrix $\mathfrak{A}_k^{(k)}$ contractible on column k, we have $\operatorname{per} \mathfrak{A}_k^{(k)} = 2^{k-1}$. □

From the Lemma 2.3, we know that

$$\operatorname{per}\mathfrak{A}_k^{(k)} = \operatorname{per}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{bmatrix} = 2^{k-1}. \tag{2.1}$$

Let G and G' denote two general graphs of order n, and let the adjacency matrices of these graphs be denoted by $\mathfrak{A}_n^{(k)}$ and $\mathfrak{F}_n^{(k)}$, respectively. From the Lemma 2.2, we know that the graph G does not isomorphic to the graph G'.

Thus, we have the following result.

Theorem 2.4 *For the positive integers n and k, $2 \le k \le n$, let $G(\mathfrak{A}_n^{(k)})$ be the bipartite graph with subadjacency matrix $\mathfrak{A}_n^{(k)}$. Then the number of 1-factors of $G(\mathfrak{A}_n^{(k)})$ is $g_{n+k-1}^{(k)}$.*

Proof For fixed k, if $n = k$ then, from Lemma 2.3, we have $\operatorname{per}\mathfrak{A}_k^{(k)} = 2^{k-1} = g_{k+k-1}^{(k)}$.

By induction on n,

$$\begin{aligned} \operatorname{per}\mathfrak{A}_n^{(k)} &= \operatorname{per}\mathfrak{A}_n^{(k)}(n|n) + \operatorname{per}\mathfrak{A}_n^{(k)}(n-1|n) \\ &= g_{n+k-2}^{(k)} + \operatorname{per}\mathfrak{A}_n^{(k)}(n-1|n) \\ &= g_{n+k-2}^{(k)} + \operatorname{per}\mathfrak{A}_n^{(k)}(n-1,n|n-1,n) + \operatorname{per}\mathfrak{A}_n^{(k)}(n-2,n-1|n-1,n) \\ &= g_{n+k-2}^{(k)} + g_{n+k-3}^{(k)} + \operatorname{per}\mathfrak{A}_n^{(k)}(n-2,n-1|n-1,n) \\ &\ \vdots \\ &= g_{n+k-2}^{(k)} + g_{n+k-3}^{(k)} + \ldots + g_n^{(k)} \\ &\quad + \operatorname{per}\mathfrak{A}_n^{(k)}(n-k,\ldots,n-2,n-1|n-k+1,\ldots,n-1,n) \end{aligned}$$

Note that, in $\mathfrak{A}_n^{(k)}(-|n-k+1,\ldots,n-1,n)$, the nth row is the same that the $(n-k)$th row.

Hence, we have

$$\begin{aligned} \operatorname{per}\mathfrak{A}_n^{(k)}(n-k, \ \ldots \ &, n-2, n-1|n-k+1,\ldots,n-1,n) \\ &= \operatorname{per}\mathfrak{A}_n^{(k)}(n-k+1,\ldots,n-1,n|n-k+1,\ldots,n-1,n) \\ &= g_{n-1}^{(k)} \end{aligned}$$

Therefore, we have

$$\begin{aligned}\operatorname{per}\mathfrak{A}_n^{(k)} &= g_{n+k-2}^{(k)} + g_{n+k-3}^{(k)} + \ldots + g_n^{(k)} \\ &\quad + \operatorname{per}\mathfrak{A}_n^{(k)}(n-k,\ldots,n-2,n-1|n-k+1,\ldots,n-1,n) \\ &= g_{n+k-2}^{(k)} + g_{n+k-3}^{(k)} + \ldots + g_n^{(k)} + g_{n-1}^{(k)} \\ &= g_{n+k-1}^{(k)},\end{aligned}$$

and the proof is completed. □

For example, since $\operatorname{per}\mathfrak{A}_8^{(3)} = 81$, the number of 1-factors of $G(\mathfrak{A}_8^{(3)})$ is $g_{8+3-1}^{(3)} = g_{10}^{(3)} = 81$. Also, we know that the number of 1-factors of $G(\mathfrak{A}_{10}^{(5)})$ is $g_{10+5-1}^{(5)} = g_{14}^{(5)} = 464$.

For the $n \times n$ $(0,1)$ matrix $\mathfrak{A}_n^{(k)} = [a_{ij}^{(k)}]$, in particular, if $k = 2$, we denote $\mathfrak{A}_n = [a_{ij}]$ instead of $\mathfrak{A}_n^{(2)} = [a_{ij}^{(2)}]$. Then we have the following result.

Corollary 2.5 *Let $G(\mathfrak{A}_n)$ be the bipartite graph with subadjacency matrix $\mathfrak{A}_n$, $2 \leq n$. Then the number of 1-factors of $G(\mathfrak{A}_n)$ is F_{n+1}.*

Let $B_n = [b_{ij}]$ be an $n \times n$ matrix of which $a_{11} = a_{nn} = 1$ and $a_{ij} = 1$ if $i < j$ or $i = j + 1$, otherwise $A_{ij} = 0$. In [10], the authors gave two matrices for which the permanent of the matrices are the $(n+1)$st Fibonacci number. That is, for the tridiagonal $(0,1)$ matrix T_n and for the matrix B_n, $\operatorname{per}T_n = \operatorname{per}B_n = F_{n+1}$, where F_{n+1} is the $(n+1)$st Fibonacci number.

Now, we define a new $(0,1)$ matrix $\mathfrak{C}_n$. Let $\mathfrak{C}_n = [c_{ij}]$ be an $n \times n$ $(0,1)$ matrix of which $c_{ij} = 0$ if and only if $j > i+1$ or $i \geq j$ and $2|(i-j+1)$ or, $i = n$ and $2|(n-j+1)$. For example,

$$T_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and

$$\mathfrak{C}_6 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then the number of 1's in T_6 is 16, the number of 1's in B_6 is 21 and the number of 1's in $\mathfrak{C}_6$ is 17. Thus, in general, we have the following result.

Lemma 2.6 *For the $n \times n$ matrices T_n, B_n and $\mathfrak{C}_n$, there do not exist permutation matrices P, Q, R and S such that $\mathfrak{C}_n = PT_nQ$ and $\mathfrak{C}_n = RB_nS$.*

Theorem 2.7 *For the $n \times n$ $(0,1)$ matrix $\mathfrak{C}_n$, $2 \leq n$, let $G(\mathfrak{C}_n)$ be the bipartite graph with subadjacency matrix $\mathfrak{C}_n$. Then the number of 1-factors of $G(\mathfrak{C}_n)$ is F_n.*

Proof For $n = 2$ and $n = 3$,

$$\mathfrak{C}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathfrak{C}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

and hence $\operatorname{per} \mathfrak{C}_2 = 1 = F_2$ and $\operatorname{per} \mathfrak{C}_3 = 2 = F_3$.

Since $\operatorname{per} \mathfrak{C}_n(1|1) = F_{n-1}$, by induction on n, we have

$$\begin{aligned} \operatorname{per} \mathfrak{C}_n &= \operatorname{per} \mathfrak{C}_n(1|1) + \operatorname{per} \mathfrak{C}_n(1|2) \\ &= F_{n-1} + \operatorname{per} \mathfrak{C}_n(1|2) \end{aligned}$$

Since the first row of $\mathfrak{C}_n(1|2)$ is the $(n-1)$ tuple such that $(0,1,0,\ldots,0)$, $\operatorname{per} \mathfrak{C}_n(1|2) = \operatorname{per} \mathfrak{C}_n(1,2|2,3)$. And, by induction on n, we have $\operatorname{per} \mathfrak{C}_n(1,2|2,3) = F_{n-2}$.

Therefore, we have

$$\begin{aligned} \operatorname{per} \mathfrak{C}_n &= F_{n-1} + \operatorname{per} \mathfrak{C}_n(1|2) \\ &= F_{n-1} + \operatorname{per} \mathfrak{C}_n(1,2|2,3) \\ &= F_{n-1} + F_{n-2} \\ &= F_n, \end{aligned}$$

and the proof is completed. □

3. k-Lucas numbers

In this section, we consider the classes of bipartite graphs whose number of 1-factors are k-Lucas number.

Let E_{ij} denote the $n \times n$ matrix with 1 in the (i,j) position and zeros elsewhere. Let $H_n^{(k)} = [h_{ij}^{(k)}]$ be the $n \times n$ $(0,1)$-matrix defined by $h_{ij}^{(k)} = 1$ if and only if $-1 \leq j - i \leq k - 1$. For $k < n$, let $\mathfrak{J}_n^{(k)} = H_n^{(k)} - \sum_{j=2}^{k} E_{1j} + E_{1k+1}$. In [7], the author found a class of bipartite graphs whose number of 1-factors is the nth k-Lucas number and the following result was proven:

Theorem 3.1 *Let $G(\mathfrak{J}_n^{(k)})$ be the bipartite graph with subadjacency matrix $\mathfrak{J}_n^{(k)}$, $n \geq 3$. Then the number of 1-factors of $G(\mathfrak{J}_n^{(k)})$ is $l_{n-1}^{(k)}$.*

We consider an $n \times n$ $(0,1)$ matrix $\mathfrak{V}_n^{(k)} = [v_{ij}^{(k)}]$. To define the $n \times n$ $(0,1)$ matrix $\mathfrak{V}_n^{(k)} = [v_{ij}^{(k)}]$, we consider two cases:(i) $i < j$ and (ii) $i \geq j$.

Case (i) If $i < j$, then

$$v_{ij}^{(k)} = \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Case (ii) If $i \geq j$, then we have two subcases: (i') $i \equiv 1 \pmod{k}$, (ii') $i \not\equiv 1 \pmod{k}$.

Subcase (i') If $i \equiv 1 \pmod{k}$, then

$$v_{ij}^{(k)} = \begin{cases} 0 & \text{if, for some positive integer } m, \ i-j+1 = km \text{ and } j \neq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Subcase (ii') If $i \not\equiv 1 \pmod{k}$, then

$$v_{ij}^{(k)} = \begin{cases} 0 & \text{if, for some positive integer } m, \ i-j+1 = km \text{ or } j = 1, \\ 1 & \text{otherwise.} \end{cases}$$

For example, if $k = 3$ and $n = 10$, then we have the 10×10 $(0,1)$ matrix $\mathfrak{V}_{10}^{(3)}$ as follows:

$$\mathfrak{V}_{10}^{(3)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Since the number of 1's in $\mathfrak{J}_{10}^{(3)}$ does not equal to the number of 1's in $\mathfrak{V}_{10}^{(3)}$. Thus, in general, we have the following result.

Lemma 3.2 *For the $n \times n$ matrices $\mathfrak{J}_n^{(k)}$ and $\mathfrak{V}_n^{(k)}$, there do not exist permutation matrices P and Q such that $\mathfrak{J}_n^{(k)} = P\mathfrak{V}_n^{(k)}Q$.*

Let G and G' denote two general graphs of order n, and let the adjacency matrices of these graphs be denoted by $\mathfrak{J}_n^{(k)}$ and $\mathfrak{V}_n^{(k)}$, respectively. From the Lemma 3.2, we know that the graph G does not isomorphic to the graph G'.

Lemma 3.3 *For $2 \leq n \leq k$,*

$$per\mathfrak{V}_n^{(k)} = 2^{n-2} = l_{n-1}^{(k)}.$$

Proof Since $n \leq k$, we have

$$\mathfrak{V}_n^{(k)} = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then $\text{per}\ \mathfrak{V}_n^{(k)} = \text{per}\ \mathfrak{V}_n^{(k)}(1|1)$. By induction on n and (1.1) and (2.1), we have

$$\text{per}\ \mathfrak{V}_n^{(k)} = \text{per}\ \mathfrak{V}_n^{(k)}(1|1) = 2^{n-2} = l_{n-1}^{(k)}.$$

Therefore, the proof is completed. □

In the following theorem, we have a class of bipartite graphs whose number of 1-factors is the $(n-1)$st k-Lucas number.

Theorem 3.4 *Let k and n be positive integers. Let $G(\mathfrak{V}_n^{(k)})$ be the bipartite graph with subadjacency matrix $\mathfrak{V}_n^{(k)}$, $n \geq 2$. Then the number of 1-factors of $G(\mathfrak{V}_n^{(k)})$ is $l_{n-1}^{(k)}$.*

Proof If $n \leq k$, then, by Lemma 3.3, we are done.

Suppose that $2 \leq k < n$.

For positive integer t, $n > t$, note that

$$\mathfrak{V}_n^{(k)}(n, n-1, \ldots, n-t+1 | n, n-1, \ldots, n-t+1) = \mathfrak{V}_{n-t}^{(k)}.$$

That is,

$$\text{per } \mathfrak{V}_n^{(k)}(n, n-1, \ldots, n-t+1 | n, n-1, \ldots, n-t+1) = \text{per } \mathfrak{V}_{n-t}^{(k)}.$$

By induction on n,

$$\begin{aligned}
\text{per}\, \mathfrak{V}_n^{(k)} &= \text{per}\, \mathfrak{V}_n^{(k)}(n|n) + \text{per}\, \mathfrak{V}_n^{(k)}(n-1|n) \\
&= l_{n-2}^{(k)} + \text{per}\, \mathfrak{V}_n^{(k)}(n-1|n) \\
&= l_{n-2}^{(k)} + \text{per}\, \mathfrak{V}_n^{(k)}(n-1, n|n, n-1) + \text{per}\, \mathfrak{V}_n^{(k)}(n-1, n-2|n, n-1) \\
&= l_{n-2}^{(k)} + l_{n-3}^{(k)} + \text{per}\, \mathfrak{V}_n^{(k)}(n-1, n-2|n, n-1) \\
&\vdots \\
&= l_{n-2}^{(k)} + l_{n-3}^{(k)} + \ldots + l_{n-k}^{(k)} \\
&\quad + \text{per}\, \mathfrak{V}_n^{(k)}(n-1, \ldots, n-k|n, \ldots, n-k+1).
\end{aligned}$$

Note that, in $\mathfrak{V}_n^{(k)}(-|n-k+1, \ldots, n-1, n)$, the nth row is the same that the $(n-k)$th row. That is,

$$\begin{aligned}
\text{per}\, \mathfrak{V}_n^{(k)}(n-1, \ldots, n-k \quad &| \quad n, \ldots, n-k+1) \\
&= \text{per}\, \mathfrak{V}_n^{(k)}(n, \ldots, n-k+1|n, \ldots, n-k+1) \\
&= l_{n-k-1}^{(k)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\text{per}\, \mathfrak{V}_n^{(k)} &= l_{n-2}^{(k)} + l_{n-3}^{(k)} + \ldots + l_{n-k}^{(k)} \\
&\quad + \text{per}\, \mathfrak{V}_n^{(k)}(n-1, \ldots, n-k|n, \ldots, n-k+1) \\
&= l_{n-2}^{(k)} + l_{n-3}^{(k)} + \ldots + l_{n-k}^{(k)} + l_{n-k-1}^{(k)} \\
&= l_{n-1}^{(k)},
\end{aligned}$$

and the proof is completed. □

For example, the number of 1-factors of the graph $G(\mathfrak{V}_{10}^{(5)})$ is $l_9^{(5)} = 244$.

For the $n \times n$ $(0,1)$ matrix $\mathfrak{V}_n^{(k)} = [v_{ij}^{(k)}]$, in particular, if $k = 2$, we denote $\mathfrak{V}_n = [v_{ij}]$ instead of $\mathfrak{V}_n^{(2)} = [v_{ij}^{(2)}]$. Then we have the following result.

Corollary 3.5 *Let $G(\mathfrak{V}_n)$ be the bipartite graph with subadjacency matrix $\mathfrak{V}_n$, $2 \le n$. Then the number of 1-factors of $G(\mathfrak{V}_n)$ is L_{n-1}.*

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