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P-strong convergence with respect to an Orlicz function

Nilay ŞAHİN BAYRAM*
Başkent University, Faculty of Engineering, Department of Electrical and Electronics Engineering, Ankara, Turkey

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Abstract: The concepts of strong convergence, statistical convergence, and uniform integrability are of some interest in convergence theories. Recently Ünver and Orhan [19] have introduced the concepts of P-strong and P-statistical convergences with the help of power series methods and established a relationship between them. In the present paper, we introduce the notion of P-strong convergence with respect to an Orlicz function and prove that all these three concepts are boundedly equivalent provided that Orlicz function satisfies $\triangle_2-$condition. We also get an improvement of this result by using the concept of uniform integrability.

Key words: Power series method, strong convergence, statistical convergence, Orlicz function

1. Introduction

Some convergence methods such as statistical convergence, strong convergence, and strong convergence with respect to Orlicz functions are of some interest in mathematical analysis. There are many studies dealing with these concepts and relationships between them (see also $[2,3,4,5,7,8,9,10,13,15,17])$. Recently Ünver and Orhan [19] have introduced the concepts of P-strong and $P-$statistical convergences via power series methods and established the following relationship between them:

"A real sequence is P-strongly convergent if and only if it is P-statistically convergent and P-uniformly integrable", which is an analog of a result of Khan and Orhan [9,10].

In the present paper motivated by that of Demirci [4], we introduce the concept of P-strong convergence with respect to an Orlicz function and prove that this concept, P-strong convergence, and P-statistical convergence are all boundedly equivalent. Our final result also shows that P-strong convergence with respect to an Orlicz function is placed between P-strong convergence and P-statistical convergence.

We pause to collect some basic concepts and notation.

Let $F:[0,\infty)\to[0,\infty)$ be a function such that it is continuous, nondecreasing, and convex with $F(0)=0$, $F(x)>0$ for $x>0$ and $F(x)\to\infty$ as $x\to\infty$. Such a function is called Orlicz function $($*see*, e.g., $[11])$. If the convexity condition of Orlicz function F is replaced by $F(x+y)\leq F(x)+F(y)$,, then F is called modulus function $($*see*, e.g., $[14])$

*Correspondence: nsbayram@baskent.edu.tr
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Throughout the paper, let e denote the sequence, which is identically 1 and let

$$\begin{aligned} w &: = \{\text{all real valued sequences}\}, \\ \ell^{\infty} &: = \left\{ x \in w : \|x\| = \sup_{n} |x_n| < \infty \right\}. \end{aligned}$$

Let (p_j) be a sequence of nonnegative real numbers such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{j=0}^{\infty} p_j t^j$$

has radius of convergence R with $0 < R \leq \infty$.

Let

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} \mid \lim_{0<t \to R^-} \frac{f(t)}{p(t)} \text{ exists} \right\}$$

and

$$C_{P_p} := \left\{ x = (x_j) \mid p_x(t) := \sum_{j=0}^{\infty} p_j t^j x_j \text{ has radius of convergence } \geq R \text{ and } p_x \in C_p \right\}.$$

The functional $P - \lim : C_{P_p} \to \mathbb{R}$ defined by

$$P - \lim x = \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and the sequence $x = (x_j)$ is said to be $P-$convergent $([1, 12])$. A power series method is called regular if $P - \lim x = \lim x$ for every convergent sequence x. Recall that a power series method P is regular if and only if

$$\lim_{0<t \to R^-} \frac{p_j t^j}{p(t)} = 0, \text{ for every } j \in \mathbb{N}_0, \text{ (see, e.g., [1])}.$$

In what follows, we assume that $t \in (0, R)$ and $0 < R \leq \infty$.

Recently Ünver and Orhan [19] have studied the concepts of strong summability and statistical convergence with respect to power series methods. Following their definitions we set the notations

$$W_0(P) = \left\{ x \in w : \lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j |x_j| = 0 \right\}$$

and

$$W(P) = \{x \in w : x - Le \in W_0(P) \text{ for some } L\},$$

where $e = (1, 1, \ldots)$. When $x \in W(P)$ we say that x is $P-$strongly convergent to L.

Recall that $x=(x_j)$ is said to be $P-$statistically convergent to L if $\chi_{K(x-Le;\varepsilon)}$ is contained in $W_0(P)$ for every $\varepsilon>0$ where $\chi_{K(x;\varepsilon)}$ is the characteristic function of the set

$$K(x;\varepsilon)=\{j\in\mathbb{N}_0:|x_j|\geq\varepsilon\},\ \text{(see, [19])}.$$

By $st(P)$, we denote the space of all $P-$statistically convergent sequences. We recall that some criteria on $P-$statistical convergence may be found in [18]. Now we give the definiton of $P-$strong convergence with respect to an Orlicz function F as in the same fashion [4]. Note that Connor [3] and Kolk [13] used similar idea for strong matrix summability with respect to a modulus function.

Let F be an Orlicz function and let P be a regular power series method. We introduce the following sequence spaces

$$W_0(P,F)=\left\{x\in w:\lim_{t\to R^-}\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jF(|x_j|)=0\right\}$$

and

$$W(P,F)=\{x\in w:x-Le\in W_0(P,F)\ \text{for some}\ L\}.$$

If $x\in W(P,F)$ we say that x is $P-$strongly convergent to L with respect to an Orlicz function F.

If there is a constant $H>0$ such that

$$F(2u)\leq HF(u),\ \ (\text{for all}\ u\geq0),$$

we say that the Orlicz function F satisfies $\triangle_2-$condition. This condition is equivalent to the condition

$$F(tu)\leq HtF(u),$$

for all $u\geq0$ and for $t\geq1$ $(see, e.g., [11])$.

2. Some inclusion results

In this section, we give some inclusion results among the spaces $W(P)$, $W(P,F)$ and $st(P)$; and show that all these concepts are equivalent for bounded sequences.

Our first result can be obtained from the technique given by Prashar and Choudhary in [16].

Proposition 2.1. Given any Orlicz function satisfying $\triangle_2-$condition we have the inclusions $W_0(P)\subset W_0(P,F)$ and $W(P)\subset W(P,F)$.

Proof. It is enough to prove $W_0(P)\subset W_0(P,F)$. Now let $x=(x_j)\in W_0(P)$ and F be an Orlicz function satisfying $\triangle_2-$condition. From the right continuity of F at zero, for a given $\varepsilon>0$, there exists δ with $0<\delta<1$ such that $F(s)<\varepsilon$ whenever $0\leq s<\delta$.

Then, we may write that

$$\begin{aligned}\frac{1}{p(t)}\sum_{j=1}^{\infty}p_jt^jF(|x_j|)&=\frac{1}{p(t)}\sum_{j:|x_j|<\delta}p_jt^jF(|x_j|)+\frac{1}{p(t)}\sum_{j:|x_j|\geq\delta}p_jt^jF(|x_j|)\\&<\varepsilon+\frac{1}{p(t)}\sum_{j:|x_j|\geq\delta}p_jt^jF(|x_j|).\end{aligned}$$

Since $0 < \delta < 1$, we get for every $j \in \mathbb{N}$, that

$$|x_j| < \frac{1}{\delta}|x_j| < 1 + \frac{|x_j|}{\delta}.$$

Also, F is an Orlicz function satisfying $\triangle_2-$condition,we observe that

$$\begin{aligned} F(|x_j|) \leq F(1 + \frac{|x_j|}{\delta}) &= F(\frac{1}{2}2 + \frac{1}{2}\frac{2|x_j|}{\delta}) \\ &\leq \frac{1}{2}F(2) + \frac{H|x_j|}{2\delta}F(2) \\ &< (1+H)F(2)\frac{|x_j|}{\delta}. \end{aligned}$$

Then, combining the above results, we obtain that

$$\frac{1}{p(t)}\sum_{j=1}^{\infty} p_j t^j F(|x_j|) < \varepsilon + \frac{(1+H)}{\delta}F(2)\frac{1}{p(t)}\sum_{j=1}^{\infty} p_j t^j |x_j|.$$

Since $x = (x_j) \in W_0(P)$, we see that

$$\lim_{0<t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j F(|x_j|) = 0,$$

which completes the proof.

The next result is an analog of Lemma 1 of Demirci [4].

Lemma 2.1. Let F be an Orlicz function satisfying $\triangle_2-$condition. Then $W_0(P,F)$ is an ideal in ℓ^∞.

Proof. Given $x \in W_0(P,F)$ and $y \in \ell^\infty$ we show that $xy \in W_0(P,F)$. Since $y \in \ell^\infty$, there is $H_1 > 1$ so that $\|y\| \leq H_1$.

Since F is nondecreasing and satisfies $\triangle_2-$condition, we have

$$F(|x_k y_k|) \leq F(H_1|x_k|) \leq H(1+H_1)F(|x_k|), \quad (H>0).$$

By hypothesis

$$\lim_{t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j F(|x_j|) = 0,$$

so we get that

$$\lim_{t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j F(|x_j y_j|) \leq H(1+H_1)\lim_{t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j F(|x_k|)$$

from which we immediately conclude that

$$\lim_{t\to R^-} \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j F(|x_j y_j|) = 0.$$

This proves the result.

We require the following lemmas:

Lemma 2.2. Let M be an ideal in ℓ^{∞} and let $x \in \ell^{\infty}$. Then, x is in the closure of M in ℓ^{∞} if and only if $\chi_{K(x;\varepsilon)} \in M$ for all $\varepsilon > 0$ $(see, [3])$.

As in Lemma 2.1 one can observe that $W_0(P) \cap \ell^{\infty}$ is an ideal in ℓ^{∞}. On the other hand using a similar idea by Freedman and Sember in [6] we get the following lemma.

Lemma 2.3. If P is a regular power series method, then $W_0(P) \cap \ell^{\infty}$ is a closed ideal in ℓ^{∞}.

The next result is also an analog of Theorem 4 of Demirci [4]. It exhibits that $P-$strong convergence and $P-$strong convergence with respect to an Orlicz function are boundedly equivalent.

Theorem 2.1. Let F be an Orlicz function that satisfies $\triangle_2-$condition and let P be a regular power series method. Then

$$W(P,F) \cap \ell^{\infty} = W(P) \cap \ell^{\infty}.$$

Proof. It is enough to prove that $W_0(P,F) \cap \ell^{\infty} = W_0(P) \cap \ell^{\infty}$. By Proposition 2.1 we have that $W_0(P) \cap \ell^{\infty} \subseteq W_0(P,F) \cap \ell^{\infty}$.

We now prove the opposite inclusion. Note that

$$\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j F\left(\chi_{K(x;\varepsilon)}(j)\right) = F(1) \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \chi_{K(x;\varepsilon)}(j). \tag{2.1}$$

Let $x \in W_0(P,F) \cap \ell^{\infty}$ and $\varepsilon > 0$. Now define a sequence $y = (y_j) \in \ell^{\infty}$ by $y_j = \dfrac{1}{x_j}$ if $|x_j| \geq \varepsilon$ and $y_j = 0$ otherwise. Since $xy = \chi_{K(x;\varepsilon)}$ and $\chi_{K(x;\varepsilon)} \in W_0(P,F) \cap \ell^{\infty}$, we have that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j F\left(\chi_{K(x;\varepsilon)}(j)\right) = 0.$$

It follows from (2.1) that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \chi_{K(x;\varepsilon)}(j) = 0.$$

Lemma 2.2 and Lemma 2.3 yield that $x \in W_0(P) \cap \ell^{\infty}$ which completes the proof.

Ünver and Orhan [19] have intrduced the concept of $P-$ uniformly integrable sequence as follows:

Let $P-$ be a power series method and let $x = (x_j)$ be a real sequence. Then, x is $P-$ uniformly integrable if there exists $t_0 \in [0,R)$ such that

$$\lim_{c \to \infty} \sup_{t \in [t_0,R)} \frac{1}{p(t)} \sum_{|x_j| \geq c} p_j t^j |x_j| = 0.$$

They have shown that a sequence x is $P-$ strongly convergent if and only if it is $P-$statistically convergent and $P-$ uniformly integrable. By $st(P)$ and $U(P)$ we respectively denote the spaces of all $P-$statistically convergent and $P-$uniformly integrable sequences. Note that any bounded sequence is $P-$uniformly bounded. Combining this result with Theorem 2.1 we immediately conclude that $P-$strong convergence, $P-$statistical convergence and $P-$strong convergence with respect to an Orlicz function satisfying $\triangle_2-$condition are boundedly equivalent. We state it formally in the next result.

Theorem 2.2. Let F be an Orlicz function that satisfies $\triangle_2-$condition. Then

$$W(P,F)\cap\ell^{\infty}=W(P)\cap\ell^{\infty}=st(P)\cap\ell^{\infty}.$$

Our final result concerns that $W(P,F)$ lies between $W(P)$ and $st(P)$.

Theorem 2.3. Let F be an Orlicz function that satisfies $\triangle_2-$condition and let P be a regular power series method. Then

$$W(P)\subset W(P,F)\subset st(P).$$

Proof. The first inclusion is given in Proposition 2.1. In order to prove the second inclusion let $x\in W_0(P,F)$ and $y\in\ell^{\infty}$. Then, it follows from Lemma 2.1 that xy is $W_0(P,F)$. Let $\varepsilon>0$ and $x\in W_0(P,F)$. Now define a bounded sequence $y=(y_j)$ by $y_j=\dfrac{1}{x_j}$, if $|x_j|\geq\varepsilon$ and $y_j=0$ otherwise. Observe that

$$xy=\chi_{K(x;\varepsilon)}\in W_0(P,F)\cap\ell^{\infty}.$$

Theorem 2.1 implies that $\chi_{K(x;\varepsilon)}\in W_0(P)\cap\ell^{\infty}$.

Hence, x is $P-$statistically convergent to zero by Theorem 3 of [19] as any bounded $P-$strongly convergent sequence is $P-$statistically convergent.

Combining this result with Theorem 3 of Ünver and Orhan [19] we immediately conclude the following result which is an ultimate improvement of Theorem 2.2.

Theorem 2.4. Let F be an Orlicz function satisfying $\triangle_2-$condition. Then we have

$$W(P)=W(P,F)\cap U(P)=st(P)\cap U(P).$$

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