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Necessary conditions for extended spectral decomposable multivalued linear operators

Yosra BARKAOUI, Maher MNIF*
Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia

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Abstract: In this paper, we use subsets of the Riemann sphere and specific types of invariant linear subspaces to introduce the extended spectral decomposable multivalued linear operators (linear relations) in Banach spaces. We also introduce the extended Bishop's property, the extended relatively single-valued extension property and the extended Dunford's property. More importantly, we show that these properties are three necessary conditions for a linear relation to be extended spectral decomposable.

Key words: Multivalued linear operators, extended spectrum, extended spectral decomposable multivalued linear operator, extended Bishop's property, extended Dunford's property

1. Introduction

The notion of decomposability has been given much attention and has led to a strong spectral theory for closed and bounded linear operators in Banach spaces. In 1963, Foias [11] was the first to define the decomposability for bounded linear operators using the spectral maximal spaces, which made the definition a little complicated. In 2000, basing on the works of E. Albercht [2], B. Nagy [19] and R. Lange [16], K. B. Laursen and Michael M. Neumann were able to make this notion more flexible [17]. According to them, a bounded linear operator T is said to be decomposable on a complex Banach space X if for every open cover $\mathbb{C} = U_1 \cup U_2$, there exist two closed invariant linear subspaces X_1 and X_2 of X under T such that

$$X = X_1 + X_2, \quad \sigma(T|X_1) \subseteq U_1 \quad \text{and} \quad \sigma(T|X_2) \subseteq U_2,$$

where $\sigma(T|X_i)$ is the spectrum of $T|X_i$, $i = 1, 2$.

In the present paper, we aim not only to extend the decomposability from the case of bounded linear operators to that of closed and continuous multivalued ones (linear relations), but also to develop several results related to the decomposability from the complex plane $\mathbb{C}$ to the extended one $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. Hence, we introduce this notion as "extended spectral decomposable linear relations". We shall see that, whenever a closed linear relation T is extended spectral decomposable, it will be decomposed as a sum of a bounded linear operator T_0 and a purely multivalued part T_1. This enables us to study T much more easily. Note that this decomposition was first studied in a Hilbert space in 1961 by Arens [3], and has then been applied by several authors such as Y. Shi[22] and Y. Liu [23], among others. This someway has motivated us to look for situations in which a linear relation can be decomposed as described before. And given the importance of the study of decomposable linear operators and the rich local spectral theory they possess, it has become necessary and even urgent to establish the concept of the decomposability for multivalued linear operators. These were first considered by V. Neumann [26] and since then, they have received an increasing interest by many

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mathematicians (e.g., [8, 10, 14, 18]) because of their extensive applications in game theory, linear differential inclusions, pseudo resolvents and other fields of applied mathematics.

The rest of this paper is outlined as follows. In Section 2, we refer to [8] to recall basic notions about linear relations and some notations which will be used in the understanding of the extended spectral decomposable linear relations. In particular, we make reference to [5] to evoke the concept of strongly and weakly invariant linear subspaces and establish certain required properties of those subspaces. Furthermore, a complex analysis framework is considered in order to facilitate the understanding of some proofs. Section 3 deals with our main objective. We introduce the extended spectral decomposable linear relations in Banach spaces and we give a direct application in Example 3.1. It is important to note that this example verifies some sufficient conditions for a linear relation to be extended spectral decomposable and this will be seen in details in our forthcoming article [4]. Enlightened by works on the local spectral theory for linear operators (cf., [1, 9, 17]), we shall establish that there are three necessary conditions for decomposable linear operators to be extended to linear relations. The first condition is the extended Bishop's property which will be introduced in the case of closed and continuous linear relations. As for the second condition, we extend the notion of the single-valued extension property from the case of linear operators and the complex plane to that of linear relations and extended complex plane, respectively. We will observe that, differently from the case of operators, there exists more than one analytic solution f verifying

$$0 \in (T - \mu I)f(\mu) \qquad \text{for all} \quad \mu \in U_\infty \cap \mathbb{C},$$

where U_∞ is an open connected neighborhood of μ in $\mathbb{C}_\infty$, T is a closed linear relation and f is given in Definition 3.5. The third condition, the extended Dunford's property, will be established after extending the concepts of the local resolvent set from $\mathbb{C}$ to $\mathbb{C}_\infty$ and defining the extended local and glocal subspaces.

2. Preliminaries

In this part, we are going to give some of the basic concepts of linear relations which will be used in the understanding of the extended spectral decomposable linear relations. In particular, some results related to the different types of invariant linear subspaces under linear relations are established. Besides, a complex analysis framework is considered to facilitate the understanding of some proofs.

Let X, Y and Z be Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Following Cross [8], a mapping $T : X \to 2^Y \setminus \emptyset$, whose domain is a linear subspace $\mathcal{D}(T)$ of X, is called a linear relation if it satisfies $\alpha Tx + \beta Ty \subseteq T(\alpha x + \beta y)$ for all $x, y \in \mathcal{D}(T)$ and $\alpha, \beta \in \mathbb{C}$. We denote by $LR(X, Y)$ the class of all linear relations from X to Y. A relation $T \in LR(X, Y)$ is entirely determined by its graph given by

$$G(T) = \{(x, y) \in X \times Y \ ; \ x \in \mathcal{D}(T) \text{ and } y \in Tx\}.$$

The inverse of T is defined as $G(T^{-1}) = \{(y, x) \in Y \times X \ ; \ (x, y) \in G(T)\}$. The kernel of T is the subspace $Ker(T) := T^{-1}(0)$. If $Ker(T) = \{0\}$, then T is said to be injective. By $T(M) := \bigcup_{x \in \mathcal{D}(T) \cap M} Tx$ and $R(T) := T(X)$, we denote the range of a subset $M \subseteq X$ and the range of T, respectively. If $R(T) = Y$, then T is said to be surjective.

Let $T, S \in LR(X, Y)$ and $B \in LR(Y, Z)$. Define $S + T$, and $S \widehat{+} T$ by

$$G(T + S) := \big\{(x, u + v) \mid (x, u) \in G(T),\ (x, v) \in G(S)\big\}.$$

$$G(T \widehat{+} S) := \big\{(x + u, y + v) \mid (x, y) \in G(T),\ (u, v) \in G(S)\big\}.$$

If $G(T) \cap G(S) = \{(0, 0)\}$, then $T \widehat{+} S$ is a direct sum denoted by $T \oplus S$. The product of B and T is defined by $G(BT) := \{(x, z) \in X \times Z \ ; (x, y) \in G(T) \text{ and } (y, z) \in G(B) \text{ for some } y \in Y\}$. The singular chain manifold of T is denoted by $R_c(T) = \big(\bigcup_{n=1}^{+\infty} T^n(0)\big) \cap \big(\bigcup_{n=1}^{+\infty} Ker(T^n)\big)$. For $x \in \mathcal{D}(T)$, the norm of Tx and T are respectively given by:

$$\|Tx\| := \|Q_T Tx\| \qquad \text{and} \qquad \|T\| := \|Q_T T\|,$$

where $Q_T : Y \longrightarrow Y/\overline{T(0)}$ denotes the natural quotient map with domain Y and kernel $\overline{T(0)}$. If $\|T\| < \infty$, we say that T is continuous. T is said to be closed (respectively open) if $G(T)$ is closed (respectively T^{-1} is continuous). We denote by $CR(X,Y)$ the set of all closed linear relations $T : X \longrightarrow Y$. If $Y = X$, then $CR(X,Y) = CR(X)$. The set of all closed and continuous linear relations $T : X \longrightarrow X$ is denoted by $CR_c(X)$.

T is said to have the property of stabilization of powers at infinity if $T \in CR(X)$ and there exists $i \in \mathbb{N}N^*$ such that $T^{i-1}(0) \subsetneq T^i(0) = T^{i+1}(0)$ and $\mathcal{D}(T^{i+1}) = \mathcal{D}(T^i) \subsetneq \mathcal{D}(T^{i-1})$. The integer i is called the order of stabilization.

We note that T is an operator whenever $T(0) = \{0\}$ and we write $T \in LO(X,Y)$. By $CO(X)$ and $End(X)$ we denote the set of all closed and bounded linear operators $A : X \longrightarrow X$, respectively. If $T \in CR(X)$ and $T^{-1} \in End(X)$, then T is said to be invertible. We note that T is invertible if and only if T is both injective and surjective.

For a closed linear subspace E of X, we define the canonical surjection $Q_E : X \longrightarrow X/E$ by $Q_E(x) := \overline{x}$ for all $x \in X$. Apart from the operator $Q_T T$, there is another operator $A \in LO(X,Y)$ that facilitates the study of T, called the linear selection of T, and defined by $T = A + T - T$ and $\mathcal{D}(A) = \mathcal{D}(T)$.

Proposition 2.1 *Let $T \in CR_c(X)$ and $\widetilde{P}$ be a bounded linear projection such that $Ker(\widetilde{P}) = T(0)$. Then, $\widetilde{P}T$ is a continuous linear selection of T.*

In the next definition, we refer to [8] to define the spectrum, the resolvent set and the resolvent function of a closed linear relation T, and we introduce the full spectrum of T.

Definition 2.1 *Let $T \in CR(X)$. The resolvent set of T is denoted by $\rho(T)$ and defined as the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is invertible. For $\lambda \in \rho(T)$, we define the resolvent function of T by $R(\lambda, T) = (T - \lambda I)^{-1}$. The spectrum of T is the set $\sigma(T) := \mathbb{C} \setminus \rho(T)$. The full spectrum of T is denoted by $\sigma_f(T)$ and defined as the union of $\sigma(T)$ and all bounded connected components of $\rho(T)$. The set $\rho_f(T) := \mathbb{C} \setminus \sigma_f(T)$ is called the full resolvent set.*

To define the extended spectral decomposable linear relations, we are going to use subsets of the extended complex plane, so we brought back to work not only the spectrum but also the extended spectrum defined by A. Baskakov in [5] as follows:

Definition 2.2 *[5, Definition1.5] The extended spectrum of a relation $T \in CR(X)$ is a subset of $\mathbb{C}_\infty$ denoted by $\widetilde{\sigma}(T)$ which coincides with the spectrum $\sigma(T)$ of T if the following conditions are fulfilled:*

1. *T(0)=0;*
2. *The resolvent $R(.,T)$ of T can be analytically continued to ∞ and $R(\mu, T) \underset{|\mu| \to +\infty}{\longrightarrow} 0$.*

Otherwise, $\widetilde{\sigma}(T) = \sigma(T) \cup \{\infty\}$. The extended resolvent set of T is defined as $\mathbb{C}_\infty \setminus \widetilde{\sigma}(T)$.

Lemma 2.1 *[5, Lemma 2.2.] For every linear relation $T \in CR(X)$, the following equivalence holds:*

$$T \in End(X) \textit{ if and only if } \infty \notin \widetilde{\sigma}(T).$$

2.1. Invariant linear subspaces

This subsection is devoted to the study of two types of invariant linear subspaces with respect to a closed linear relation; these two types were introduced by A. G. Baskakov and A. S. Zagorskii in [7]. We recall that, for a relation $T \in LR(X)$, a linear subspace M of X is said to be T-invariant if $T(M) \subseteq M$.

Definition 2.3 *Let $T \in CR(X)$ and let M be a closed linear subspace of X. Then,*

1. *M is said to be T-weakly invariant subspace if $T(x) \cap M \neq \emptyset$ for all $x \in \mathcal{D}(T) \cap M$.*
2. *M is said to be T-strongly invariant subspace if $\rho(T) \neq \emptyset$ and $R(\lambda, T)M \subseteq M$ for all $\lambda \in \rho(T)$.*

We note that the T-weakly invariant subspaces are the T-invariant ones if $T \in CO(X)$.

Definition 2.4 *Let $T \in CR(X)$. Then,*

1. *The* weak restriction *of the relation T to a T-weakly invariant subspace $M \subseteq X$ is the relation $T|_w M \in CR(M)$ whose graph is given by $G(T|_w M) = G(T) \cap (M \times M)$.*
2. *The* strong restriction *of the relation T to a T-strongly invariant subspace $M \subseteq X$ is the relation $T|_s M \in CR(M)$ whose resolvent is* $\begin{array}{rccc} R(.,T|_s M): & \rho(T) & \longrightarrow & End(M) \\ & \lambda & \longrightarrow & R(\lambda,T)|M. \end{array}$

Proposition 2.1 *Let $T \in CR(X)$. Then, for every T-weakly and strongly invariant linear subspace $X_1 \subseteq X$, we have*

$$\mathcal{D}(T|_s X_1) = X_1 \cap \mathcal{D}(T).$$

Proof As X_1 is a T-weakly invariant subspace, we have $T(X_1) \subseteq X_1 + T(0)$, and therefore $(T - \lambda I)(X_1 \cap \mathcal{D}(T)) \subseteq X_1 + T(0)$ for all $\lambda \in \rho(T)$. Hence, for all $\lambda \in \rho(T)$, we have

$$X_1 \cap \mathcal{D}(T) = R(\lambda, T)(T - \lambda I)(X_1 \cap \mathcal{D}(T)) \subseteq R(\lambda, T)(X_1 + T(0)) \subseteq X_1 \cap \mathcal{D}(T).$$

Therefore, for all $\lambda \in \rho(T)$, we obtain that $X_1 \cap \mathcal{D}(T) = R(\lambda, T)X_1 = R(\lambda, T|_s X_1)X_1 = \mathcal{D}(T|_s X_1)$, as desired. □

An essential link between a closed linear relation and its strong restriction is given in the following proposition.

Proposition 2.2 *Consider $T \in CR(X)$ for which $\rho(T) \neq \emptyset$ and let $Y \subseteq X$ be a T-strongly invariant subspace. Then, for all $x \in \mathcal{D}(T|_s Y)$, we have*

$$T|_s Yx \subseteq Tx$$

with equality if $T(0) \subseteq Y$.

Proof We first observe that $\rho(T) \subseteq \rho(T|_s Y)$, by [6, Remark 2.7]. Let $x \in \mathcal{D}(T|_s Y)$, $y \in T|_s Yx$ and let $\lambda \in \rho(T)$ be arbitrarily given. Then, $y - \lambda x \in (T|_s Y - \lambda I)x$, and therefore

$$R(\lambda, T)(y - \lambda x) = R(\lambda, T)(T|_s Y - \lambda I)x = R(\lambda, T|_s Y)(T|_s Y - \lambda I)x = x. \tag{2.1}$$

Applying $T - \lambda I$ on (2.1), we obtain that $y \in Tx$. Hence, $T|_s Yx \subseteq Tx$. To establish the other inclusion, suppose that $T(0) \subseteq Y$. We claim that $T(0) = T|_s Y(0)$. Indeed, it follows from the above that $T|_s Y(0) \subseteq T(0)$. To prove the other inclusion, consider an arbitrary $y \in T(0) \subseteq Y$. Then, for all $\lambda \in \rho(T)$, we have $y \in (T - \lambda I)(0)$, and hence $R(\lambda, T)y = 0$. Since $y \in Y$, it follows that $R(\lambda, T)y = R(\lambda, T|_s Y)y = 0$ for all $\lambda \in \rho(T)$. Consequently, for all $\lambda \in \rho(T)$, we have $y \in (T|_s Y - \lambda I)(0) = T|_s Y(0)$. We conclude that $T(0) \subseteq T|_s Y(0)$, as claimed. Since $T|_s Yx \subseteq Tx$, it follows that $Tx = T|_s Yx + T(0) = T|_s Yx + T_s|Y(0) = T|_s Yx$. □

2.2. Complex analysis framework

This subsection is devoted to recalling some basic definitions and properties about the extended complex plane $\mathbb{C}_\infty$ which can be found, essentially, in O. Forster [12], R. M. Timoney [24] and [13]. The typical distance used is called the chordal distance or chordal metric on the Riemann sphere, denoted by d_∞ and defined as

$$d_\infty(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}} \quad \text{for all } z_1, z_2 \in \mathbb{C};$$

$$d_\infty(z, \infty) = \frac{2}{\sqrt{(1+|z|^2)}} \quad \text{for all } z \in \mathbb{C}.$$

With reference to [12], the neighborhoods in $\mathbb{C}_\infty$ correspond exactly to the usual definitions of neighborhoods in $\mathbb{C}$ and add that neighborhoods of ∞ are defined as sets of the form $U_\infty = \{z \in \mathbb{C}\ ;\ |z| > r'\} \cup \{\infty\}$, where $r' > 0$. According to [13], the open sets in $\mathbb{C}_\infty$ are the usual open sets in $\mathbb{C}$ and those of the form $O \cup \{\infty\}$, where $\mathbb{C}\backslash O$ is a compact subset of $\mathbb{C}$. Moreover, it follows from [24, Proposition 6.7] that, a set $U \subseteq \mathbb{C}_\infty$ is open in $(\mathbb{C}_\infty, d_\infty)$ if and only if $U \cap \mathbb{C}$ is open in $(\mathbb{C}, |.|)$, and whenever $\infty \in U$, there exists $r' > 0$ such that $\{z \in \mathbb{C}\ ;\ |z| > r'\} \subseteq U$.
Given arbitrary $\lambda_0 \in \mathbb{C}$, $\lambda_1 \in \mathbb{C}_\infty$ and $r > 0$, we write $B_{\mathbb{C}}(\lambda_0, r) := \{\lambda \in \mathbb{C}\ ;\ |\lambda_0 - \lambda| < r\}$, $\overline{B}_{\mathbb{C}}(\lambda_0, r) := \{\lambda \in \mathbb{C}\ ;\ |\lambda_0 - \lambda| \leqslant r\}$, and $B_{\mathbb{C}_\infty}(\lambda_1, r) := \{\lambda \in \mathbb{C}_\infty\ ;\ d_\infty(\lambda_1, \lambda) < r\}$.

Lemma 2.2 *Let $\lambda_0 \in \mathbb{C}_\infty$ and let $r \in]0, 2[$. Then,*

1. $B_{\mathbb{C}_\infty}(\lambda_0, r) = B_{\mathbb{C}}\left(\dfrac{4\ \lambda_0}{4 - r^2(1+|\lambda_0|^2)}, \dfrac{r(1+|\lambda_0|^2)\sqrt{(4-r^2)}}{4 - r^2(1+|\lambda_0|^2)}\right)$, *if* $\lambda_0 \in \mathbb{C}$ *and* $|\lambda_0| < \sqrt{\dfrac{4}{r^2} - 1}$.

2. $B_{\mathbb{C}_\infty}(\lambda_0, r) = \mathbb{C} \setminus \overline{B}_{\mathbb{C}}\left(0, \sqrt{\frac{4}{r^2} - 1}\right) \cup \{\infty\}$, *if* $\lambda_0 = \infty$.

Definition 2.5 *[24, Defintion 6.12] Given an arbitrary open set $U \subseteq \mathbb{C}_\infty$, we say that a function $f : U \to \mathbb{C}$ is analytic if*

(i) *f is continuous (from U with the metric d_∞ to $\mathbb{C}$);*

(ii) *f is analytic on $(U \cap \mathbb{C}, |.|)$;*

(iii) *$z \to f(\frac{1}{z})$ is analytic on $\{z \in \mathbb{C};\ \dfrac{1}{z} \in U\}$.*

For an open set $U \subseteq \mathbb{C}$, we denote by $H(U, \mathbb{C})$ the space of all analytic functions $f : U \subseteq (\mathbb{C}, |.|) \longrightarrow \mathbb{C}$ and by $H_\infty(U, \mathbb{C})$ the space of all analytic functions $f : U \subseteq (\mathbb{C}, d_\infty) \longrightarrow \mathbb{C}$. For an open $U_\infty \subseteq \mathbb{C}_\infty$, the space of all analytic functions $f : U_\infty \subseteq (\mathbb{C}_\infty, d_\infty) \longrightarrow \mathbb{C}$ is denoted by $H^\sharp(U_\infty, \mathbb{C})$.

Let Z be a complex topological vector space. With reference to [20], for an open set U of $(\mathbb{C}, |.|)$, a function $f : U \to Z$ is said to be analytic if $\varphi \circ f$ is analytic for every $\varphi \in Z^*$, where Z^* denotes the topological dual space of Z. We denote by $H(U, Z)$ the space of all analytic functions from an open set $U \subseteq (\mathbb{C}, |.|)$ into Z. According to [17], if X is a Banach space, $H(U, X)$ becomes a Fréchet space with respect to pointwise vector space operations and the topology of locally uniform convergence. Furthermore, if $f \in H(U, X)$, we infer from [17, Theorem A.3.1] that, for each $\varepsilon > 0$ and each $\mu_0 \in U$ such that $B_{\mathbb{C}}(\mu_0, \varepsilon) \subseteq U$, we have $f(\mu) = \displaystyle\sum_{n=0}^{\infty} x_n(\mu - \mu_0)^n$ for all $\mu \in B_{\mathbb{C}}(\mu_0, \varepsilon)$ with $(x_n)_{n \in \mathbb{N}} \subseteq X$. We denote by $H_\infty(U, X)$ the space of all analytic functions $f : U \subseteq (\mathbb{C}, d_\infty) \to X$. This space has the same properties as $H(U, X)$.

Given an open set U_∞ in $(\mathbb{C}_\infty, d_\infty)$, a function $f : U_\infty \to X$ is said to be analytic if $\varphi \circ f$ is analytic for all $\varphi \in X^*$. We denote by $H^\sharp(U_\infty, X)$ the space of all analytic functions $f : U_\infty \to X$ and we define the space $P(U_\infty, X)$ as follows: $P(U_\infty, X) = \{f \in H^\sharp(U_\infty, X) \mid f(\infty) = 0 \text{ if } \infty \in U_\infty\}$. According to [17], topologized by means of uniform convergence on all compact subsets of the open set U_∞, the space $H^\sharp(U_\infty, X)$ is a Fréchet space whenever X is a Banach space.

Lemma 2.3 *Let X and Y be two Banach spaces and let X_0 and X_1 be two closed linear subspaces of X. Then, for all open disc $E := B_{\mathbb{C}_\infty}(\lambda_0, r) \subseteq \mathbb{C}_\infty$, where $\lambda_0 \in \mathbb{C}_\infty$, $0 < r < 2$ and $r \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$, the following assertions hold:*

i/ If $A : X \to Y$ is a bounded and surjective linear operator then the operator $\widetilde{A} : P(E, X) \to P(E, Y)$ defined by:

$$\widetilde{A}(f)(\lambda) = Af(\lambda) \quad \text{for all } \lambda \in E \text{ and } f \in P(E, X),$$

is a surjective linear operator.

ii/ $P(E, X/X_0) \cong P(E, X)/P(E, X_0)$.

iii/ If $X = X_0 + X_1$, then the operator $\psi : P(E, X_0) \times P(E, X_1) \to P(E, X)$ defined by

$$\psi(f, g) = f + g \qquad \text{for all } (f, g) \in P(E, X_0) \times P(E, X_1)$$

is a surjective linear operator.

Proof

i/ The linearity of $\widetilde{A}$ follows immediately from that of A. To prove the surjectivity of $\widetilde{A}$, consider $h \in P(E, Y)$ and discuss all the possible cases of E, studied in Lemma 2.2:

Case 1 : $E = \left(\mathbb{C} \setminus \overline{B}_{\mathbb{C}}(0, \sqrt{\frac{4}{r^2} - 1})\right) \cup \{\infty\}$. In this case, for all $\lambda \in E \cap \mathbb{C}$, we have $\frac{1}{|\lambda|} < \frac{1}{\sqrt{\frac{4}{r^2} - 1}}$, and therefore, for all $\lambda \in E$, we obtain that

$$h(\lambda) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{y_n}{\lambda^n}, & \text{if } \lambda \in \mathbb{C} \setminus \overline{B}_{\mathbb{C}}(0, \sqrt{\frac{4}{r^2} - 1}) \\ 0, & \text{if } \lambda = \infty, \end{cases}$$

where $(y_n)_{n \in \mathbb{N}} \subseteq Y$. Since $h(\infty) = 0$, we conclude that $y_0 = 0$, and hence

$$h(\lambda) = \sum_{i=1}^{\infty} \frac{y_n}{\lambda^n} \quad \text{for all } \lambda \in E.$$

Because A is a bounded surjective linear operator, it follows from [8, Theorem.III.4.2] that A is open. Therefore there exists $c_0 > 0$ such that, for all $y \in Y$, there exists $x \in X$ for which $Ax = y$ and $\|x\| \leqslant c_0 \|y\|$. Hence, there exists $(x_n)_{n \in \mathbb{N}^*} \subseteq X$ with the property that $Ax_n = y_n$ and $\|x_n\| \leqslant c_0 \|y_n\|$ for all $n \in \mathbb{N}^*$. Let f be the function given by $f(\lambda) := \displaystyle\sum_{n=1}^{\infty} \frac{x_n}{\lambda^n}$. Then,

$$\limsup_{n \to \infty} \|x_n\|^{\frac{1}{n}} \leqslant \limsup_{n \to \infty} c_0^{\frac{1}{n}} \|y_n\|^{\frac{1}{n}} \leqslant \limsup_{n \to \infty} \|y_n\|^{\frac{1}{n}} < +\infty,$$

which entails that $f \in P(E,X)$. Moreover, for all $\lambda \in E$, we have

$$(\widetilde{A}f)(\lambda) = A(\sum_{n=1}^{\infty}\frac{x_n}{\lambda^n}) = \sum_{n=1}^{\infty}\frac{Ax_n}{\lambda^n} = \sum_{n=1}^{\infty}\frac{y_n}{\lambda^n} = h(\lambda).$$

Consequently, $\widetilde{A}$ is surjective.

Case 2 : $E = B_{\mathbb{C}}\left(\frac{4\lambda_0}{4-r^2(1+|\lambda_0|^2)}, r_1\right)$ where $|\lambda_0| < \sqrt{\frac{4}{r^2}-1}$ and $r_1 = \frac{r(1+|\lambda_0|^2)\sqrt{(4-r^2)}}{4-r^2(1+|\lambda_0|^2)}$. In this case, the result follows immediately from [17, Proposition 1.2.1].

ii/ Since X_0 is a closed subspace of X, and $Q_{X_0} : X \to X/X_0$ is a bounded and surjective linear operator, then it follows from $i/$ that $\widetilde{Q_{X_0}} : P(E,X) \to P(E,X/X_0)$ is a surjective linear operator. On the other hand, we have $Ker(\widetilde{Q_{X_0}}) = P(E,X_0)$. Consequently, the induced operator $\widehat{Q_{X_0}} : P(E,X)/P(E,X_0) \to P(E,X/X_0)$ is an isomorphism, and therefore $P(E,X)/P(E,X_0) \cong P(E,X/X_0)$.

iii/ If $X = X_0 + X_1$, then the operator $J : X_0 \times X_1 \to X$ given by $J(x_0,x_1) := x_0 + x_1$ for all $(x_0,x_1) \in X_0 \times X_1$ is surjective and bounded. Thus, by $i/$, it follows that $\widetilde{J} : P(E, X_0 \times X_1) \to P(E,X)$ is a surjective linear operator. As $P(E, X_0 \times X_1) \cong P(E,X_0) \times P(E,X_1)$, we obtain that ψ is surjective.

□

3. Extended spectral decomposable linear relations

Enlightened by the work of [11] and [15], we not only extend the notion of the decomposability to the case of multivalued linear operators and the extended complex plane, but we also weaken certain conditions of the invariant linear subspaces in question. In fact, we are going to consider strongly invariant linear subspaces instead of spectral maximal spaces (see [11] for more details).

Definition 3.1 *Let $T \in CR_c(X)$ be a linear relation such that $\rho(T) \neq \emptyset$. We say that T is extended spectral decomposable if every open cover $\mathbb{C}_\infty = U \cup V$ of the extended complex plane by a bounded open subset U of $\mathbb{C}$ and an open subset V of $\mathbb{C}_\infty$ effects a splitting of the extended spectrum $\widetilde{\sigma}(T)$ and of the space X, in the sense that there exist T-strongly invariant closed linear subspaces Y and Z of X with the following properties:*

i/ $\widetilde{\sigma}(T|_sY) = \sigma(T|_sY) \subseteq U$, $\widetilde{\sigma}(T|_sZ) \subseteq V$;

ii/ $T(Z) \subseteq Z$ *and* $\widetilde{P}(Z) \subseteq Z$ *for some bounded linear projection* $\widetilde{P} : X \longrightarrow X$ *verifying* $R(\widetilde{P}) = \mathcal{D}(T)$;

iii/ $X = Y + Z$.

It is very important to note that every decomposable bounded operator T in the sense of [11] is extended spectral decomposable in the sense of Definition 3.1. To see this, it suffices to verify that the restrictions of T coincide with the strong restrictions and then, combine Theorem 1 of [15] with Lemma 2.1 to obtain the desired result.

Remark 3.1 *If T is extended spectral decomposable then one can prove, by using Lemma 2.1 and Proposition 2.1, that the subspaces Y and Z still check the following properties:*

(1) $\mathcal{D}(T|_sZ) = Z \cap \mathcal{D}(T)$; (2) $\mathcal{D}(T) = \mathcal{D}(T|_sY) + \mathcal{D}(T|_sZ)$; (3)$T = T|_sY \widehat{+} T|_sZ$.

Example 3.1 *If $P \in End(X)$ is a bounded linear projection such that $0 < dim(Ker(P)) < \infty$, then P^{-1} is an extended spectral decomposable linear relation.*

Proof Let $T = P^{-1}$. We first observe that $\sigma(P) = \{0,1\}$ so $\widetilde{\sigma}(T) = \{1\} \cup \{\infty\}$, by [5, Theorem 2.4], and hence $\rho(T) \neq \emptyset$. Also, T has the property of stabilization of powers at infinity with order of stabilization 1. From a combination of [5, Theorem 4.1] and [5, Theorem 4.2], we deduce that there exist two T-strongly invariant closed linear subspaces $X_0 = \overline{\mathcal{D}(T)}$ and $X_1 = T(0)$, such that

$$T|_s X_0(0) = \{0\}, \quad \widetilde{X} = X_0 \oplus X_1, \quad \sigma(T|_s X_0) = \sigma(T) = \{1\}, \quad \text{and} \quad \widetilde{\sigma}(T|_s X_1) = \{\infty\},$$

where $\widetilde{X} = \{x \in X \ ; \ \exists \lim_{n\to\infty} (I - (-\lambda_n R(\mu_n, T)))\, x\}$. But $T \in CR_c(X)$, and hence $\mathcal{D}(T) = \overline{\mathcal{D}(T)} = X_0$ thanks to [8, Theorem.III.4.2]. Consequently, $T|_s X_0 \in End(X_0)$, or equivalently, $\widetilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0)$. On the other hand, we have $X = T(0) \oplus \mathcal{D}(T)$. Then, from the fact that $dim(Ker(P)) < \infty$ and from [5, Theorem 4.3], we conclude that $X = \widetilde{X}$. Now, consider an open cover $\{U, V\}$ of $\mathbb{C}_\infty$ with the property that U is an open bounded subset of $\mathbb{C}$ and V is an open subset of $\mathbb{C}_\infty$. To establish the extended spectral decomposability of T, we separate the following two cases.

Case.1: $1 \in U$. We have $X = X_0 + X_1$ and we observe that $P(X_1) = P(T(0)) = P(Ker(P)) = \{0\} \subseteq X_1$. In addition, we have $T(X_1) = T(T(0)) = T^2(0) = T(0) \subseteq X_1$. On the other hand, we have $\widetilde{\sigma}(T|_s X_0) = \sigma(T|_s X_0) = \{1\} \subseteq U$ and $\widetilde{\sigma}(T|_s X_1) = \{\infty\} \subseteq V$.

Case.2: $1 \in V$. In this case, it is sufficient to consider the two T-strongly invariant linear subspaces $X'_0 = \{0\}$ and $X'_1 = X$. We have, $X = X'_0 + X'_1$ and it is easy to observe that $P(X'_1) \subseteq X'_1$ and $T(X'_1) \subseteq X'_1$. It is also easy to verify that $\widetilde{\sigma}(T|_s X'_0) = \sigma(T|_s X'_0) = \emptyset \subseteq U$ and $\widetilde{\sigma}(T|_s X'_1) = \widetilde{\sigma}(T) = \{1\} \cup \{\infty\} \subseteq V$.

Hence, T is an extended spectral decomposable linear relation. □

3.1. Necessary conditions for the extended spectral decomposable linear relations

In this subsection, we give three necessary conditions of an extended spectral decomposable linear relation. Let us first begin with the extended Bishop's property.

3.1.1. Extended Bishop's property ($\mathcal{E}_\beta$)

As an extension of definitions [17, Definition 1.2.5] and [1, Definition 2.168] of Bishop's property relative to the bounded linear operators, the extended Bishop's property is defined for continuous closed linear relations, not necessary bounded as follows.

Definition 3.2 *Let $T \in CR_c(X)$. We say that T has the extended Bishop's property ($\mathcal{E}_\beta$) if, for every open set $U \subseteq \mathbb{C}_\infty$ and all sequences of analytic functions $(f_n)_{n\in\mathbb{N}} \subseteq P(U, \mathcal{D}(T)), (g_n)_{n\in N} \subseteq H^\sharp(U, X)$ with the property that $g_n(\lambda) \in (T - \lambda I) f_n(\lambda)$ for all $\lambda \in U \cap \mathbb{C}$, we have the following implication*

$$g_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } H^\sharp(U, X) \quad \Longrightarrow f_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } P(U, \mathcal{D}(T)).$$

Next, we present two statements which are related to the extended Bishop's property ($\mathcal{E}_\beta$) :

Statement 3.1 *For $T \in CR_c(X)$ and for all $\lambda_0 \in \mathbb{C}_\infty$, there exists a small enough $0 < \varepsilon_{\lambda_0} \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$ such that for every open subset $U \subseteq \mathbb{C}_\infty$ of $B_{\mathbb{C}_\infty}(\lambda_0, \varepsilon_{\lambda_0})$ and for all sequences of analytic functions $(f_n) \subseteq P(U, \mathcal{D}(T)), (g_n) \subseteq H^\sharp(U, X)$ with the property that $g_n(\lambda) \in (T - \lambda I) f_n(\lambda)$ for al $\lambda \in U \cap \mathbb{C}$, the following implication holds:*

$$g_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } H^\sharp(U, X) \quad \Longrightarrow f_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } P(U, \mathcal{D}(T)). \tag{3.1}$$

Statement 3.2 *For $T \in CR_c(X)$ and for every open disc $D \subseteq \mathbb{C}_\infty$ centered at $\lambda_0 \in \mathbb{C}_\infty$ with radius $0 < r \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$ chosen sufficiently small, and for all sequences of analytic functions $(f_n) \subseteq P(D, \mathcal{D}(T)), (g_n) \subseteq H^\sharp(D, X)$ with the property that $g_n(\lambda) \in (T - \lambda I)f_n(\lambda)$ for all $\lambda \in D \cap \mathbb{C}$, we have the following implication*

$$g_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } H^\sharp(D, X) \quad \Longrightarrow f_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } P(D, \mathcal{D}(T)).$$

In the following lemma, we are going to show that the two statements given above are equivalent to each other and to the definition of the extended Bishop's property given in Definition 3.2.

Lemma 3.1 *Let $T \in CR_c(X)$. Then, the following statements are equivalent:*

1/ *Statement 3.2 holds.*

2/ *T has the extended Bishop's property $(\mathcal{E}_\beta)$.*

3/ *Statement 3.1 holds.*

Proof

Proving 1/ implies 2/. Suppose that the statement 3.2 holds and let U be an open subset of $\mathbb{C}_\infty$ and $(f_n) \subseteq P(U, \mathcal{D}(T)), (g_n) \subseteq H^\sharp(U, X)$ be two sequences of analytic functions such that

$$g_n(\lambda) \in (T - \lambda I)f_n(\lambda) \quad \text{for all } \lambda \in U \cap \mathbb{C}, \quad \text{ and } \quad g_n \underset{n\to+\infty}{\longrightarrow} 0 \text{ in } H^\sharp(U, X).$$

Consider a closed subset $F \subseteq \mathbb{C}_\infty$ of U. Then, because U is open, for all $\lambda_0 \in F$, there exists $\varepsilon_{\lambda_0} > 0$ small enough such that $\lambda_0 \in B_{\mathbb{C}_\infty}(\lambda_0, \varepsilon_{\lambda_0}) \subseteq U$ and $\varepsilon_{\lambda_0} \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$. Hence, $F \subseteq \bigcup_{\lambda_0 \in F} B_{\mathbb{C}_\infty}(\lambda_0, \varepsilon_{\lambda_0}) \subseteq U$.

Since F is a compact set in $\mathbb{C}_\infty$, there exists $1 \leqslant p < \infty$ for which $F \subseteq \bigcup_{\substack{i=1\\ \lambda_i \in F}}^{p} B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i}) \subseteq U$. Moreover, because $B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i})$ is open for all λ_i, we may choose a sufficiently small $0 < r_{\lambda_i} \neq \frac{2}{\sqrt{1+|\lambda_i|^2}}$ for which

$$\overline{B_{\mathbb{C}_\infty}(\lambda_i, r_{\lambda_i})} \subseteq B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i})$$

and

$$F \subseteq \bigcup_{\substack{i=1\\ \lambda_i \in F}}^{p} \overline{B_{\mathbb{C}_\infty}(\lambda_i, r_{\lambda_i})} \subseteq \bigcup_{\substack{i=1\\ \lambda_i \in F}}^{p} B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i}) \subseteq U.$$

Therefore,

$$\sup_{\lambda \in F} \|f_n(\lambda)\| \leqslant \sum_{i=1}^{p} \sup_{\lambda \in \overline{B_{\mathbb{C}_\infty}(\lambda_i, r_{\lambda_i})}} \|f_n(\lambda)\|. \tag{3.2}$$

In addition, we know that, for all $(\lambda_i)_{\{1 \leqslant i \leqslant p\}} \subseteq F$ we have

$$g_n(\lambda_i) \in (T - \lambda_i)f_n(\lambda_i) \quad \text{for all} \quad \lambda_i \in D_i \cap \mathbb{C}, \quad \text{ and } \quad g_n \underset{n\to+\infty}{\longrightarrow} 0 \text{ in } H^\sharp(D_i, X),$$

where $D_i = B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i})$. Since the statement 3.2 holds and $\overline{B_{\mathbb{C}_\infty}(\lambda_i, r_{\lambda_i})}$ is a compact subset of D_i, for λ_i defined above, it follows that $f_n \underset{n\to+\infty}{\longrightarrow} 0$ uniformly on F, by (3.2). This entails that T has the extended Bishop's property. The implication 2/ leads to 3/ is obvious. Finally, let us show that 3/ implies 1/. Let $\lambda_0 \in \mathbb{C}_\infty$, $0 < r \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$

be sufficiently small and let $(h_n) \subseteq P(B_{\mathbb{C}_\infty}(\lambda_0, r), \mathcal{D}(T))$, $(t_n) \subseteq H^\sharp(B_{\mathbb{C}_\infty}(\lambda_0, r), X)$ be two sequences of functions such that $t_n \underset{n\to+\infty}{\longrightarrow} 0$ in $H^\sharp(B_{\mathbb{C}_\infty}(\lambda_0, r), X)$ and $t_n(\lambda) \in (T - \lambda I)h_n(\lambda)$ for all $\lambda \in B_{\mathbb{C}_\infty}(\lambda_0, r) \cap \mathbb{C}$. To show Statement 3.2, it remains to be shown that $h_n \underset{n\to+\infty}{\longrightarrow} 0$ in $P(B_{\mathbb{C}_\infty}(\lambda_0, r), \mathcal{D}(T))$. To this end, let F be a closed subset of $B_{\mathbb{C}_\infty}(\lambda_0, r)$. Because $B_{\mathbb{C}_\infty}(\lambda_0, r)$ is open and Statement 3.1 holds, we conclude that, for all $\lambda \in F$, there exists a sufficiently small $0 < \varepsilon_\lambda \neq \frac{2}{\sqrt{1+|\lambda_0|^2}}$ such that

$$\lambda \in B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda) \subseteq B_{\mathbb{C}_\infty}(\lambda_0, r), \tag{3.3}$$

and for every $((f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}) \in P(B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda), \mathcal{D}(T)) \times H^\sharp(B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda), X)$ verifying

$$g_n(\mu) \in (T - \mu I)f_n(\mu) \quad \text{for all } \mu \in B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda) \cap \mathbb{C},$$

the implication (3.1) of Statement 3.1 holds for $U = B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda)$. By (3.3), we see that $F \subseteq \bigcup_{\lambda\in F} B_{\mathbb{C}_\infty}(\lambda, \varepsilon_\lambda) \subseteq B_{\mathbb{C}_\infty}(\lambda_0, r)$. As F is compact in $\mathbb{C}_\infty$, we conclude that

$$F \subseteq \bigcup_{\substack{i=1 \\ \lambda_i\in F}}^{p} U_i \cap F \subseteq B_{\mathbb{C}_\infty}(\lambda_0, r),$$

where $U_i = B_{\mathbb{C}_\infty}(\lambda_i, \varepsilon_{\lambda_i}) \cap B_{\mathbb{C}_\infty}(\lambda_0, r)$. Consequently,

$$\sup_{\lambda\in F} \|h_n(\lambda)\| \leqslant \sum_{i=1}^{p} \sup_{\lambda\in F\cap U_i} \|h_n(\lambda)\|. \tag{3.4}$$

From the fact that $F \cap U_i$ is a closed subset of U_i for all $\{\lambda_i\}_{1\leqslant i\leqslant p} \subseteq F$ and from the hypothesis on the sequence $(t_n)_{n\in\mathbb{N}}$, we deduce that

$$\sup_{\lambda\in F\cap U_i} \|h_n(\lambda)\| \underset{n\to+\infty}{\longrightarrow} 0.$$

This proves that $h_n \underset{n\to+\infty}{\longrightarrow} 0$ uniformly on F, by (3.4). Therefore, Statement 3.2 holds.

□

Definition 3.3 *For every $T \in CR(X)$ and for every open set $U \subseteq \mathbb{C}_\infty$, we define the relation T_U by*

$$\begin{array}{rccl} T_U : & P(U, \mathcal{D}(T)) & \longrightarrow & H^\sharp(U, X) \\ & f & \longrightarrow & T_U(f) = \{g \in H^\sharp(U, X) \mid g(\lambda) \in (T - \lambda I)f(\lambda)\ \forall \lambda \in U \cap \mathbb{C}\}. \end{array}$$

Proposition 3.1 *For all open set U in $\mathbb{C}_\infty$ and for every $T \in CR(X)$, we have $T_U \in LR\left(P(U, \mathcal{D}(T)), H^\sharp(U, X)\right)$. Moreover, if T is continuous and $T(0)$ is complemented in X, then the following assertions hold:*

(1) $\mathcal{D}(T_U) = P(U, \mathcal{D}(T))$.

(2) *$G(T_U)$ is closed.*

Proof Let $f_1, f_2 \in \mathcal{D}(T_U)$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned} \alpha T_U(f_1) + \beta T_U(f_2) =& \big\{\alpha g_1 + \beta g_2 \in H^\sharp(U, X) \mid g_1 \in T_U(f_1) \text{ and } g_2 \in T_U(f_2)\big\} \\ \subseteq& \big\{\alpha g_1 + \beta g_2 \in H^\sharp(U, X) \mid (\alpha g_1 + \beta g_2)(\lambda) \in (T - \lambda I)(\alpha f_1 + \beta f_2)(\lambda) \forall \lambda \in U \cap \mathbb{C}\big\} \\ \subseteq& \ T_U(\alpha f_1 + \beta f_2). \end{aligned}$$

This proves that $T_U \in LR(P(U, \mathcal{D}(T)), H^\sharp(U, X))$. Suppose now that $T \in CR_c(X)$ and $T(0)$ is complemented.

(1) Let $f \in P(U, \mathcal{D}(T))$. We separate the following two cases.

If $\infty \notin U$, then $f \in H_\infty(U, \mathcal{D}(T))$. Because $T(0)$ is complemented, we conclude from Proposition 2.1 that T admits a continuous selection A. Therefore, the function $g : U \to X$ given by $g(\lambda) := (A - \lambda I)f(\lambda)$ for all $\lambda \in U$ belongs to $H_\infty(U, X) \subseteq H^\sharp(U, X)$ and verifies $g(\lambda) \in (T - \lambda I)f(\lambda)$ for all $\lambda \in U \cap \mathbb{C}$. This means that $g \in T_U(f)$, and hence $f \in \mathcal{D}(T_U)$.

If $\infty \in U$, then there exists an open neighborhood V_∞ of ∞ in U with the property that, for all $\lambda \in V_\infty$, we have $f(\lambda) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda^n}$, where $(a_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$. As $f(\infty) = 0$, it follows that, for all $\lambda \in V_\infty$, we have $f(\lambda) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda^n}$. This yields that

$$(A - \lambda I)f(\lambda) = Af(\lambda) - \lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda^n} = Af(\lambda) - \lambda \sum_{n=0}^{\infty} \frac{a_{n+1}}{\lambda^{n+1}} = Af(\lambda) - \sum_{n=0}^{\infty} \frac{a_{n+1}}{\lambda^n},$$

and therefore $\lim_{\lambda\to\infty} (A - \lambda I)f(\lambda) = a_1$. As a result, the function

$$\begin{array}{rccl} g: & U & \to & X \\ & \lambda & \mapsto & \begin{cases} (A - \lambda I)f(\lambda) & \text{if } \lambda \in U \cap \mathbb{C} \\ a_1 & \text{if } \lambda = \infty \end{cases} \end{array}$$

belongs to $H^\sharp(U, X)$. Also, for all $\lambda \in U \cap \mathbb{C}$, we have $g(\lambda) = (A - \lambda I)f(\lambda) \in (T - \lambda I)f(\lambda)$, which entails that $g \in T_U(f)$. Thus, $f \in \mathcal{D}(T_U)$.

(2) Since $T \in CR_c(X)$, it follows from [8, Theorem.III.4.2] that $\mathcal{D}(T)$ is closed, so $P(U, \mathcal{D}(T))$ and $H^\sharp(U, \mathcal{D}(T))$ are Fréchet spaces. Thus, to show that $G(T_U)$ is closed, it suffices to prove that $G(T_U)$ is sequentially closed. To this end, consider $(f_n, g_n)_{n\in\mathbb{N}} \subseteq G(T_U)$ such that $(f_n, g_n) \underset{n\to+\infty}{\longrightarrow} (f, g)$ locally uniformly on U. Then, f_n converges pointwise to f on U and, for all $\lambda \in U \cap \mathbb{C}$, we have

$$g_n(\lambda) \in (T - \lambda I)f_n(\lambda). \tag{3.5}$$

Now let $\lambda \in U \cap \mathbb{C}$. Then, we infer from (3.5) that $Q_{T-\lambda I}g_n(\lambda) = Q_{T-\lambda I}(T - \lambda I)f_n(\lambda)$. Also, since T is continuous, it follows that $Q_{T-\lambda I}g_n(\lambda)$ converges to $Q_{T-\lambda I}(T - \lambda I)f(\lambda)$ on U. On the other hand, $Q_{T-\lambda I}g_n(\lambda)$ converges to $Q_{T-\lambda I}g(\lambda)$ on U, and hence $Q_{T-\lambda I}(T - \lambda I)f(\lambda) = Q_{T-\lambda I}g(\lambda)$. Consequently, $g(\lambda) \in (T - \lambda I)f(\lambda)$ for all $\lambda \in U \cap \mathbb{C}$, which proves that $g \in T_U(f)$, and hence $(f, g) \in G(T_U)$.

□

In the next theorem, we give a characterization of a linear relation having the extended Bishop's property.

Theorem 3.1 *For a linear relation $T \in CR_c(X)$, the following assertions hold:*

i/ If T has property $(\mathcal{E}_\beta)$ then, for every open set $U \subseteq \mathbb{C}_\infty$, the relation T_U on $P(U, \mathcal{D}(T))$ is injective.

ii/ If $T(0)$ is complemented, then T has property $(\mathcal{E}_\beta)$ if and only if, for every open set $U \subseteq \mathbb{C}_\infty$, the relation T_U is injective and has closed range.

Proof

i/ Suppose that T has property $(\mathcal{E}_\beta)$ and let $f \in Ker(T_U)$. Then, $\widetilde{0} \in T_U f$ where $\widetilde{0}$ denotes the zero function. Let $(g_n)_{n\in\mathbb{N}}$ be the null sequence of functions in $H^\sharp(U, X)$ and $(f_n)_{n\in\mathbb{N}}$ be the constant sequence f. Hence, for all $\lambda \in U \cap \mathbb{C}$, we have $g_n(\lambda) \in (T - \lambda I)f_n(\lambda)$. Since $g_n \underset{n\to+\infty}{\longrightarrow} 0$ in $H^\sharp(U, X)$ and T has property $(\mathcal{E}_\beta)$, it follows that $f_n \underset{n\to+\infty}{\longrightarrow} \widetilde{0}$ in $P(U, \mathcal{D}(T))$. Consequently, $f = \widetilde{0}$, and therefore T_U is injective.

ii/ Assume that $T(0)$ is complemented and let U be an open subset of $\mathbb{C}_\infty$. Then, it follows from Proposition 3.1 that $G(T_U)$ is closed. Now, suppose that T_U is injective and has closed range. Then, T_U^{-1} is a closed linear operator defined on $R(T_U)$ which is closed. Hence, using the closed graph theorem [25, Theorem 1.8], it follows that T_U^{-1} is continuous. To prove property $(\mathcal{E}_\beta)$, consider two sequences of analytic functions $f_n \in P(U, \mathcal{D}(T))$ and $g_n \in H^\sharp(U, X)$ such that, for all $\lambda \in U \cap \mathbb{C}$, we have $g_n(\lambda) \in (T - \lambda)f_n(\lambda)$ and $g_n \underset{n\to+\infty}{\longrightarrow} 0$ uniformly on all closed subsets of U. Because T_U^{-1} is a continuous operator, this entails that $T_U^{-1}g_n \underset{n\to+\infty}{\longrightarrow} 0$ in $P(U, \mathcal{D}(T))$. On the other hand, we have

$$f_n = T_U^{-1}T_U f_n = T_U^{-1}(g_n + T_U(0)) = T_U^{-1}g_n.$$

Consequently, $f_n \underset{n\to+\infty}{\longrightarrow} 0$ in $P(U, \mathcal{D}(T))$, which proves that T has property $(\mathcal{E}_\beta)$.

Conversely, suppose that T has property $(\mathcal{E}_\beta)$. Then, it follows from *i/* that T_U is injective. It remains to be shown that $R(T_U)$ is closed. To this end, consider $(g_n)_{n\in\mathbb{N}} \subseteq R(T_U)$ such that $g_n \underset{n\to+\infty}{\longrightarrow} g$ in $H^\sharp(U, X)$. Hence, there exists $f_n \in P(U, \mathcal{D}(T))$ for which $g_n(\lambda) \in (T - \lambda I)f_n(\lambda)$ for all $\lambda \in U \cap \mathbb{C}$. Moreover $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $P(U, \mathcal{D}(T))$. Indeed, if $(f_n)_{n\in\mathbb{N}}$ is not a Cauchy sequence then we can construct a subsequence $(\varphi_k)_{k\in\mathbb{N}}$ of $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} \varphi_k := f_{n(k+1)} - f_{n(k)} \neq 0$ in $P(U, \mathcal{D}(T))$. On the other hand, for all $\lambda \in U \cap \mathbb{C}$, $(g_{n(k+1)} - g_{n(k)})(\lambda) \in T_U\varphi_k(\lambda)$ and $g_{n(k+1)} - g_{n(k)} \underset{n\to+\infty}{\longrightarrow} 0$ in $H^\sharp(U, X)$. But, T has property $(\mathcal{E}_\beta)$, so $\varphi_k \underset{n\to+\infty}{\longrightarrow} 0$ in $H^\sharp(U, X)$ which is contradictory. As a consequence, there exists $f \in P(U, \mathcal{D}(T))$ for which $\lim\limits_{n\to\infty} f_n = f$ in $P(U, \mathcal{D}(T))$. Because $G(T_U)$ is closed, $(f_n, g_n)_{n\in\mathbb{N}} \subseteq G(T_U)$ and $(f_n, g_n) \underset{n\to+\infty}{\longrightarrow} (f, g)$, we obtain that $(f, g) \in G(T_U)$. Hence, $g \in T_U(f) \subseteq R(T_U)$. This proves that $R(T_U)$ is closed.

□

Lemma 3.2 *Let $T \in CR(X)$ be such that $\rho(T)$ contains an unbounded component and let Y and Z be two T-strongly invariant subspaces with the property that $T|_sY \in End(Y)$. Then*

$$R(\lambda, T|_sY)(Y \cap Z) \subseteq Y \cap Z \quad \textit{for all } \lambda \in \rho_f(T|_sY).$$

Proof Let $U := \rho_f(T|_sY)$, $x \in Y \cap Z$, $S := \rho(T) \cap \rho_f(T|_sY)$ and

$$\begin{array}{rcl} f_x : U & \to & Y \\ \lambda & \mapsto & R(\lambda, T|_sY)x. \end{array}$$

Our goal is to show that $f_x(U) \subseteq Y \cap Z$. We first observe that, for all $\lambda \in S$, we have

$$f_x(\lambda) = R(\lambda, T|_sY)x = R(\lambda, T)x \in R(\lambda, T)Y \cap R(\lambda, T)Z \subseteq Y \cap Z.$$

This means that $f_x(S) \subseteq Y \cap Z$. We note that $S \neq \emptyset$ and S is an open subset of U, and therefore it clusters in U. Because U is an open and connected subset of $\mathbb{C}$, $Y \cap Z$ is a closed subset of Y, and f is an analytic function verifying $f_x(S) \subseteq Y \cap Z$, it follows from [17, Theorem A.3.2] that $f_x(U) \subseteq Y \cap Z$, which completes the proof. □

Given a closed linear relation $T : X \to X$ and a T-invariant closed subspace Y of X, we define the coinduced linear operator $T/Y : X/Y \longrightarrow X/Y$ induced by T on the quotient space X/Y as follows:

$$\mathcal{D}(T/Y) = \frac{\mathcal{D}(T) + Y}{Y},$$

and, for all $\overline{x} \in \mathcal{D}(T/Y)$, we have $T/Y\overline{x} = \overline{T\alpha} = Q_Y T(\alpha)$, where $x = \alpha + y$, $\alpha \in \mathcal{D}(T)$ and $y \in Y$.

Lemma 3.3 *Let $T \in CR_c(X)$ be such that $\rho(T)$ contains an unbounded component. Consider two T-strongly invariant closed linear subspaces Y and Z of X with the property that $X = Y + Z$, $T|_sY \in End(Y)$, $T(Z) \subseteq Z$ and $\widetilde{P}(Z) \subseteq Z$, where $\widetilde{P}$ is a bounded linear projection on X which verifies $R(\widetilde{P}) = \mathcal{D}(T)$. Then the following assertions hold:*

- *i/* $T/Z \in End(X/Z)$;
- *ii/* $\sigma\big(T|_sY/(Y \cap Z)\big) \subseteq \sigma_f(T|_sY)$;
- *iii/* $\sigma(T/Z) \subseteq \sigma_f(T|_sY)$.

Proof We first observe that $\mathcal{D}(T/Z) = X/Z$. Indeed, we have $\mathcal{D}(T|_sY) = Y \subseteq \mathcal{D}(T)$, and therefore $X = Y + Z = \mathcal{D}(T) + Z$. Hence, $\mathcal{D}(T/Z) = \big(\mathcal{D}(T) + Z\big)/Z = X/Z$.

i/ To prove that $T/Z \in CO(X/Z)$, consider a sequence $(\overline{x_n})_{n\in\mathbb{N}} \subseteq \mathcal{D}(T/Z)$ such that $\overline{x_n} \underset{n\to+\infty}{\longrightarrow} \overline{x}$ and $T/Z(\overline{x_n}) \underset{n\to+\infty}{\longrightarrow} \overline{y}$, where $\overline{x}, \overline{y} \in X/Z$. Then, there exist $((\alpha_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}) \in \mathcal{D}(T) \times Z$ and $(\alpha, z) \in \mathcal{D}(T) \times Z$ with the property that $x_n = \alpha_n + z_n$ and $x = \alpha + z$. Consequently,

$$\overline{x_n} = \overline{\alpha_n}, \ \overline{x} = \overline{\alpha}, \ \ \overline{\alpha_n} \underset{n\to+\infty}{\longrightarrow} \overline{\alpha} \text{ and } T/Z(\overline{\alpha_n}) \underset{n\to+\infty}{\longrightarrow} \overline{y}. \tag{3.6}$$

It follows from (3.6) that there exists $(y_n)_{n\in\mathbb{N}} \subseteq Z$ such that $\alpha_n - y_n \underset{n\to+\infty}{\longrightarrow} \alpha$, and then $\widetilde{P}(\alpha_n - y_n) \underset{n\to+\infty}{\longrightarrow} \widetilde{P}(\alpha)$. But both α_n and α are in $\mathcal{D}(T) = R(\widetilde{P})$, and hence $\widetilde{P}(\alpha_n - y_n) = \alpha_n - \widetilde{P}(y_n) \underset{n\to+\infty}{\longrightarrow} \widetilde{P}(\alpha) = \alpha$. Because T is continuous, it follows that Q_ZT is continuous, and hence $Q_ZT(\alpha_n - \widetilde{P}(y_n)) \underset{n\to+\infty}{\longrightarrow} Q_ZT(\alpha)$. Thus, $T/Z\left(\overline{\alpha_n - \widetilde{P}(y_n)}\right) \underset{n\to+\infty}{\longrightarrow} T/Z(\overline{\alpha})$. Since $\widetilde{P}(y_n) \in \widetilde{P}(Z) \subseteq Z$, it follows that $\widetilde{P}(\overline{y_n}) = \overline{0}$, and therefore

$$T/Z(\overline{\alpha_n}) \underset{n\to+\infty}{\longrightarrow} T/Z(\overline{\alpha}). \tag{3.7}$$

(3.7) leads to $\overline{y} = T/Z(\overline{\alpha}) = T/Z(\overline{x})$, which means that T/Z is closed. In addition, since $\mathcal{D}(T/Z) = X/Z$, we conclude from [8, Theorem.III.4.2] that $T/Z \in End(X/Z)$.

ii/ Let $\lambda \in \rho_f(T|_sY)$, and let $\bar{x} \in Ker(T|_sY/(Y \cap Z) - \lambda I)$. Then, $Q_{Y\cap Z}(T|_sY - \lambda I)x = \bar{0}$, and therefore $(T|_sY - \lambda I)x \in Y \cap Z$. Since $\rho_f(T|_sY) \subseteq \rho(T|_sY)$, it follows that $R(\lambda, T|_sY)(T|_sY - \lambda I)x \in R(\lambda, T|_sY)(Y \cap Z)$. Hence, $x \in R(\lambda, T|_sY)(Y\cap Z)$. By Lemma 3.2, we infer that $x \in Y\cap Z$, which entails that $Ker(T|_sY/(Y\cap Z)-\lambda I) = \{\bar{0}\}$. Now, let $\bar{y} \in Y/(Y\cap Z)$. Because $\lambda \in \rho(T|_sY)$, there exists $y_0 \in Y$ such that $y = (T|_sY-\lambda I)y_0$, and therefore $\bar{y} = Q_{Y\cap Z}(T|_sY - \lambda I)y_0 = (T|_sY/(Y \cap Z) - \lambda I)\overline{y_0}$. Consequently, $R\left(T|_sY/(Y \cap Z) - \lambda I\right) = Y/(Y \cap Z)$, which completes the proof of *ii/*.

iii/ We claim that $\sigma(T/Z) = \sigma\big((T|_sY)/(Y \cap Z)\big)$. To verify this, let

$$\begin{array}{rccc} \hat{R}: & Y/(Y \cap Z) & \longrightarrow & X/Z \\ & \widetilde{x} & \longrightarrow & \hat{R}\widetilde{x} = \overline{x} = Q_Zx \end{array}$$

be such that $\mathcal{D}(\hat{R}) = Y/(Y \cap Z)$. Then, $\hat{R}$ is a bounded linear operator. Indeed, for every $x \in Y$, $\|\hat{R}(\widetilde{x})\| = \|Q_Zx\| = d(x, Z) \leqslant d(x, Z \cap Y) = \|\widetilde{x}\|$. Moreover, it is clear that $\hat{R}$ is injective and since $X = Y + Z$ it becomes invertible. On the other hand, we can see that $(T/Z)\hat{R} = \hat{R}\, \widetilde{T|_sY}$ where $\widetilde{T|_sY} = (T|_sY)/(Y \cap Z)$. Moreover, we have the following equivalences:

$$\lambda \in \rho(T/Z) \Leftrightarrow \ T/Z - \lambda I \ \text{ is invertible} \Leftrightarrow \ (T/Z - \lambda I)\hat{R} \ \text{ is invertible} \Leftrightarrow \ (T/Z)\hat{R} - \lambda\hat{R}$$

$$\text{is invertible} \Leftrightarrow \ \hat{R}\ \widetilde{T|_sY} - \lambda\hat{R} \ \text{ is invertible} \Leftrightarrow \ \lambda \in \rho(\widetilde{T|_sY}).$$

This means that $\sigma(T/Z) = \sigma(\widetilde{T|_sY}) = \sigma\big((T|_sY)/(Y \cap Z)\big)$, as claimed. By $ii/$, we conclude that $\sigma(T/Z) \subseteq \sigma_f(T|_sY)$.

□

Now, we can state the main result of this subsection:

Theorem 3.2 *Let $T \in CR_c(X)$ be a linear relation such that $\rho(T)$ contains an unbounded component. If T is extended spectral decomposable, then T has property $(\mathcal{E}_\beta)$.*

Proof Suppose that T is extended spectral decomposable. Let $U \subseteq \mathbb{C}_\infty$ be an open set and let $(f_n)_{n\in\mathbb{N}} \in P(U, \mathcal{D}(T))$, $(t_n)_{n\in\mathbb{N}} \in H^\sharp(U, X)$ be two sequences of analytic functions with the property that

$$t_n(\lambda) \in (T-\lambda)f_n(\lambda) \ \text{ for all } \lambda \in U \cap \mathbb{C},\ n \in \mathbb{N} \quad \text{ and } \quad t_n \underset{n\to+\infty}{\longrightarrow} 0 \ \text{ in } H^\sharp(U, X).$$

To show that $f_n \underset{n\to+\infty}{\longrightarrow} 0$ in $P(U, \mathcal{D}(T))$, we have to distinguish two cases:

Case 1: $\infty \notin U$. In this case, U becomes an open subset of $\mathbb{C}$ and $(f_n)_{n_in\mathbb{N}} \subseteq H(U, \mathcal{D}(T))$, so it suffices to prove that $f_n \underset{n\to+\infty}{\longrightarrow} 0$ on every closed disc $G \subseteq U \subseteq \mathbb{C}$. Given an arbitrary closed disc $G \subseteq U$, we choose an open disc E_1 for which $G \subseteq E_1 \subseteq \overline{E_1} \subseteq U$, and we apply the extended spectral decomposability of T to the open cover $\{E_1, \mathbb{C}_\infty \setminus G\}$ of $\mathbb{C}_\infty$. Then, we obtain T-strongly invariant subspaces $Y, Z \subseteq X$ for which $\widetilde{\sigma}(T|_sY) = \sigma(T|_sY) \subseteq E_1$, $\widetilde{\sigma}(T|_sZ) \subseteq \mathbb{C}_\infty \setminus G$, $X = Y + Z$, $T(Z) \subseteq Z$, and $\widetilde{P}(Z) \subseteq Z$ for some bounded linear projection $\widetilde{P}$ which verifies $R(\widetilde{P}) = \mathcal{D}(T)$. We also have $\mathcal{D}(T) = \mathcal{D}(T|_sY) + \mathcal{D}(T|_sZ)$, by Remark 3.1. From Proposition 1.2.2 of [17], we conclude that there exist functions $a_n \in H(U, \mathcal{D}(T|_sY))$ and $b_n \in H(U, \mathcal{D}(T|_sZ))$ such that

$$f_n = a_n + b_n \qquad \text{for all } n \in \mathbb{N}.$$

Moreover, by Lemma 3.3, we have $\sigma(T/Z) \subseteq \sigma_f(T|_sY) \subseteq E_1$. Then $\partial E_1 = \overline{E_1} \setminus E_1 \subseteq \mathbb{C} \setminus E_1 \subseteq \rho(T/Z)$, and hence $T/Z - \lambda I$ is invertible for all $\lambda \in \partial E_1$. Thus, we obtain

$$Q_Z f_n(\lambda) = Q_Z a_n(\lambda) = (T/Z - \lambda I)^{-1} Q_Z (T - \lambda I) f_n(\lambda) \qquad \text{for all } \lambda \in \partial E_1.$$

Because $T/Z - \lambda I$ is invertible, it follows from the continuity of $\lambda \to (T/Z - \lambda)^{-1}$ on the compact set ∂E_1 that there exists $\widehat{c} > 0$ for which $\|(T/Z - \lambda I)^{-1}\| \leqslant \widehat{c}$ for every $\lambda \in \partial E_1$. Consequently, for every $\lambda \in \partial E_1$, we obtain

$$\begin{aligned} \|Q_Z a_n(\lambda)\| = \ & \|(T/Z - \lambda I)^{-1} Q_Z (T - \lambda I) f_n(\lambda)\| \leqslant \hat{c}\, \|Q_Z (T - \lambda I) f_n(\lambda)\| \\ & \leqslant \hat{c}\, d(t_n(\lambda), Z) \leqslant \ \hat{c} \|t_n(\lambda)\|. \end{aligned}$$

By the assumption on the functions t_n, this implies that $Q_Z a_n \in H(U, X/Z)$ converges to zero uniformly on ∂E_1, and therefore uniformly on E_1, by Theorem $A.3.3$ of [17]. However, we can see from [17, Proposition 1.2.2] that $H\big(E_1, X/Z\big) \cong H(E_1, X)/H(E_1, Z)$, which implies that there is a sequence of analytic functions $(k_n)_{n\in\mathbb{N}} \subseteq H(E_1, Z)$ such that $a_n - k_n \underset{n\to+\infty}{\longrightarrow} 0$ locally uniformly on E_1, and hence uniformly on G. But, for all $\lambda \in G$, we have $a_n(\lambda) \in \mathcal{D}(T|_sY) \subseteq \mathcal{D}(T)$, and therefore $\widetilde{P}(a_n(\lambda)) = a_n(\lambda)$. As a consequence, $a_n - \widetilde{P}(k_n) \underset{n\to+\infty}{\longrightarrow} 0$ uniformly on G.

Because $f_n = a_n + b_n = \Big(a_n - \widetilde{P}(k_n)\Big) + \Big(\widetilde{P}(k_n) + b_n\Big)$ for all $n \in \mathbb{N}$, it remains to prove that $b_n + \widetilde{P}(k_n)$ converges uniformly to zero on G. We first observe that, for all $\lambda \in G$, we have $\widetilde{P}(k_n)(\lambda) \in \widetilde{P}(Z) \cap \mathcal{D}(T) \subseteq Z \cap \mathcal{D}(T)$, and therefore $\widetilde{P}(k_n(\lambda)) \in \mathcal{D}(T|_s Z)$ thanks to Remark 3.1. Moreover, since $\widetilde{\sigma}(T|_s Z) \cap G = \emptyset$, it follows that $\sigma(T|_s Z) \cap G = \emptyset$. Hence, for every $\lambda \in G$, the relation $T|_s Z - \lambda I$ is invertible or, equivalently, $\lambda \in \rho(T|_s Z)$. Consequently, $\lambda \to (T|_s Z - \lambda I)^{-1}$ is continuous on G. Since the latter set is compact, there exists a constant $c > 0$ with the property that

$$\|(T|_s Z - \lambda I)^{-1}\| \leqslant c \quad \text{for all } \lambda \in G.$$

As a result, for all $\lambda \in G$, we have

$$\begin{aligned} \|(b_n + \widetilde{P}(k_n))(\lambda)\| &= \|(T|_s Z - \lambda I)^{-1}(T|_s Z - \lambda I)(b_n + \widetilde{P}(k_n))(\lambda)\| \\ &\leqslant c\, \|(T|_s Z - \lambda I)(b_n + \widetilde{P}(k_n))(\lambda)\|. \end{aligned}$$

From Proposition 2.2 and from the fact that $T(0) \subseteq T(Z) \subseteq Z$, we obtain that

$$(T|_s Z - \lambda I)(b_n + \widetilde{P}(k_n))(\lambda) = (T - \lambda I)(b_n + \widetilde{P}(k_n))(\lambda) \quad \text{for all } \lambda \in G.$$

Thus, for all $\lambda \in G$, we have

$$\begin{aligned} \|(b_n + \widetilde{P}(k_n))(\lambda)\| &\leqslant c\, \|(T - \lambda I)(b_n + \widetilde{P}(k_n))(\lambda)\| \\ &\leqslant c\, \|(T - \lambda I)(f_n - (a_n - \widetilde{P}(k_n))(\lambda)\| \\ &\leqslant c\, \|(T - \lambda I) f_n(\lambda)\| + c\, \|(T - \lambda I)(a_n - \widetilde{P}(k_n))(\lambda)\| \\ &\leqslant c\, \|t_n(\lambda)\| + c\, \|(T - \lambda I)(a_n - \widetilde{P}(k_n))(\lambda)\|. \end{aligned}$$

Since $\lambda \to \|T - \lambda I\|$ is continuous on G, there exists $b > 0$ such that $\|T - \lambda I\| \leqslant b$. Consequently, for all $\lambda \in G$, we have

$$\|(b_n + \widetilde{P}(k_n))(\lambda)\| \leqslant c\, \|t_n(\lambda)\| + bc\, \|(a_n - \widetilde{P}(k_n))(\lambda)\|,$$

and therefore $b_n + \widetilde{P}(k_n) \underset{n \to +\infty}{\longrightarrow} 0$ uniformly on G, as desired.

Case 2: $\infty \in U$. In this case, from Lemma 3.1, we can suppose that $U := B_{\mathbb{C}_\infty}(\lambda_0, \varepsilon)$ for arbitrary $\lambda_0 \in \mathbb{C}_\infty$ and $\frac{2}{\sqrt{1+|\lambda_0|^2}} \neq \varepsilon < 2$. It remains to show that $f_n \underset{n \to +\infty}{\longrightarrow} 0$ on every closed set $F \subseteq U$. If $\infty \notin F$, then there exists a compact set $K \subseteq \mathbb{C}$ such that $F = \mathbb{C}_\infty \setminus (\mathbb{C} \setminus K \cup \{\infty\}) = K$. By proceeding as in Case 1, we obtain the result. If $\infty \in F$, then there exists an open set $U_1 \subseteq \mathbb{C}$ such that $F = \mathbb{C}_\infty \setminus U_1 = \mathbb{C} \setminus U_1 \cup \{\infty\}$. Consider an open disc $E \subseteq \mathbb{C}$ such that $F \subseteq E \cup \{\infty\} \subseteq \overline{E} \cup \{\infty\} \subseteq U$. Then, E is an open bounded subset of $\mathbb{C}$ and $\mathbb{C}_\infty \setminus F \cup \{\infty\}$ is an open subset of $\mathbb{C}_\infty$, and therefore we apply the extended spectral decomposability of T to the open cover $\{\mathbb{C}_\infty \setminus F \cup \{\infty\}, E\}$ of $\mathbb{C}_\infty$. We obtain T-strongly invariant subspaces $Y, Z \subseteq X$ for which $\widetilde{\sigma}(T|_s Y) = \sigma(T|_s Y) \subseteq E$, $\widetilde{\sigma}(T|_s Z) \subseteq \mathbb{C}_\infty \setminus F \cup \{\infty\}$, $X = Y + Z$, $T(Z) \subseteq Z$, and $\widetilde{P}(Z) \subseteq Z$ for some bounded linear projection $\widetilde{P}$ which verifies $R(\widetilde{P}) = \mathcal{D}(T)$. In addition, we have $\mathcal{D}(T) = \mathcal{D}(T|_s Y) + \mathcal{D}(T|_s Z)$ thanks to Remark 3.1. Since $f_n \in P(U, \mathcal{D}(T))$, it follows from Lemma 2.3 that there exist two functions $g_n \in P(U, \mathcal{D}(T|_s Y))$ and $h_n \in P(U, \mathcal{D}(T|_s Z))$ such that $f_n = g_n + h_n$ for all $n \in \mathbb{N}$. Proceeding exactly as in Case 1, we conclude that there exists a sequence of analytic functions $(k_n)_{n \in \mathbb{N}} \subseteq P(E, Z)$ such that $g_n - \widetilde{P}(k_n) \underset{n \to +\infty}{\longrightarrow} 0$, locally uniformly on E, and hence uniformly on $F \setminus \{\infty\}$. Because $f_n = \left(g_n - \widetilde{P}(k_n)\right) + \left(h_n + \widetilde{P}(k_n)\right)$ for all $n \in \mathbb{N}$, it remains to be seen that $h_n + \widetilde{P}(k_n)$ converges to zero uniformly on $F \setminus \{\infty\}$.

Since $\widetilde{\sigma}(T|_s Z) \subseteq \mathbb{C}_\infty \setminus F \cup \{\infty\}$, we obtain that $F \cap \mathbb{C} \subseteq \rho(T|_s Z)$, and therefore, for every $\lambda \in F \cap \mathbb{C}$, the relation $T|_sZ - \lambda I$ is invertible or, equivalently, $\lambda \in \rho(T|_sZ)$. Thus, from [8, Proposition VI.3.2], we conclude that $\lambda \to (T|_sZ - \lambda I)^{-1}$ is a continuous function which converges to zero, and therefore there is a constant $c > 0$ for which $\|(T|_sZ - \lambda I)^{-1}\| \leqslant c$ for all $\lambda \in F \cap \mathbb{C}$. As a result, for every $\lambda \in F \cap \mathbb{C}$, we have

$$\|(h_n + \widetilde{P}(k_n))(\lambda)\| = \|(T|_sZ - \lambda I)^{-1}(T|_sZ - \lambda I)(h_n + k_n)(\lambda)\| \leqslant c\| \ (T|_sZ - \lambda I)(h_n + \widetilde{P}(k_n))(\lambda)\|.$$

Again, we proceed as in the proof of Case 1 to conclude that, for all $\lambda \in F \cap \mathbb{C}$, we have

$$\|(h_n + \widetilde{P}(k_n))(\lambda)\| \leqslant c \ \| \ (T - \lambda I)(h_n + \widetilde{P}(k_n))(\lambda)\| \leqslant c \ \|t_n(\lambda)\| + c \ \| \ (T - \lambda I)(g_n - \widetilde{P}(k_n))(\lambda)\|.$$

We claim that the function $\lambda \to \|T - \lambda I\|$ is continuous on $(F \cap \mathbb{C}, d_\infty)$. To see this, consider $\lambda_0 \in F \cap \mathbb{C}$, $0 < r < \frac{2}{\sqrt{1+|\lambda_0|^2}}$ such that $d_\infty(\lambda, \lambda_0) < r$ and let $\varepsilon > 0$. Then, by Lemma 2.2 it follows that

$$|\lambda - (\frac{4Re(\lambda_0)}{4 - r^2(1 + |\lambda_0|^2)} + i \ \frac{4Im(\lambda_0)}{4 - r^2(1 + |\lambda_0|^2)})| \leqslant \frac{r(1 + |\lambda_0|^2)\sqrt{(4 - r^2)}}{4 - r^2(1 + |\lambda_0|^2)}.$$

Hence, $|\lambda| \leqslant r' + |\lambda_0'|$ with $r' = \dfrac{r(1 + |\lambda_0|^2)\sqrt{(4 - r^2)}}{4 - r^2(1 + |\lambda_0|^2)}$ and $\lambda_0' = \dfrac{4\lambda_0}{4 - r^2(1 + |\lambda_0|^2)}$, and therefore we obtain the following implications:

$$\begin{aligned} 1 + |\lambda|^2 \leqslant 1 + (r' + |\lambda_0'|)^2 &\Rightarrow \ \frac{1}{\sqrt{1 + (r' + |\lambda_0'|)^2}} \leqslant \frac{1}{\sqrt{1 + |\lambda|^2}} \\ &\Rightarrow \ \frac{2|\lambda_0 - \lambda|}{\sqrt{1 + (r' + |\lambda_0'|)^2}\sqrt{1 + |\lambda_0|^2}} \leqslant \frac{2|\lambda_0 - \lambda|}{\sqrt{1 + |\lambda|^2}\sqrt{1 + |\lambda_0|^2}} \\ &\Rightarrow \ |\lambda_0 - \lambda| \leqslant \ M \ d_\infty(\lambda, \lambda_0), \end{aligned} \tag{3.8}$$

where $M = \dfrac{\sqrt{1 + (r' + |\lambda_0'|)^2}\sqrt{1 + |\lambda_0|^2}}{2}$. On the other hand, we have

$$\|T - \lambda I\| \leqslant \|(T - \lambda I) - (T - \lambda_0)\| + \|(T - \lambda_0)\|,$$

and hence

$$\big|\|T - \lambda I\| - \|(T - \lambda_0)\|\big| \leqslant \|(\lambda_0 - \lambda)I + T - T\| \leqslant \|Q_T\| \ |\lambda - \lambda_0|.$$

We obtain from (3.8) that

$$\big|\|T - \lambda I\| - \|(T - \lambda_0)\|\big| \leqslant M \ \|Q_T\| d_\infty(\lambda, \lambda_0) \leqslant Mr.$$

This proves that $\lambda \to \|T - \lambda I\|$ is continuous, as claimed.

On the other hand, $F \cap \mathbb{C}$ is a compact subset of (U, d_∞), so there exists $b > 0$ for which $\|T - \lambda I\| \leqslant b$ for all $\lambda \in F \cap \mathbb{C}$. Consequently, for all $\lambda \in F \cap \mathbb{C}$, we have

$$\|(h_n + \widetilde{P}(k_n))(\lambda)\| \leqslant \ c \ \|t_n(\lambda)\| + bc \ \|(g_n - \widetilde{P}(k_n))(\lambda)\|.$$

As a result, $h_n + \widetilde{P}(k_n)$ converges to zero uniformly on $F \cap \mathbb{C}$, as desired. But we know that $f_n \in P(U, \mathcal{D}(T))$ so $f_n(\infty) = 0$, and therefore $f_n \underset{n \to +\infty}{\longrightarrow} 0$ uniformly on F. This proves that T has property $(\mathcal{E}_\beta)$.

□

3.1.2. Extended relatively single-valued extension property

We recall that in the case of bounded linear operators and according to Pavla Vrbovà [27], the analytic residuum of $T \in End(X)$ is denoted by S_T and defined as the set of all $\lambda \in \mathbb{C}$ for which, in every neighborhood U_λ of λ, there exist an open set $G \subseteq U_\lambda$ and a nonzero analytic function f which satisfies $(T - \mu I)f(\mu) = 0$ in G. Also, T is said to have the single-valued extension property whenever $S_T = \emptyset$.
As we are going to consider the local spectral theory for linear relations in the extended complex plane, we shall reformulate the notions stated above to obtain the following definitions.

Definition 3.4 *The extended analytic residuum of a relation $T \in CR(X)$ at $\lambda_0 \in \mathbb{C}_\infty$ is denoted by $S_T(\lambda_0)$ and defined as the union of all open connected neighborhoods $U_{\lambda_0} \subseteq \mathbb{C}_\infty$ of λ_0 for which there exists a function $f \in P(U_{\lambda_0}, \mathcal{D}(T))$ satisfying*

$$0 \in (T - \mu I)f(\mu) \text{ for all } \mu \in U_{\lambda_0} \cap \mathbb{C},$$

and there exists $\mu_0 \in U_{\lambda_0}$ such that $f(\mu_0) \notin T(0) \cap Ker(T)$. The extended analytic residuum of T is given by $S_T := \bigcup_{\lambda \in \mathbb{C}_\infty} S_T(\lambda)$.

Definition 3.5 *Let $T \in CR(X)$. The relation T is said to have the extended relatively single-valued extension property, abbreviated ER-SVEP, at $\lambda_0 \in \mathbb{C}_\infty$ if $S_T(\lambda_0) = \emptyset$, i.e. for every open connected neighborhood $U_{\lambda_0} \subseteq \mathbb{C}_\infty$ of λ_0 and for all function $f \in P(U_{\lambda_0}, \mathcal{D}(T))$ which satisfies*

$$0 \in (T - \mu I)f(\mu) \text{ for all } \mu \in U_{\lambda_0} \cap \mathbb{C}, \tag{3.9}$$

we have $f(\mu) \in T(0) \cap Ker(T)$ for all $\mu \in U_{\lambda_0}$. The relation T is said to have the ER-SVEP if T has the ER-SVEP at every $\lambda \in \mathbb{C}_\infty$.

Remark 3.2 *Let $T \in CR(X)$ be such that $R_c(T) = \{0\}$ and let $\lambda \in \mathbb{C}_\infty$. Then, T has the ER-SVEP at λ if and only if, for all neighborhood $U_\lambda \subseteq \mathbb{C}_\infty$ of λ, the only function $f \in P(U_\lambda, \mathcal{D}(T))$ satisfying $0 \in (T - \mu I)f(\mu)$ for all $\mu \in U_\lambda \cap \mathbb{C}$ is the constant function $f \equiv 0$.*

The next theorem shows that beside the extended Bishop's property, the extended relatively single-valued extension property is also a necessary condition for a linear relation to be extended spectral decomposable.

Theorem 3.3 *Let $T \in CR_c(X)$ be a linear relation such that $\rho(T)$ contains an unbounded component. If T is extended spectral decomposable, then T has the ER-SVEP.*

Proof Let U be an open connected set in $\mathbb{C}_\infty$ and $f \in P(U, \mathcal{D}(T))$ be a function such that

$$0 \in (T - \mu I)f(\mu) \qquad \text{for all } \mu \in U \cap \mathbb{C}.$$

Then, $f \in Ker(T_U)$ as $f(\infty) = 0$ whenever $\infty \in U$. Since T is extended spectral decomposable, then it has property $(\mathcal{E}_\beta)$, by Theorem 3.2. Hence, using Theorem 3.1, we obtain that T_U is injective, which entails that $f = 0$. □

3.1.3. Extended Dunford's property $(\mathcal{E}_C)$

In this part, we give some definitions related to the local spectral theory in the Riemann sphere to introduce the extended Dunford's property in Definition 3.7.

Definition 3.6 *Let $T \in CR(X)$ and $x \in X$. We define the extended local resolvent set of T at the point x, denoted by $\widetilde{\rho_T}(x)$, as the set of all $\lambda \in \mathbb{C}_\infty$ for which there are an open neighborhood U_λ in $\mathbb{C}_\infty$ of λ and an analytic function $\widetilde{f}_{x,\lambda} \in P(U_\lambda, \mathcal{D}(T))$ such that $x \in (\mu I - T)\widetilde{f}_{x,\lambda}(\mu)$ for all $\mu \in U_\lambda \cap \mathbb{C}$. Such a function is called an extended local resolvent function of T at x. The extended local spectrum of T at x is defined by $\widetilde{\sigma_T}(x) := \mathbb{C}_\infty \backslash \widetilde{\rho_T}(x)$.*

Let $F \subseteq \mathbb{C}_\infty$. The extended local spectral subspace of T at F is the set $\widetilde{X_T}(F) := \{w \in X \ ; \ \widetilde{\sigma_T}(w) \subseteq F\}$.

Let $F \subseteq \mathbb{C}_\infty$ be a closed set. The extended glocal spectral subspace of T at F is the set $\widetilde{\mathcal{X}_T}(F)$ of all $x \in X$ for which there exists $\widetilde{f} \in P(\mathbb{C}_\infty \setminus F, \mathcal{D}(T))$ such that $x \in (T - \mu I)\widetilde{f}(\mu)$ for all $\mu \in (\mathbb{C}_\infty \setminus F) \cap \mathbb{C}$.

Remark 3.3 *Let $T \in CR(X)$ and F be a closed subset of $\mathbb{C}_\infty$. Then, the extended local and glocal subspaces of T at F are both linear subspaces of X which verify the inclusion $\widetilde{\mathcal{X}_T}(F) \subseteq \widetilde{X_T}(F)$.*

Definition 3.7 *A relation $T \in CR_c(X)$ is said to have the extended Dunford's property $(\mathcal{E}_C)$, shortly property $(\mathcal{E}_C)$, if the extended local spectral subspace $\widetilde{X_T}(F)$ is closed for every closed subset $F \subseteq \mathbb{C}_\infty$.*

Lemma 3.4 *Let $T \in CR_c(X)$ be a relation that has the ER-SVEP such that $R_c(T) = \{0\}$. Then, for every closed subset F of $\mathbb{C}_\infty$, we have*

$$\widetilde{X_T}(F) = \widetilde{\mathcal{X}_T}(F).$$

Proof By Remark 3.3, we obtain $\widetilde{\mathcal{X}_T}(F) \subseteq \widetilde{X_T}(F)$. To prove the other inclusion, let $x \in \widetilde{X_T}(F)$. Then, for all $\lambda \in U := \mathbb{C}_\infty \setminus F$, there exist an open neighborhood $U_\lambda \subseteq U$ of λ in $\mathbb{C}_\infty$ and a function $\widetilde{f}_{\lambda,x} \in P(U_\lambda, \mathcal{D}(T))$ such that $x \in (T - \mu I)\widetilde{f}_{\lambda,x}(\mu)$ for all $\mu \in U_\lambda \cap \mathbb{C}$. Hence, $U = \bigcup_{\lambda \in U} U_\lambda$ and there exists a function $\widetilde{f} \in P(U, \mathcal{D}(T))$ defined by

$$\begin{array}{rcll} \widetilde{f} : U & \to & \mathcal{D}(T) & \\ \mu & \to & \widetilde{f}_{\lambda,x}(\mu) & \quad \text{if } \mu \in U_\lambda, \lambda \in U \end{array}$$

which verifies

$$x \in (T - \mu I)\widetilde{f}(\mu) \qquad \text{for all } \mu \in U \cap \mathbb{C}. \tag{3.10}$$

To prove that $\widetilde{f}$ is well defined, we shall show that, for $\lambda_1, \lambda_2 \in U$, we have $\widetilde{f}_{\lambda_1,x}(\mu) = \widetilde{f}_{\lambda_2,x}(\mu)$ for all $\mu \in U \cap \mathbb{C}$. To do this, let $\lambda_1, \lambda_2 \in U$, be arbitrarily given. Then, there exist open neighborhoods $U_{\lambda_1}, U_{\lambda_2} \subseteq \mathbb{C}_\infty$ of λ_1 and λ_2, respectively and $(\widetilde{f}_{\lambda_1,x}, \widetilde{f}_{\lambda_2,x}) \in P(U_{\lambda_1}, \mathcal{D}(T)) \times P(U_{\lambda_2}, \mathcal{D}(T))$, verifying

$$x \in \quad (T - \mu I)\widetilde{f}_{\lambda_i,x}(\mu) \qquad \text{for all } \mu \in U_{\lambda_i} \cap \mathbb{C} \ , \ i = 1, 2. \tag{3.11}$$

(3.11) implies that $0 \in (T - \mu I)(\widetilde{f}_{\lambda_1,x} - \widetilde{f}_{\lambda_2,x})(\mu)$ for all $\mu \in U_{\lambda_1} \cap U_{\lambda_2} \cap \mathbb{C}$. Since T has the ER-SVEP and $R_c(T) = \{0\}$, it follows from Remark 3.2 that $\widetilde{f}_{\lambda_1,x}(\mu) = \widetilde{f}_{\lambda_2,x}(\mu)$ for all $\mu \in U_{\lambda_1} \cap U_{\lambda_2} \cap \mathbb{C}$. By the connectedness of U, we obtain that $\widetilde{f}_{\lambda_1,x}(\mu) = \widetilde{f}_{\lambda_2,x}(\mu)$ for all $\mu \in U \cap \mathbb{C}$. Therefore, $\widetilde{f}$ is well defined as required. We conclude from (3.10) that $x \in \widetilde{\mathcal{X}_T}(F)$, which implies that $\widetilde{X_T}(F) \subseteq \widetilde{\mathcal{X}_T}(F)$. □

Beside the two before-stated necessary conditions, we can also consider the extended Dunford's property as a third necessary condition of an extended spectral decomposable linear relation.

Theorem 3.4 *Let $T \in CR_c(X)$ be a linear relation such that $T(0)$ is complemented and $\rho(T)$ contains an unbounded component. If T is extended spectral decomposable, then T has property $(\mathcal{E}_C)$.*

Proof Let $F \subseteq \mathbb{C}_\infty$ be a closed set. Since, by Theorem 3.2, T has property $(\mathcal{E}_\beta)$, then, using Theorem 3.3, we see that T has the ER-SVEP. Moreover, because $\rho(T) \neq \emptyset$, we conclude from Lemma 3.4 and [21, Lemma 6.1] that $\widetilde{X_T}(F) = \widetilde{\mathcal{X}_T}(F)$. Hence, it suffices to prove that $\widetilde{\mathcal{X}_T}(F)$ is closed. To this end, let $(x_n)_{n\in\mathbb{N}} \subseteq \widetilde{\mathcal{X}_T}(F)$ be an arbitrary sequence which converges to $x \in X$. As $(x_n)_{n\in\mathbb{N}} \subseteq \widetilde{\mathcal{X}_T}(F)$, there exists $(f_n)_{n\in\mathbb{N}} \subseteq P(U, \mathcal{D}(T))$, with $U := \mathbb{C}_\infty \setminus F$, such that

$$x_n \in (T - \mu I) f_n(\mu) \qquad \text{for all } n \in \mathbb{N},\ \mu \in U \cap \mathbb{C}.$$

Consider the constant function $g := x \in H^\sharp(U, X)$ and the sequence of analytic functions $(g_n)_{n\in\mathbb{N}} \subseteq H^\sharp(U, X)$ such that

$$g_n : \begin{array}{ccc} U & \longrightarrow & X \\ \mu & \longmapsto & x_n, \quad n \in \mathbb{N}. \end{array}.$$

Then, $g_n \underset{n\to+\infty}{\longrightarrow} g$ in $H^\sharp(U, X)$. Also, we have $g_n(\mu) \in (T - \mu I) f_n(\mu)$ for all $\mu \in U \cap \mathbb{C}$ and $n \in \mathbb{N}$. This entails that $g_n \in T_U(f_n) \subseteq R(T_U)$ for all $n \in \mathbb{N}$. Since T has property $(\mathcal{E}_\beta)$, we infer from Theorem 3.1 that $g \in R(T_U)$, and therefore there exists $f \in P(U, \mathcal{D}(T))$ such that $g \in T_U(f)$. Consequently, for all $\mu \in U \cap \mathbb{C}$, we have $g(\mu) \in (T - \mu I) f(\mu)$, or equivalently, $x \in (T - \mu I) f(\mu)$. This proves that $x \in \widetilde{\mathcal{X}_T}(F)$, and hence $\widetilde{\mathcal{X}_T}(F)$ is closed. □

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