Turkish Journal of Mathematics

Volume 46 | Number 3 Article 10

1-1-2022

A Köthe-Toeplitz dual of a generalized Cesaro difference sequence space, a degenerate Lorentz space, their corresponding function spaces and fpp

VEYSEL NEZİR

NİZAMİ MUSTAFA

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 810 – 831
© TÜBİTAK
doi:10.55730/1300-0098.3125

A Köthe–Toeplitz dual of a generalized Cesàro difference sequence space, a degenerate Lorentz space, their corresponding function spaces and fpp

Veysel NEZİR*, **Nizami MUSTAFA**
Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey

Received: 15.11.2021 • **Accepted/Published Online:** 17.01.2022 • **Final Version:** 11.03.2022

Abstract: In 1970, Cesàro sequence spaces was introduced by Shiue. In 1981, Kızmaz defined difference sequence spaces for ℓ^∞, c_0 and c. Then, in 1983, Orhan introduced Cesàro difference sequence spaces. Both works used difference operator and investigated the Köthe–Toeplitz dual for the new Banach spaces they introduced. Later, various authors generalized these new spaces, especially the one introduced by Orhan. In this study, first we discuss the fixed point property for these spaces and for the corresponding function space of the Köthe–Toeplitz dual. Moreover, we consider another generalized space which is a degenerate Lorentz space because the spaces we consider are somehow related in terms of their construction. In fact, the Köthe–Toeplitz dual is contained in ℓ^1, the corresponding function space contains Lebesgue space, the Banach space of Lebesgue integrable functions on $[0,1]$, $L_1[0,1]$ but for the degenerate Lorentz space and its corresponding function space, this is reversed completely. Then, as an important result, for both function spaces we show that they fail the weak fixed point property for isometries and even for contractions. As the second main result, by passing to counting measure, we take the corresponding sequence spaces, and for both spaces we get large classes of closed, bounded and convex subsets satisfying the fixed point property for nonexpansive mappings as a Goebel and Kuczumow analogy after noting that in ℓ^1 Goebel and Kuczumow found a large class with the fixed point property for nonexpansive mappings.

Key words: Fixed point property, nonexpansive mapping, weak fixed point property, contractive mapping, Cesàro difference sequences, Köthe–Toeplitz dual

1. Introduction and preliminaries

A Banach space is said to have a certain fixed point property if every self-map of any closed, bounded and convex (c.b.c.) domain in that space satisfying the given metric condition has a fixed point. For example, a c.b.c. domain, D, has the fixed point property for nonexpansive maps [fpp(ne)] if every nonexpansive map on D has a fixed point. Here, by nonexpansive map we mean a map $T : D \to D$ such that $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in D$.

Maurey and others wondered if all weakly compact domains had the fpp(ne). His result [24] on c_0 indicated that they might. However, Alspach [1] provided the first example of a fixed point free map on a weakly compact, convex set.

Alspach used Baker's transformation for his example in $L_1[0,1]$. His construction is given as in the

*Correspondence: veyselnezir@yahoo.com
2010 *AMS Mathematics Subject Classification:*

following.

Define $C{:=}\{f{\in}L_1[0,1] \ : \ 0{\le}f{\le}1, \ t{\in}[0,1]\}$ and consider the nonexpansive mapping

$$Tf(t)=\begin{cases} \min\{2f(2t),1\} & \text{if } \ 0{\le}t{<}\frac{1}{2} \\ \max\{2f(2t{-}1)-1,0\} & \text{if } \ \frac{1}{2}{\le}t{<}\,1. \end{cases} \tag{1.1}$$

Then, set $C_{\frac{1}{2}}{:=}\{f{\in}C \ : \ \int_0^1 \ fdm=\frac{1}{2}\}$ where m is Lebesque measure. So he shows that T is a fixed point free isometry on $C_{\frac{1}{2}}$.

Later, different examples like his by some researchers have been given such as [10, 11] by Dowling, Lennard and Turett, [22] by Llorens-Fuster and Sims, and [30] by Sine.

For example, using the similar set to Alspach's, Sine constructed a composite isometry given by

$$Sf{=}\chi_{[0,1]}{-}Tf, \ \ \forall f{\in}C_{\frac{1}{2}} \ . \tag{1.2}$$

On the other hand, Dowling, Lennard and Turett's study [10] showed that there exists a fixed point free isometry on $C_{\frac{1}{2}}$ as in the following:

Let

$$\Delta f(t):=\begin{cases} f(2t) & \text{if } 0\le t<\frac{1}{2} \\ 1-f(2t-1) & \text{if } \frac{1}{2}\le t<1. \end{cases} \tag{1.3}$$

Then, they showed that $T\Delta{:}C_{\frac{1}{2}}{\to}C_{\frac{1}{2}}$ is an isometry such that it does not have any fixed point.

So all the examples we mentioned above are isometries. As we recall the famous Banach contraction principle, contractions defined on complete metric spaces or closed subsets of those have fixed points. Then, researchers wondered if there exist fixed point free contractions defined on weakly compact sets. Note that if a map $f{:}D{\to}D$ satisfies the condition $\|fx{-}fy\|{<}\|x{-}y\|$ for all $x,y{\in}D$, then we call it a contraction.

The first example of a fixed point free such map on a weakly compact convex set was introduced in [4]. Later, Casini, Miglierina, and Piasecki in [6] provided the second example. There are more examples found in Sivek's Ph.D. thesis [31] written under supervision of Chris Lennard.

In this study, firstly, we consider two different Lebesgue $L_1[0,1]$-like Banach function spaces. The spaces we take have some relevance in terms of their construction. The first one is originated from the Köthe–Toeplitz dual of a generalized Cesàro difference sequence space. It is actually the corresponding function space containing $L_1[0,1]$ where counting measure produces the dual. The second one is originated from a degenerate Lorentz–Marcinkiewicz function space. As our first important result, we show that both spaces fail the weak fixed point property for isometries and even for contractions; that is, we show for both of them that there exist weakly compact, convex subsets and invariant fixed point free isometries and contractions defined on them. As the second main result, passing to the counting measure, we work on their corresponding sequence spaces, one contained in ℓ^1 and the other one containing ℓ^1, respectively. Noting that Goebel and Kuczumow in 1970 [16] found a very large class of nonweakly c.b.c. subsets in ℓ^1 with fixed point property for nonexpansive mappings and Everest, in his Ph.D. thesis [14], written under supervision of Chris Lennard, extended Goebel and with Kuczumow's result by finding larger classes in ℓ^1 with fixed point property for nonexpansive mappings, we get an analogue result for the corresponding sequence spaces we take care of and we obtain large classes of closed, bounded and convex subsets satisfying the fixed point property for nonexpansive mappings.

Now, let us recall the definition of the Cesàro sequence spaces as the starting point of our interest. The Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| \; < \infty \right\}$$

were introduced by Shiue [29] in 1970, where $1 \leq p < \infty$. It has been shown that $\ell^p \subset \mathrm{ces}_p$ for $1 < p \leq \infty$. Moreover, it has been shown that Cesàro sequence spaces ces_p for $1 < p < \infty$ are separable reflexive Banach spaces. Furthermore, it was also proved by Cui and Hudzik [7], Cui, Hudzik and Li [8] and Cui, Meng and Pluciennik [9] that Cesàro sequence spaces ces_p for $1 < p < \infty$ have the fixed point property. They prove this result using different methods. One method is to calculate Garcia-Falset coefficient. It is known that if Garcia-Falset coefficient is less than 2 for a Banach space, then it has the fixed point property for nonexpansive mappings [15]. Using this fact, since they calculate this coeeficient for ces_p as $2^{1/p}$ similary to what it is for ℓ^p, they point the result for the Cesàro sequence spaces. Another fact is that they see that the space has normal structure for $1 < p < \infty$. Then using the fact via Kirk [19] that reflexive Banach spaces with normal structure has the fixed point property, they easily deduce that the space has the fixed point property for $1 < p < \infty$. Their results on Cesàro sequence spaces as a survey can be seen in [6].

Later, in 1981, Kızmaz [18] introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences $x = (x_n)_n$, respectively. As it is seen below, his definitions for these spaces were given using difference operator applied to the sequence x, $\triangle\, x = (x_k - x_{k+1})_k$.

$$\ell^\infty(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\triangle\, x \in \ell^\infty \},$$

$$\mathrm{c}(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\triangle\, x \in \mathrm{c} \},$$

$$\mathrm{c}_0(\triangle) = \{ x = (x_n)_n \subset \mathbb{R} \,|\triangle\, x \in \mathrm{c}_0 \}.$$

Kızmaz investigated Köthe–Toeplitz duals and some properties of these spaces.

Furthermore, Cesàro sequence spaces X^p of nonabsolute type were defined by Ng and Lee [25] in 1977 as follows:

$$X^p = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| \; < \infty \right\},$$

where $1 \leq p < \infty$. They prove that X^p is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Later, in 1983, Orhan [26] introduced Cesàro difference sequence spaces by the following definitions:

$$C_p = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty \right\},$$

where $1 \leq p < \infty$ and $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$. He noted that their norms are given as below for any $x = (x_n)_n$:

$$\|x\|_p^* = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty^* = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|$$

respectively.

Orhan showed that there exists a linear bounded operator $S : C_p \to C_p$ for $1 \leq p \leq \infty$ such that Köthe–Toeplitz $\beta-$Duals of these spaces are given respectively as follows:

$$S(C_p)^\beta = \{ a = (a_n)_n \subset \mathbb{R} \mid (na_n)_n \in \ell^q \} \text{ where } 1 < p < \infty \text{ and } q = \frac{p}{p-1},$$

$$S(C_1)^\beta = \{ a = (a_n)_n \subset \mathbb{R} \mid (na_n)_n \in \ell^\infty \} \text{ and}$$

$$S(C_\infty)^\beta = \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; (na_n)_n \in \ell^1 \right\}.$$

It might be better to use the notation $X^p(\triangle)$ instead of C_p for $1 \leq p \leq \infty$ since we also recalled the difference sequence spaces and used similar type of notation.

We note that Orhan also proved that $X^p \subset X^p(\triangle)$ for $1 \leq p \leq \infty$ strictly. Also, one can clearly see that $X^p(\triangle)$ is linearly isomorphic and isometric to ℓ^p for $1 \leq p \leq \infty$. Thus, one would easily deduce that they have similar properties in terms of the fixed point theory. That is, for $1 < p < \infty$ they have the fixed point property for nonexpansive mappings but for other two cases they fail.

Note also that Köthe–Toeplitz Dual for $p = \infty$ case in Orhan's study and ℓ^∞ case in Kızmaz study coincides.

Furthermore, Et and Çolak [12] generalized the spaces introduced in Kızmaz's work [18] in the following way for $m \in \mathbb{N}$.

$$\ell^\infty(\triangle^m) = \{ x = (x_n)_n \subset \mathbb{R} \mid \triangle^m x \in \ell^\infty \},$$

$$\mathrm{c}(\triangle^m) = \{ x = (x_n)_n \subset \mathbb{R} \mid \triangle^m x \in \mathrm{c} \},$$

$$\mathrm{c}_0(\triangle^m) = \{ x = (x_n)_n \subset \mathbb{R} \mid \triangle^m x \in \mathrm{c}_0 \}$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})_k$, $\triangle^0 x = (x_k)_k$, $\triangle^m x = (\triangle^m x_k) = \left(\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}\right)_k$ and $\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$.

Also, Et [13] and Tripathy et al. [33] generalized the space introduced by Orhan in the following way for $m \in \mathbb{N}$.

$$X^p(\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right|^p \right)^{1/p} < \infty \right\}$$

and

$$X^{\infty}\ (\triangle^m) = \left\{ x = (x_n)_n \subset \mathbb{R} \;\middle|\; \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle^m x_k \right| < \infty \right\}.$$

Then, it is seen that that Köthe–Toeplitz dual for $p = \infty$ case in Et's study [13] and ℓ^{∞} case in Et and Çolak's study [12] coincides such that Köthe–Toeplitz dual was given as below for any $m \in \mathbb{N}$.

$$\begin{aligned} \Upsilon_m : &= \left\{ a = (a_n)_n \subset \mathbb{R} \mid (n^m a_n)_n \in \ell^1 \right\} \\ &= \left\{ a = (a_k)_k \subset \mathbb{R} : \ \|a\|_{\triangle} = \sum_{k=1}^{\infty} k^m |a_k| < \infty \right\}. \end{aligned} \tag{1.4}$$

Note that $\Upsilon_m \subset \ell^1$.

One can see that corresponding function space for these duals can be given as below:

$$\Sigma_m := \left\{ \begin{array}{l} f\colon [0,1] \to \mathbb{R} \\ \text{measurable} \end{array} \ : \ \|f\| = \int_0^1 t^m |f(t)|\, dt < \infty \right\}. \tag{1.5}$$

Note that $L_1[0,1] \subset \Sigma_m$ and Υ_m is the space when counting measure is used for Σ_m.

Before giving some preliminaries, we can note that related with the different generalizations of the spaces in this study, one can find applied ideas using contractions about studying the existence of solutions for hybrid and nonhybrid k-dimensional sequential inclusion systems or the existence of solution for a multiterm fractional integro-differential system with special boundary conditions under some different conditions (see, for example, [2, 27]).

Now, the followings are needed as preliminaries.

Definition 1.1 *Let* $(\mathrm{X}, \|\cdot\|)$ *be a Banach space ,* E *be a nonempty c.b.c. subset and* $T : E \to \mathrm{E}$ *be a mapping.*

1. T *is called an affine mapping if for every* $\lambda \in [0,1]$ *and* $\mathrm{x}, \mathrm{y} \in \mathrm{E}$*,* $\mathrm{T}((1-\lambda)\,\mathrm{x} + \lambda\ \mathrm{y}) = (1-\lambda)\,\mathrm{T}(\mathrm{x}) + \lambda\ \mathrm{T}(\mathrm{y})$*.*

2. T *is called a* $\|\cdot\|-$ *nonexpansive mapping if for every* $\mathrm{x}, \mathrm{y} \in \mathrm{E}$*,* $\|\mathrm{T}(\mathrm{x}) - \mathrm{T}(\mathrm{y})\| \leq \|\mathrm{x} - \mathrm{y}\|$*.*

Furthermore, if every $\|\cdot\|-$ *nonexpansive mapping* $\mathrm{T} : \mathrm{E} \to \mathrm{E}$ *has a fixed point; i.e. if there exists a* $\mathrm{u} \in \mathrm{E}$ *such that* $\mathrm{T}(\mathrm{u}) = \mathrm{u}$*, then we say that* E *has the fpp(ne).*

3. T *is called an isometry (or isometric mapping) if for every* $\mathrm{x}, \mathrm{y} \in \mathrm{E}$*,* $\| T(x) - T(y) \| = \| x - y \|$*. Note that every isometry is nonexpansive.*

Furthermore, if every isometry $\mathrm{T} : \mathrm{E} \to \mathrm{E}$ *has a fixed point; i.e. if there exists a* $\mathrm{u} \in \mathrm{E}$ *such that* $\mathrm{T}(\mathrm{u}) = \mathrm{u}$*, then we say that* E *has the fixed point property for isometries [fpp(i)]. Note that if* E *fails the fpp(i), then it fails the fpp(ne).*

4. T *is called a strictly contractive mapping if there exists a scalar* $\lambda \in [0,1)$ *such that for every* $\mathrm{x}, \mathrm{y} \in \mathrm{E}$*,* $\| T(x) - T(y) \| \leq \lambda \| x - y \|$ *(a map that shrinks distances by a uniform factor).*

5. T *is called a contractive mapping if for every* $\mathrm{x},\mathrm{y}\in\mathrm{E}$, $\| T(x)-T(y)\| < \| x-y\|$. *Please note the difference between this and the strict contractions of Banach's contraction mapping theorem, which proves any invariant strict contraction defined on complete metric spaces must have a fixed point [3].*

Furthermore, if every contractive mapping $\mathrm{T}:\mathrm{E}\to\mathrm{E}$ *has a fixed point; i.e. if there exists a* $\mathrm{u}\in\mathrm{E}$ *such that* $\mathrm{T(u)}=\mathrm{u}$, *then we say that* E *has the fixed point property for contractive mappings [fpp(c)].*

Definition 1.2 *Let* $(X,\|\cdot\|)$ *be a Banach space and* C *be a nonempty weakly compact, convex subset.*

1. If for every nonexpansive mapping $T:C\to C$, *there exists* $z\in C$ *with* $T(z)=z$, *then* C *is said to have the weak fixed point property for nonexpansive mappings [w-fpp(ne)].*

2. If for every isometry $T:C\to C$, *there exists* $z\in C$ *with* $T(z)=z$, *then* C *is said to have the weak fixed point property for isometries [w-fpp(i)].*

3. If for every contractive mapping $T:C\to C$, *there exists* $z\in C$ *with* $T(z)=z$, *then* C *is said to have the weak fixed point property for contractive mappings [w-fpp(c)].*

We also note that through the study, the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1, where i^{th} term is 1 and others 0 for the e_i.

2. Main results

We will give our results in two subsections. As we stated in the previous section, firstly, we consider two different Lebesgue $L_1[0,1]$-like Banach function spaces. The spaces we take have some relevance in terms of their construction. The first one is originated from the Köthe–Toeplitz dual of a generalized Cesàro difference sequence space. It is actually the corresponding function space containing $L_1[0,1]$ where counting measure produces the dual. The second one is originated from a degenerate Lorentz–Marcinkiewicz function space. As our first important result, we show that both spaces fail the weak fixed point property for isometries and even for contractions; so they fail the fixed point property as well. As the second main result, passing to the counting measure, we work on the corresponding sequence spaces, one contained in ℓ^1 and the other one containing ℓ^1, respectively. Noting that Goebel and Kuczumow in 1970 [16] found a very large class of nonweakly c.b.c. subsets in ℓ^1 with fixed point property for nonexpansive mappings and Everest, in his Ph.D. thesis [14], written under supervision of Chris Lennard, extended Goebel and Kuczumow's result by finding larger classes in ℓ^1 with fixed point property for nonexpansive mappings, we get an analogue result and obtain large classes of closed, bounded and convex subsets satisfying the fixed point property for nonexpansive mappings for both spaces we take care of.

2.1. Failure of the weak fixed point property for isometries and contractions in two Lebesgue-like spaces

In this section, we consider two different Lebesgue $L_1[0,1]$-like Banach function spaces. The first one is the corresponding function space for the Köthe–Toeplitz dual of a generalized Cesàro difference sequence space. The second space we consider is related with a degenerate Lorentz–Marcinkiewicz function space but its construction also resembles to our first function space we take care of. We show that both spaces fail the weak fixed point property for nonexpansive mappings and so they fail the fixed point property. In fact, we show for both of them that there exist weakly compact, convex subsets and invariant fixed point free contractive mappings defined on them. Now, let us see these spaces and our findings.

2.1.1. A Lebesgue-like space containing $L_1[0,1]$

Now firstly we consider the corresponding function space for a Köthe–Toeplitz dual of a generalized Cesàro difference sequence space. Then, we give the following theorem.

Theorem 2.1 *Fix $m \in \mathbb{N}$ and consider the corresponding function space for Σ_m, the Köthe–Toeplitz dual of a generalized Cesàro difference sequence space, given as in (1.5). Then, Σ_m fails the weak fixed point property; that is, there exists an invariant fixed point free isometry defined on a weakly compact subset of Σ_m. In fact, there exists an invariant fixed point free contractive mapping defined on the same set.*

Proof Fix $m \in \mathbb{N}$ and first consider the set

$$D_m := \{f \in L_1[0,1]: \quad 0 \le f(t) \le 1, \quad t \in [0,1]\}$$

and define

$$D_{m\frac{1}{2}} := \left\{ f \in D_m \ : \ \int_0^1 \mathrm{t}^{\mathrm{m}} f(t)\, dt = \frac{1}{2} \right\}.$$

Then, consider the composite mapping which uses Alspach's mapping in (1.1) as below defined on $D_{m\frac{1}{2}}$.

$$(\sigma(f))(t) := \frac{T(t^m f)}{t^m}, \ \forall f \in D_{m\frac{1}{2}} \text{ and } \forall t \in [0,1]. \tag{2.1}$$

Then, for any $f, g \in D_{m\frac{1}{2}}$, we have

$$\begin{aligned}
\|\sigma(f) - \sigma(g)\| &= \int_0^1 t^m \left|\sigma(f) - \sigma(g)\right| dt \\
&= \int_0^1 \left|T(t^m f) - T(t^m g)\right| dt \\
&= \|\mathrm{T}(t^m f) - \mathrm{T}(t^m g)\|_1 \\
&= \|t^m f - t^m g\|_1 \\
&= \int_0^1 \left|t^m f(t) - t^m g(t)\right| dt \\
&= \int_0^1 t^m \left|f(t) - g(t)\right| dt \\
&= \|f - g\|.
\end{aligned}$$

Then, σ is a fixed point free isometry on $D_{m\frac{1}{2}}$ by the proof of Alspach's theorem in [1].

Also, analogously to Sine's function given by the formula (1.2), we can define

$$(\rho f)(t) = \frac{\chi_{[0,1]}}{t^m} - (\sigma f)(t), \ \ \forall f \in D_{m\frac{1}{2}} \text{ and } \forall t \in [0,1]. \tag{2.2}$$

Hence, similarly to Sine's result [30], it is easy to see the mapping ρ is another fixed point free isometry on $D_{m\frac{1}{2}}$.

Moreover, analogously to Dowling, Lennard and Turett's mapping given by the formula (1.3), we can define

$$\hat{\Delta} f(t) := \begin{cases} (2t)^{\mathrm{m}} f(2t) & \text{if } \ 0 \leq t \leq \frac{1}{2} \\ 1-(2t-1)^{\mathrm{m}} \mathrm{f}(2t-1) & \text{if } \ \frac{1}{2} < t < 1 \end{cases}, \tag{2.3}$$

and next define $\hat{\Delta}^* : D_{m\frac{1}{2}} \rightarrow D_{m\frac{1}{2}}$ by $\hat{\Delta}^*(f)(t) := \frac{\hat{\Delta} f(t)}{t^m} = \frac{\Delta(t^m f(t))}{t^m}$, $\forall t \in [0,1]$. Then, we have the following claim.

Claim: $\sigma \hat{\Delta}^*: D_{m\frac{1}{2}} \rightarrow D_{m\frac{1}{2}}$ is a fixed point free isometry where σ is given as in the formula (2.1).

Proof of the claim: For any $f, g \in D_{m\frac{1}{2}}$, we have

$$\begin{aligned}
\left\| \sigma \hat{\Delta}^*(f) - \sigma \hat{\Delta}^*(g) \right\| &= \int_0^1 t^m \left| \frac{T\left(t^m \hat{\Delta}^*(f)(t) \right)}{t^m} - \frac{T\left(t^m \hat{\Delta}^*(g)(t) \right)}{t^m} \right| dt \\
&= \int_0^1 \left| T\left(t^m \hat{\Delta}^*(f)(t) \right) - T\left(t^m \hat{\Delta}^*(g)(t) \right) \right| dt \\
&= \left\| \mathrm{T}\left(\hat{\Delta}(f) \right) - \mathrm{T}\left(\hat{\Delta}(g) \right) \right\|_1 \\
&= \left\| \hat{\Delta}(f) - \hat{\Delta}(g) \right\|_1 \\
&= \int_0^1 \left| \Delta\left(t^m f(t) \right) - \Delta\left(t^m g(t) \right) \right| dt \\
&= \left\| \Delta\left(t^m f(t) \right) - \Delta\left(t^m g(t) \right) \right\|_1 \\
&= \left\| t^m f - t^m g \right\|_1 \\
&= \int_0^1 \left| t^m f - t^m g \right| dt \\
&= \| f - g \|.
\end{aligned}$$

In conclusion, for each $m \in \mathbb{N}$, we obtain three different fixed point free invariant isometries σ, ρ and $\hat{\Delta}^*$ defined on $D_{m\frac{1}{2}}$. Then, firstly, using the ideas of the study by Burns, Lennard and Sivek [4], we can show that if T is taken as the invariant mapping σ defined on $D_{m\frac{1}{2}}$, then the following mapping is fixed point free contractive: Define $R : D_{m\frac{1}{2}} \rightarrow D_{m\frac{1}{2}}$ by

$$R(f) = \sum_{n=0}^{\infty} \frac{T^n(f)}{2^{n+1}}.$$

Secondly, using the ideas in Ph.D. thesis of Sivek [31, Section I.C.], written under supervision of Chris Lennard, we can show that if T is one of the invariant mappings σ or $\sigma \hat{\Delta}^*$ defined on $D_{m\frac{1}{2}}$, then the following mapping is fixed point free contractive: Define $R : D_{m\frac{1}{2}} \rightarrow D_{m\frac{1}{2}}$ by

$$R(f) = \sum_{n=0}^{\infty} \frac{T^n(f)}{2^{n+1}}.$$

To reach these results, we only need to imitate those works. So we will not include them.

□

2.1.2. A Lebesgue-like space contained in $L_1[0,1]$

Now, we consider another Lebesgue like Banach space which is actually contained in Lebesgue space $L_1[0,1]$. It can be said that the space we consider is a degenerate Lorentz space. So firstly we recall Lorentz space.

Definition 2.2 *Let $\alpha \in (0,1)$.*

$$L_{\alpha,1}[0,1] := \left\{ f : [0,1] \to \mathbb{R} \ \ measurable \ \middle| \ \| x\|_{\alpha,1} := \int_0^1 \frac{\alpha f^*(t)}{t^{1-\alpha}} dt < \infty \right\},$$

where $f^*(t)$ is the decreasing rearrangement of $|f(t)|$; that is, $f^*(t)$ is ordered decreasing and equimeasurable with $|f(t)|$.

Note that rearrangement is a nonexpansive mapping with respect to Lebesgue norms. The nonincreasing rearrangement of a function was first studied by Steiner [32] and it is defined as a certain generalized inverse of the distribution function. There have been many researches on these type of functions but first most known properties were given by Hardy, Littlewood and Pólya in 1930's [17].

Another $L_1[0,1]$ analogue, Lorentz–Marcinkiewicz space, is defined as follows:

Definition 2.3 *Let $w : [0,1] \to \mathbb{R}$ be the weight function which is nonincreasing such that $\int_0^\infty w dm = \infty$ where m is the Lebesgue measure. Then,*

$$L_{w,1}[0,1] := \left\{ \begin{array}{c} f\colon [0,1]\to\mathbb{R} \\ \text{measurable} \end{array} \ \middle| \ \|x\|^{w,1} := \int_0^1 w(t) f^*(t) dt < \infty \right\},$$

where $f^*(t)$ is the decreasing rearrangement of $|f(t)|$.

But in this study we will be working on the Banach space defined below. Note that this space is originated from the space introduced in previous two definitions but here we do not use decreasing rearrangements instead we use the absolute value. One may consider subspaces of decreasing functions in Lorentz spaces or Lorentz–Marcinkiewicz spaces. So our space would generalize this type of subspaces. Standard references for Lorentz–Marcinkiewicz spaces are [20, 21, 23].

Definition 2.4 *Let $m \in \mathbb{N}$.*

$$\mathcal{M}_m := \left\{ \begin{array}{c} f\colon [0,1] \to\mathbb{R} \\ \text{measurable} \end{array} \ : \ \|f\|^\sim = \int_0^1 \frac{|f(t)|}{t^m} dt < \infty \right\}.$$

Then, it is easy to see that for any $m \in \mathbb{N}$, $\mathcal{M}_m$ is a Banach space contained in Lebesgue space $L_1[0,1]$.

The following theorem shows that for any $m \in \mathbb{N}$, $\mathcal{M}_m$ does not have weak fixed point property and in fact there exist a weakly compact subset and invariant fixed point free contractive mappings defined on it.

Theorem 2.5 *For any $m \in \mathbb{N}$, there exists a weakly compact subsets of $\mathcal{M}_m$ such that there exist invariant fixed point free isometries and invariant fixed point free contractive mappings defined on that set.*

Proof Fix $m \in \mathbb{N}$. In $\mathcal{M}_m$, consider the subset
$K_m{:=}\left\{f{\in}\mathcal{M}_m \ : \ 0{\leq}f{\leq}1, \ t{\in}[0,1], \ \int_0^1 \frac{f(t)}{t^m}dt{\leq}1\right\}$, and next consider the set

$$K_{m\frac{1}{2}}{:=}\left\{f{\in}K_m : \int_0^1 \frac{f(t)}{t^m}dt{=}\frac{1}{2}\right\}$$

then also consider the mapping

$$(\psi f)(t) {=} t^m T\left(\frac{f(t)}{t^m}\right), \ \forall f \in K_{m\frac{1}{2}} \text{ and } \forall t \in [0,1] \tag{2.4}$$

where T is Alspach's mapping with the formula (1.1).
Then, for any $f, g{\in}K_{m\frac{1}{2}}$, we have

$$\begin{aligned}
\|\psi(f){-}\psi(g)\|^{\sim} &= \int_0^1 \frac{\left|t^m T\left(\frac{f(t)}{t^m}\right) - t^m T\left(\frac{g(t)}{t^m}\right)\right|}{t^m}dt \\
&= \int_0^1 \left|T\left(\frac{f(t)}{t^m}\right) - T\left(\frac{g(t)}{t^m}\right)\right|dt \\
&= \left\|T\left(\frac{f}{t^m}\right) - T\left(\frac{g}{t^m}\right)\right\|_1 \\
&= \left\|\frac{f}{t^m} - \frac{g}{t^m}\right\|_1 \\
&= \int_0^1 \left|\frac{f(t)}{t^m} - \frac{g(t)}{t^m}\right|dt \\
&= \int_0^1 \frac{|f(t) - g(t)|}{t^m}dt \\
&= \|f{-}g\|^{\sim}.
\end{aligned}$$

Then, ψ is a fixed point free isometry on $K_{m\frac{1}{2}}$ by the proof of Alspach's theorem in [1]. Also, similarly to Sine's result [30], the mapping defined by $(\Omega f)(t){:=}t^m \chi_{[0,1]}{-}\psi f, \ \forall f{\in}K_{m\frac{1}{2}}, \ \forall t \in [0,1]$ is another fixed point free isometry on $K_{m\frac{1}{2}}$

Now, define $R^{\sim}{:}K_{m\frac{1}{2}}{\to}K_{m\frac{1}{2}}$ by

$$R^{\sim}(f) = \sum_{n=0}^{\infty} \frac{\psi^n(f)}{2^{n+1}},$$

then using the strategy in the proof of the main theorem in [5], it is seen that $R^{\sim}$ is a fixed point free contractive invariant mapping on $K_{m\frac{1}{2}}$ and so $\mathcal{M}_m$ fails w-fpp for contractive mappings (so does for isometries).

Now, we will provide another invariant fixed point free isometry and a different invariant fixed point free contractive mapping defined on the set $K_{m\frac{1}{2}}$ using ideas in [12].

Claim: Using Dowling, Lennard and Turett's mapping given in the formula (1.3), define a composite mapping by

$$\tilde{\Delta}f(t) := \begin{cases} \frac{f(2t)}{(2t)^m} & \text{if } \ 0 \leq t \leq \frac{1}{2} \\ 1 - \frac{f(2t-1)}{(2t-1)^m} & \text{if } \ \frac{1}{2} < t < 1 \end{cases} ,$$

and next define $\tilde{\Delta}^* : K_{m\frac{1}{2}} \to K_{m\frac{1}{2}}$ by $\tilde{\Delta}^*(f)(t) := t^m \tilde{\Delta} f(t) = t^m \Delta\left(\frac{f(t)}{t^m}\right)$, $\forall t \in [0,1]$.

Then, $\psi\tilde{\Delta}^*{:}K_{m\frac{1}{2}} \to K_{m\frac{1}{2}}$ is a fixed point free isometry where ψ is given as in the formula (2.4). Moreover, there exists a fixed point free contractive mapping $\varphi^{\sim}{:}K_{m\frac{1}{2}} \to K_{m\frac{1}{2}}$.

Proof of the claim: For any $f, g \in K_{m\frac{1}{2}}$, we have

$$\begin{aligned} \left\| \psi\tilde{\Delta}^*(f) - \psi\tilde{\Delta}^*(g) \right\|^{\sim} &= \int_0^1 \frac{\left| t^m T\left(\frac{\tilde{\Delta}^*(f)(t)}{t^m}\right) - t^m T\left(\frac{\tilde{\Delta}^*(g)(t)}{t^m}\right) \right|}{t^m} dt \\ &= \int_0^1 \left| T\left(\frac{\tilde{\Delta}^*(f)(t)}{t^m}\right) - T\left(\frac{\tilde{\Delta}^*(g)(t)}{t^m}\right) \right| dt \\ &= \left\| \mathrm{T}\left(\tilde{\Delta}(f)\right) - \mathrm{T}\left(\tilde{\Delta}(g)\right) \right\|_1 \\ &= \left\| \tilde{\Delta}(f) - \tilde{\Delta}(g) \right\|_1 \\ &= \int_0^1 \left| \Delta\left(\frac{f}{t^m}\right)(t) - \Delta\left(\frac{g}{t^m}\right)(t) \right| dt \\ &= \left\| \Delta\left(\frac{f}{t^m}\right) - \Delta\left(\frac{g}{t^m}\right) \right\|_1 \\ &= \left\| \frac{f}{t^m} - \frac{g}{t^m} \right\|_1 \\ &= \int_0^1 \left| \frac{f(t)}{t^m} - \frac{g(t)}{t^m} \right| dt \\ &= \|f - g\|^{\sim}. \end{aligned}$$

In conclusion, for each $m \in \mathbb{N}$, we obtain three different fixed point free invariant isometries ψ, Ω and $\tilde{\Delta}^*$ defined on $K_{m\frac{1}{2}}$. Then, firstly, using the ideas of the study by Burns, Lennard and Sivek [4], we can show that if T is taken as the invariant mapping ψ defined on $K_{m\frac{1}{2}}$, then the following mapping is fixed point free contractive: Define $P : K_{m\frac{1}{2}} \to K_{m\frac{1}{2}}$ by

$$P(f) = \sum_{n=0}^{\infty} \frac{T^n(f)}{2^{n+1}}.$$

Secondly, using the ideas in Ph.D. thesis of Sivek [31, Section I.C.], written under supervision of Chris Lennard, we can show that if T is one of the invariant mappings ψ or $\tilde{\Delta}^*$ defined on $K_{m\frac{1}{2}}$, then the following mapping is fixed point free contractive: Define $P : K_{m\frac{1}{2}} \to K_{m\frac{1}{2}}$ by

$$P(f) = \sum_{n=0}^{\infty} \frac{T^n(f)}{2^{n+1}}.$$

To reach these results, we only need to imitate those works. So we will not include them. □

2.2. Large classes with fixed point property for nonexpansive mappings

In 1979, Goebel and Kuczumow [16] showed there exists a large class of closed, bounded and convex subsets of ℓ^1 using a key lemma they obtained. Their lemma says that if $\{\mathrm{x_n}\}$ is a sequence in ℓ^1 converging to x in weak-star topology, then for any $\mathrm{y}\in\ell^1$,

$$\mathrm{r}\,(\mathrm{y}) = \mathrm{r}\,(\mathrm{x}) + \|\mathrm{y}-\mathrm{x}\|_1 \quad \text{where} \quad \mathrm{r}\,(\mathrm{y}) = \limsup_{\mathrm{n}} \|\mathrm{x_n} - \mathrm{y}\|_1 \quad .$$

Since the Köthe–Toeplitz dual Υ_m for $X^\infty(\triangle^m)$ is contained in ℓ^1 for any $m \in \mathbb{N}$ and in fact ℓ^1 is isometrically isomorphic to Υ_m, for any $m \in \mathbb{N}$, Goebel and Kuczumow's lemma above (Lemma 1 in [16]) applies in Köthe–Toeplitz dual for $X^\infty(\triangle^m)$. We will call this fact $\heartsuit$.

When we consider the degenerate Lorentz space $\mathcal{M}_m$ for any $m \in \mathbb{N}$ and take counting measure, we obtain a Banach space containing ℓ^1. Let us call this space η_m for each $m \in \mathbb{N}$. Clearly we can write this space as below and it is a space contained in ℓ^1 for any $m \in \mathbb{N}$. In fact, η_m is isometrically isomorphic to ℓ^1, for any $m \in \mathbb{N}$.

$$\begin{aligned} \eta_m : \quad &= \quad \left\{ a = (a_n)_n \subset \mathbb{R} \;\middle|\; \left(\frac{a_n}{n^m}\right)_n \in \ell^1 \right\} \\ &= \quad \left\{ a = (a_k)_k \subset \mathbb{R} : \; \|a\|_{\triangleleft} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^m} < \infty \right\}. \end{aligned} \tag{2.5}$$

Since η_m is isometrically isomorphic to ℓ^1, for any $m \in \mathbb{N}$, Goebel and Kuczumow's lemma above (Lemma 1 in [16]) applies in η_m for each $m \in \mathbb{N}$. We will call this fact $\diamond$.

2.2.1. Large classes in Υ_m with fixed point property for nonexpansive mappings

In this section, we consider Goebel and Kuczumow [16] analogy, for any $m \in \mathbb{N}$, for the Köthe–Toeplitz dual Υ_m of a generalized Cesàro difference sequence space. We show that for any $m \in \mathbb{N}$ there exists a large class of closed, bounded and convex subsets of the Köthe–Toeplitz dual Υ_m for X^∞ $(\triangle^m)$ with fixed point property for nonexpansive mappings. We note that case $m = 1$ has recently been done by Nezir and Cankurt while Nezir, Dutta and Yıldırım provided a larger class for the general case $m \in \mathbb{N}$ submitted to a refereed journal but here estending all results, we provide a lot larger classes for the general case $m \in \mathbb{N}$.

Now, we consider the following class of closed, bounded and convex subsets. Note that here we will be using the ideas similar to those in the section 2 of Ph.D. thesis of Everest [14], written under supervision of Chris Lennard, where Everest firstly provides Goebel and Kuczumow's proofs in detailed.

Example 2.6 *Fix $m \in \mathbb{N}$ and let $0 < b_1 \le b_2 < 1$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b_1\ e_1$, $f_2 := \frac{b_2\ e_2}{2^m}$, and $f_n := \frac{1}{n^m} e_n$ for all integers $n \ge 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both*

c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of the Köthe-Toeplitz Dual Υ_m for $X^{\infty}\ (\triangle^m)$ for arbitrary $m \in \mathbb{N}$ by

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \quad \forall n \in \mathbb{N},\ \ t_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} t_n = 1 \right\} \ .$$

Theorem 2.7 *For any $m \in \mathbb{N}$ and $0 < b_1 \leq b_2 < 1$, the set $E^{(m)} \subset \Upsilon_m$ defined as in the example above has the fixed point property for $\|.\|_{\triangle}$-nonexpansive mappings.*

Proof Fix $m \in \mathbb{N}$ and let $0 < b_1 \leq b_2 < 1$. Let also $T{:}E^{(m)}{\to}E^{(m)}$ be a nonexpansive mapping. Then, since T is nonexpansive mapping, there exists an approximate fixed point sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}{\in}E^{(m)}$ such that $\left\|Tx^{(n)}{-}x^{(n)}\right\|_{\triangle} \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $x{\in}\Upsilon_m$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ♡ given in the last part of the previous section, we can define a function $s{:}\Upsilon_m{\to}[0,\infty)$ by

$$s\left(y\right) = \limsup_{n} \left\|x^{(n)}{-}y\right\|_{\triangle} \quad , \ \forall y{\in}\Upsilon_m$$

and so

$$s\left(y\right) = s\left(x\right) + \left\|x-y\right\|_{\triangle} \quad , \ \forall y{\in}\Upsilon_m.$$

Now define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W{:=}\overline{E^{(m)}}^{w^*} = \left\{ \sum_{\mathrm{n}=1}^{\infty} t_n\ f_n : \quad each \ \ t_n \geq 0 \ \ and \ \sum_{n=1}^{\infty} t_n{\leq}1 \right\}$$

Since T is nonexpansive mapping, $\forall \mathrm{x},y{\in}\mathrm{E}^{(\mathrm{m})}$,

$$\|T\mathrm{x}{-}T\mathrm{y}\|_{\triangle} \leq \|\mathrm{x}-\mathrm{y}\|_{\triangle}.$$

Case 1: $x{\in}E^{(m)}$.

Then, $\forall \mathrm{n} \in \mathbb{N}$, we have $s(Tx) = s(x) + \|Tx{-}x\|_{\triangle}$ and

$$\begin{aligned} s\left(Tx\right) &= \limsup_{n} \left\|Tx{-}x^{(n)}\right\|_{\triangle} \\ &\leq \limsup_{n} \left\|Tx{-}T\left(x^{(n)}\right)\right\|_{\triangle} + \limsup_{n} \left\|x^{(n)}{-}T\left(x^{(n)}\right)\right\|_{\triangle} \\ &\leq \limsup_{n} \left\|x{-}x^{(n)}\right\|_{\triangle} + 0 \\ &= s\left(x\right). \end{aligned} \tag{2.6}$$

Therefore, $s\left(Tx\right) = s\left(x\right) + \|Tx{-}x\|_{\triangle} \leq s(x)$ and so $\|Tx{-}x\|_{\triangle} = 0$. Thus, $Tx{=}x$.

Case 2: $x{\in}W\backslash E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty}\gamma_n f_n$ suchthat $\sum_{n=1}^{\infty}\gamma_n < 1$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$.

Define $\delta := 1 - \sum_{n=1}^{\infty}\gamma_n$ and for $\alpha \in \left[\frac{-\gamma_1}{\delta}, \frac{\gamma_2}{\delta}+1\right]$ define

$$\mathrm{h}_\alpha := (\gamma_1 + \alpha\delta) f_1 + (\gamma_2 + (1-\alpha)\delta) f_2 + \sum_{n=3}^{\infty}\gamma_n f_n.$$

Then,

$$\|\mathrm{h}_\alpha - x\|_\triangle = \left\|\alpha b_1 \delta e_1 + (1-\alpha)\delta \frac{b_2\ e_2}{2^m}\right\|_\triangle = \begin{cases} \delta(b_2 - \alpha[b_1 + b_2]) & \text{if } \alpha \in \left[\frac{-\gamma_1}{\delta}, 0\right) \\ \delta(b_2 - \alpha[b_2 - b_1]) & \text{if } \alpha \in [0,1] \\ \delta(-b_2 + \alpha[b_1 + b_2]) & \text{if } \alpha \in \left(1, \frac{\gamma_2}{\delta}+1\right] \end{cases}. \tag{2.7}$$

Therefore, $\|\mathrm{h}_\alpha - x\|_\triangle$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b_1\delta$ occurring at $\alpha = 1$.

Now fix $y \in E^{(m)}$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then,

$$\begin{aligned} \|\mathrm{y} - x\|_\triangle &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty}\gamma_k f_k\right\|_\triangle \\ &= b_1 |t_1 - \gamma_1| + b_2 |t_2 - \gamma_2| + \sum_{k=3}^{\infty} |t_k - \gamma_k| \\ &= b_1 \left|1 - \sum_{k=2}^{\infty} t_k - \sum_{k=1}^{\infty}\gamma_k + \sum_{k=2}^{\infty}\gamma_k\right| + b_2 |t_2 - \gamma_2| + \sum_{k=3}^{\infty} |t_k - \gamma_k| \\ &= b_1 \left|\delta - \sum_{k=2}^{\infty} (t_k - \gamma_k)\right| + b_2 |t_2 - \gamma_2| + \sum_{k=3}^{\infty} |t_k - \gamma_k| \\ &\geq b_1\delta - b_1 \sum_{k=2}^{\infty} |t_k - \gamma_k| + b_2 |t_2 - \gamma_2| + \sum_{k=3}^{\infty} |t_k - \gamma_k| \\ &= b_1\delta + (1 - b_1) \sum_{k=3}^{\infty} |t_k - \gamma_k| + (b_2 - b_1) |t_2 - \gamma_2| \ . \end{aligned}$$

Hence,

$$\|\mathrm{y} - x\|_\triangle \geq b_1\delta$$

and we have the equality if and only if $(1 - b_1)\sum_{k=3}^{\infty} |t_k - \gamma_k| = 0$ which means we have $\|\mathrm{y} - x\|_\triangle = b_1\delta$ if and only if $t_k = \gamma_k$ for every $k \geq 3$; or say, $\|\mathrm{y} - x\|_\triangle = b\delta$ if and only if $\mathrm{y} = \mathrm{h}_\alpha$ for some $\alpha \in [0,1]$.

Now, define $\Lambda := \{\mathrm{h}_\alpha \ : \ \alpha \in [0,1]\}$. Clearly, Λ is the contiunous image of a compact set and so it is a compact subset of $E^{(m)}$. It is also easy to see that it is convex.

Now for any $h \in \Lambda$, since $\|\mathrm{y} - x\|_\triangle$ achieves its minimum value at $\mathrm{y} = \mathrm{h}_\alpha$, firstly we have

$$
\begin{aligned}
s(h) = s(x) + \|\mathrm{h}-x\|_{\triangle} &\leq s(x) + \|T\mathrm{h}-x\|_{\triangle} \\
&= s(T\mathrm{h}) \text{ but this follows} \\
&= \limsup_n \left\|Th-x^{(n)}\right\|_{\triangle} \text{ then similarly to the inequality 2.8} \\
&\leq \limsup_n \left\|Th-T\left(x^{(n)}\right)\right\|_{\triangle} + \limsup_n \left\|x^{(n)}-T\left(x^{(n)}\right)\right\|_{\triangle} \\
&\leq \limsup_n \left\|h-x^{(n)}\right\|_{\triangle} + \limsup_n \left\|x^{(n)}-T\left(x^{(n)}\right)\right\|_{\triangle} \\
&\leq \limsup_n \left\|h-x^{(n)}\right\|_{\triangle} + 0 \\
&= s(h).
\end{aligned}
$$

Hence, $s(h) \leq s(T\mathrm{h}) \leq s(h)$ and so $s(T\mathrm{h}) = s(h)$. Hence, $s(x) + \|T\mathrm{h}-x\|_{\triangle} = s(x) + \|\mathrm{h}-x\|_{\triangle}$.

Therefore,

$$\|T\mathrm{h}-x\|_{\triangle} = \|\mathrm{h}-x\|_{\triangle}$$

and so $Th \in \Lambda$ but this means $T(\Lambda) \subseteq \Lambda$ and since T is continuous, Schauder's fixed point theorem [28] tells us that T has a fixed point such that h is the unique minimizer of $\|\mathrm{y}-x\|_{\triangle}$: $y \in E$ and $Th = h$.

Thus, $E^{(m)}$ has fpp(ne) as desired. □

Example 2.8 *Fix $m \in \mathbb{N}$ and let Q be an integer larger than 2. Also, let $0 < b_1 \leq b_2 < 1$. Define a sequence $(f_n)_{n \in \mathbb{N}}$ by setting $f_1 := b_1\ e_1, \cdots, \ f_{(Q-1)} := \frac{b_1\ e_{(Q-1)}}{(Q-1)^m}$, $f_Q := \frac{b_2}{Q^m} e_Q$, and $f_n := \frac{1}{n^m} e_n$ for all integers $n \geq Q+1$ where the sequence $(e_n)_{n \in \mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $E^{(m)}$ of the Köthe–Toeplitz dual Υ_m for $X^\infty\ (\triangle^m)$ for arbitrary $m \in \mathbb{N}$ by*

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \quad \forall n \in \mathbb{N},\ \ t_n \geq 0 \ \ \text{and} \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

Theorem 2.9 *For any $m \in \mathbb{N}$ and $0 < b_1 \leq b_2 < 1$, the set $E^{(m)} \subset \Upsilon_m$ defined as in the example above has the fixed point property for $\|.\|_{\triangle}$-nonexpansive mappings.*

Proof Let Q be an integer larger than 2. Fix $m \in \mathbb{N}$ and let $0 < b_1 \leq b_2 < 1$. Let also $T{:}E^{(m)} \to E^{(m)}$ be a nonexpansive mapping. Then, since T is nonexpansive mapping, there exists an approximate fixed point sequence $\left(x^{(n)}\right)_{n \in \mathbb{N}} \in E^{(m)}$ such that $\left\|Tx^{(n)}-x^{(n)}\right\|_{\triangle} \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $x \in \Upsilon_m$ such that $x^{(n)}$ converges to x in weak* topology. Then, by Goebel Kuczumow analog fact ♡ given in the last part of the previous section, we can define a function $s{:}\Upsilon_m \to [0,\infty)$ by

$$s(y) = \limsup_n \left\|x^{(n)}-y\right\|_{\triangle} \quad , \ \forall y \in \Upsilon_m$$

and so

$$s(y) = s(x) + \|x-y\|_{\triangle} \quad , \ \forall y \in \Upsilon_m .$$

Now define the weak* closure of the set $E^{(m)}$ as it is seen below.

$$W{:=}\overline{E^{(m)}}^{w^*}=\left\{\sum_{n=1}^{\infty} t_n\ f_n :\quad each\ \ t_n\geq 0\ \ and\ \ \sum_{n=1}^{\infty} t_n{\leq}1\right\}$$

Since T is nonexpansive mapping, $\forall \mathrm{x},y{\in}\mathrm{E}^{(\mathrm{m})}$,

$$\|T\mathrm{x}{-}T\mathrm{y}\|_{\triangle}\leq\|\mathrm{x}-\mathrm{y}\|_{\triangle}.$$

Case 1: $x{\in}E^{(m)}$.

Then, $\forall \mathrm{n}\in\mathbb{N}$, we have $s(Tx)=s(x)+\|Tx{-}x\|_{\triangle}$ and

$$\begin{aligned} s\left(Tx\right) &= \limsup_n \left\|Tx{-}x^{(n)}\right\|_{\triangle} \\ &\leq \limsup_n \left\|Tx{-}T\left(x^{(n)}\right)\right\|_{\triangle}+\limsup_n \left\|x^{(n)}{-}T\left(x^{(n)}\right)\right\|_{\triangle} \\ &\leq \limsup_n \left\|x{-}x^{(n)}\right\|_{\triangle}+0 \\ &= s\left(x\right). \end{aligned} \tag{2.8}$$

Therefore, $s\left(Tx\right)=s\left(x\right)+\|Tx{-}x\|_{\triangle}\leq s(x)$ and so $\|Tx{-}x\|_{\triangle}=0$. Thus, $Tx{=}x$.

Case 2: $x{\in}W\backslash E^{(m)}$.

Then, x is of the form $\sum_{n=1}^{\infty}\gamma_n f_n$ suchthat $\sum_{n=1}^{\infty}\gamma_n\ <1$ and $\gamma_n{\geq}0,\ \forall n{\in}\mathbb{N}$.

Define $\delta{:=}\ 1{-}\sum_{n=1}^{\infty}\gamma_n$ and for $\alpha\in\left[\frac{-\gamma_1}{\delta},\frac{\gamma_2}{\delta}+1\right]$ define

$$h_\alpha:=\left(\gamma_1+\frac{\alpha}{Q-1}\delta\right)f_1+\cdots+\left(\gamma_{(Q-1)}+\frac{\alpha}{Q-1}\delta\right)f_{(Q-1)}+\left(\gamma_{\mathrm{Q}}+(1-\alpha)\delta\right)f_Q+\sum_{n=Q+1}^{\infty}\gamma_n f_n.$$

Then,

$$\|\mathrm{h}_\alpha{-}x\|_{\triangle}=\left\|\frac{\alpha}{Q-1}b_1\delta e_1+\cdots+\frac{\alpha}{Q-1}\delta\frac{b_1 e_{(Q-1)}}{(Q-1)^m}+(1-\alpha)\delta\frac{b_2\ e_Q}{Q^m}\right\|_{\triangle}=\begin{cases}\delta\left(b_2-\alpha\left[b_2+b_1\right]\right) & \text{if}\ \ \alpha\in\left[\frac{-\gamma_1}{\delta},0\right)\\ \delta\left(b_2+\alpha\left[b_1-b_2\right]\right) & \text{if}\ \ \alpha\in[0,1]\\ \delta\left(\alpha\left[b_1+b_2\right]-b_2\right) & \text{if}\ \ \alpha\in\left(1,\frac{\gamma_2}{\delta}+1\right]\end{cases}. \tag{2.9}$$

Therefore, $\|\mathrm{h}_\alpha{-}x\|_{\triangle}$ is minimized for $\alpha\in[0,1]$ and its minimum value would be $b_1\delta$ occurring at $\alpha=1$.

Now fix $y{\in}E^{(m)}$ of the form $\sum_{n=1}^{\infty}t_n f_n$ such that $\sum_{n=1}^{\infty}t_n\ =1$ with $t_n{\geq}0,\ \forall n{\in}\mathbb{N}$.

Then,

$$\begin{aligned}
\|\mathrm{y}-x\|_{\triangle} &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right\|_{\triangle} \\
&= b_1\left|t_1-\gamma_1\right| + \cdots + b_1\left|t_{(Q-1)}-\gamma_{(Q-1)}\right| + b_2\left|t_Q-\gamma_Q\right| + \sum_{k=\mathrm{Q}+1}^{\infty}\left|t_k-\gamma_k\right| \\
&\geq b_1\left|t_1-\gamma_1+t_2-\gamma_2+\cdots+b_1(t_{(Q-1)}-\gamma_{(Q-1)})\right| + b_2\left|t_Q-\gamma_Q\right| + \sum_{k=Q+1}^{\infty}\left|t_k-\gamma_k\right| \\
&= b_1\left|1-\sum_{k=\mathrm{Q}}^{\infty} t_k - \sum_{k=1}^{\infty}\gamma_k + \sum_{k=\mathrm{Q}}^{\infty}\gamma_k\right| + b_2\left|t_Q-\gamma_Q\right| + \sum_{k=\mathrm{Q}+1}^{\infty}\left|t_k-\gamma_k\right| \\
&\geq b_1\delta - b_1\sum_{k=\mathrm{Q}}^{\infty}\left|t_k-\gamma_k\right| + b_2\left|t_Q-\gamma_Q\right| + \sum_{k=\mathrm{Q}+1}^{\infty}\left|t_k-\gamma_k\right| \\
&= b_1\delta + (1-b_1)\sum_{k=\mathrm{Q}+1}^{\infty}\left|t_k-\gamma_k\right| + (b_2-b_1)\left|t_Q-\gamma_Q\right| .
\end{aligned}$$

Hence,

$$\|\mathrm{y}-x\|_{\triangle} \geq b_1\delta$$

and we have the equality if and only if $(1-b_1)\sum_{k=\mathrm{Q}+1}^{\infty}\left|t_k-\gamma_k\right| = 0$ which means we have $\|\mathrm{y}-x\|_{\triangle} = b_1\delta$ if and only if $t_k = \gamma_k$ for every $k \geq Q+1$; or say, $\|\mathrm{y}-x\|_{\triangle} = b\delta$ if and only if $\mathrm{y} = \mathrm{h}_\alpha$ for some $\alpha \in [0,1]$.

Now, define $\Lambda := \{\mathrm{h}_\alpha \ : \ \alpha \in [0,1]\}$. Clearly, Λ is the contiunous image of a compact set and so it is a compact subset of $E^{(m)}$. It is also easy to see that it is convex.

Now for any $h \in \Lambda$, since $\|\mathrm{y}-x\|_{\triangle}$ achieves its minimum value at $\mathrm{y} = \mathrm{h}_\alpha$, firstly we have

$$\begin{aligned}
s(h) = s(x) + \|\mathrm{h}-x\|_{\triangle} &\leq s(x) + \|T\mathrm{h}-x\|_{\triangle} \\
&= s(T\mathrm{h}) \text{ but this follows} \\
&= \limsup_n \left\|Th - x^{(n)}\right\|_{\triangle} \text{ then similarly to the inequality 2.8} \\
&\leq \limsup_n \left\|Th - T\left(x^{(n)}\right)\right\|_{\triangle} + \limsup_n \left\|x^{(n)} - T\left(x^{(n)}\right)\right\|_{\triangle} \\
&\leq \limsup_n \left\|h - x^{(n)}\right\|_{\triangle} + \limsup_n \left\|x^{(n)} - T\left(x^{(n)}\right)\right\|_{\triangle} \\
&\leq \limsup_n \left\|h - x^{(n)}\right\|_{\triangle} + 0 \\
&= s(h).
\end{aligned}$$

Hence, $s(h) \leq s(T\mathrm{h}) \leq s(h)$ and so $s(T\mathrm{h}) = s(h)$. Hence, $s(x) + \|T\mathrm{h}-x\|_{\triangle} = s(x) + \|\mathrm{h}-x\|_{\triangle}$.

Therefore,

$$\|T\mathrm{h}-x\|_{\triangle} = \|\mathrm{h}-x\|_{\triangle}$$

and so $Th \in \Lambda$ but this means $T(\Lambda) \subseteq \Lambda$ and since T is continuous, Schauder's fixed point theorem [28] tells us that T has a fixed point such that h is the unique minimizer of $\|y - x\|_{\triangle}\ :\ y \in E$ and $Th = h$.

Thus, $E^{(m)}$ has fpp(ne) as desired. □

The reader may notice that we use a technique exactly same as in the works given at the second section of Everest's Ph.D. thesis [14]. The largest class in ℓ^1 with the fixed point property in his thesis given as below:

Theorem 2.10 *[14] Let $(b_{i,k})_{i,k\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Define for each $i \in \mathbb{N}$,*

$$f_i := \sum_{k=1}^{\infty} b_{i,k} e_k \ .$$

Suppose that there exists a finite nonempty subset $M \subset \mathbb{N}$ such that $\forall i \in M$ and $\forall k \notin M$, $b_{i,k} = 0$. That is,

$$f_i := \sum_{k\in M}^{\infty} b_{i,k} e_k \ .$$

Equivalently, without loss of generality, we may say that $\exists \nu \in \mathbb{N}$ such that $M = \{1, 2, 3, \cdots, \nu - 1, \nu\}$ and

$$f_i := \sum_{k=1}^{\nu+1} b_{i,k} e_k \ .$$

Suppose that

$$0 < \max_{1 \leq i < \nu+1} \sum_{k=1}^{\infty} |b_{i,k}| < |b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}|,\ \forall m \in \mathbb{N} \setminus M$$

and that

$$0 < |b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}|,\ \forall m \in M\ .$$

Next, define a subset of ℓ^1 as below.

$$E := \left\{ \sum_{n=1}^{\infty} t_n\ f_n :\quad \forall n \in \mathbb{N},\ \ t_n \geq 0 \ \ \mathit{and} \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} \ .$$

Then, E is a closed, bounded and convex subset in ℓ^1 and it has the fixed point property for $\|\cdot\|_1$-nonexpansive mappings.

Then, using the same strategy we used in the proof of our theorems which actually apply the same method as in the works of the second section in Everest's thesis but combined with our construction, by looking at the proof of the previous theorem [14, Theorem 2.3.18], one can obtain the proof of the following corollary which gives us the largest class in Υ_m with fixed point property for $\|\dot{|}\|_{\triangle}$-nonexpansive mappings in our paper.

Corollary 2.11 *Fix $m \in \mathbb{N}$. Let $(b_{i,k})_{i,k\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Define for each $i \in \mathbb{N}$,*

$$f_i := \sum_{k=1}^{\infty} b_{i,k} \frac{e_k}{k^m} \ .$$

Suppose that there exists a finite nonempty subset $M \subset \mathbb{N}$ such that $\forall i \in M$ and $\forall k \notin M$, $b_{i,k} = 0$. That is,

$$f_i := \sum_{k \in M}^{\infty} b_{i,k} \frac{e_k}{k^m} \ .$$

Equivalently, without loss of generality, we may say that $\exists \nu \in \mathbb{N}$ such that $M = \{1, 2, 3, \cdots, \nu - 1, \nu\}$ and

$$f_i := \sum_{k=1}^{\nu+1} b_{i,k} \frac{e_k}{k^m} \ .$$

Suppose that

$$0 < \max_{1 \le i < \nu+1} \sum_{k=1}^{\infty} |b_{i,k}| < |b_{m,m}| - \sum_{k \in \mathbb{N} \setminus \{m\}} |b_{m,k}|, \ \forall m \in \mathbb{N} \setminus M$$

and that

$$0 < |b_{m,m}| - \sum_{k \in \mathbb{N} \setminus \{m\}} |b_{m,k}|, \ \forall m \in M \ .$$

Next, define a subset of Υ_m as below.

$$E^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n \ f_n : \quad \forall n \in \mathbb{N}, \ \ t_n \ge 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Then, $E^{(m)}$ is a closed, bounded and convex subset in Υ_m and it has the fixed point property for $\|\cdot\|_{\triangle}$-nonexpansive mappings.

2.2.2. Large classes in η_m with fixed point property for nonexpansive mappings

In the beginning of the section two, taking counting measure at the definition of the degenerate Lorentz function space space $\mathcal{M}_m$ for any $m \in \mathbb{N}$, we got the Banach space η_m containing ℓ^1 and in fact, isometrically isomorphic to ℓ^1. Its definiton was given by the formula 2.5 as below.

$$\eta_m = \left\{ a = (a_k)_k \subset \mathbb{R} : \ \|a\|_{\triangleleft} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^m} < \infty \right\}.$$

So in this section, we consider Goebel and Kuczumow [16] analogy, for η_m, for each $m \in \mathbb{N}$ and we show that there exists a large class of closed, bounded and convex subsets with fixed point property for nonexpansive mappings.

Now, we consider the following class of closed, bounded and convex subsets. Note that here again we will be using the ideas similar to those in the section 2 of Ph.D. thesis of Everest [14], written under supervision of Chris Lennard. In fact, we will not give the proofs because the proofs of the theorems will be exactly similar to the proofs of the theorems in the previous section but keeping the same computations, we only need to replace $\|\cdot\|_{\triangle}$ with $\|\cdot\|_{\triangleleft}$ and use the fact $\diamond$ instead of $\heartsuit$ where it was used in earlier section.

Example 2.12 *Fix $m \in \mathbb{N}$ and let $0 < b_1 \leq b_2 < 1$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b_1\ e_1$, $f_2 := b_2\ 2^m e_2$, and $f_n := n^m e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $K^{(m)}$ of η_m for arbitrary $m \in \mathbb{N}$ by*

$$K^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \quad \forall n \in \mathbb{N},\ \ t_n \geq 0 \ \ \text{and} \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Theorem 2.13 *For any $m \in \mathbb{N}$ and $0 < b_1 \leq b_2 < 1$, the set $K^{(m)} \subset \eta_m$ defined as in the example above has the fixed point property for $\|.\|_{\triangleleft}$-nonexpansive mappings.*

Example 2.14 *Fix $m \in \mathbb{N}$ and let Q be an integer larger than 2. Also, let $0 < b_1 \leq b_2 < 1$. Define a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_1 := b_1\ e_1$, $\cdots$, $f_{(Q-1)} := b_1\ (Q-1)^m e_{(Q-1)}$, $f_Q := b_2 Q^m e_Q$, and $f_n := n^m e_n$ for all integers $n \geq Q+1$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Next, we can define a closed, bounded, convex subset $K^{(m)}$ of the Köthe–Toeplitz dual η_m for $X^{\infty}\ (\triangle^m)$ for arbitrary $m \in \mathbb{N}$ by*

$$K^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \quad \forall n \in \mathbb{N},\ \ t_n \geq 0 \ \ \text{and} \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Theorem 2.15 *For any $m \in \mathbb{N}$ and $0 < b_1 \leq b_2 < 1$, the set $K^{(m)} \subset \eta_m$ defined as in the example above has the fixed point property for $\|.\|_{\triangleleft}$-nonexpansive mappings.*

Now similarly to the Corollary for the space Υ_m in the previous section, we get the largest class in η_m with fixed point property for $\|\dot{\|}_{\triangleleft}$-nonexpansive mappings in our paper by using the same strategy we used in the proof of our theorems that apply the same method as in the works of the second section in Everest's thesis but combined with our construction and following the steps in the proof of the theorem 2.10 [14, Theorem 2.3.18].

Corollary 2.16 *Fix $m \in \mathbb{N}$. Let $(b_{i,k})_{i,k\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. Define for each $i \in \mathbb{N}$,*

$$f_i := \sum_{k=1}^{\infty} b_{i,k} k^m e_k \ .$$

Suppose that there exists a finite nonempty subset $M \subset \mathbb{N}$ such that $\forall i \in M$ and $\forall k \notin M$, $b_{i,k} = 0$. That is,

$$f_i := \sum_{k\in M}^{\infty} b_{i,k} k^m e_k \ .$$

Equivalently, without loss of generality, we may say that $\exists \nu \in \mathbb{N}$ such that $M = \{1, 2, 3, \cdots, \nu - 1, \nu\}$ and

$$f_i := \sum_{k=1}^{\nu+1} b_{i,k} k^m e_k \ .$$

Suppose that

$$0 < \max_{1\leq i<\nu+1} \sum_{k=1}^{\infty} |b_{i,k}| < |b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}|, \ \forall m \in \mathbb{N} \setminus M$$

and that

$$0 < |b_{m,m}| - \sum_{k\in\mathbb{N}\setminus\{m\}} |b_{m,k}|, \ \forall m \in M \ .$$

Next, define a subset of η_m *as below.*

$$K^{(m)} := \left\{ \sum_{n=1}^{\infty} t_n \ f_n : \quad \forall n \in \mathbb{N}, \ \ t_n \geq 0 \ \ and \ \ \sum_{n=1}^{\infty} t_n = 1 \right\} \quad .$$

Then, $K^{(m)}$ *is a closed, bounded and convex subset in* η_m *and it has the fixed point property for* $\|\cdot\|_{\triangleleft}$*-nonexpansive mappings.*

References

[1] Alspach D. A fixed point free nonexpansive mapping. Proceedings of the American Mathematical Society 1981; 82: 423-424.

[2] Aydogan SM, Sakar FM, Fatehi M, Rezapour S, Masiha HP. Two hybrid and non-hybrid k-dimensional inclusion systems via sequential fractional derivatives. Advances in Difference Equations 2021; 449: 1-21.

[3] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae 1922; 3: 133-181.

[4] Burns J, Lennard C, Sivek J. A contractive fixed point free mapping on a weakly compact convex set. Studia Mathematica 2014; 3 (223): 275-283.

[5] Casini E, Miglierina E, Piasecki Ł. Separable Lindenstrauss spaces whose duals lack the weak* fixed point property for nonexpansive mappings. Studia Mathematica 2017; 238: 1-16.

[6] Chen S, Cui Y, Hudzik H, Sims B. Geometric properties related to fixed point theory in some Banach function lattices. In: Kirk WA, Sims B (editors). Handbook of metric fixed point theory. Dordrecht: Springer, 2001, pp. 339-389.

[7] Cui Y. Some geometric properties related to fixed point theory in Cesàro spaces. Collectanea Mathematica 1999; 277-288.

[8] Cui Y, Hudzik H, Li Y. On the Garcia-Falset Coefficient in Some Banach Sequence Spaces. In: Hudzik H, Skrzypcazk L (editors). Function Spaces. CRC Press, 2000, pp. 163-170.

[9] Cui Y, Meng C, Płuciennik R. (2000). Banach–Saks Property and Property (β) in Cesàro Sequence Spaces. Southeast Asian Bulletin of Mathematics 2000; 24 (2): 201-210.

[10] Dowling PN, Lennard CJ, Turett B. Some more examples of subsets of c_0 and $L_1[0,1]$ failing the fixed point property. Contemporary Mathematics 2003; 328: 171-176.

[11] Dowling PN, Lennard CJ, Turett B. New fixed point free nonexpansive maps on weakly compact, convex subsets of $L_1[0,1]$. Studia Math 2007; 180 (3): 271-284.

[12] Et M, Çolak R. On some generalized difference sequence spaces. Soochow Journal of Mathematics 1995; 21 (4): 377-386.

[13] Et M. (1996). On some generalized Cesàro difference sequence spaces. İstanbul University Science Faculty The Journal Of Mathematics, Physics and Astronomy 1996; 55: 221-229.

[14] Everest T. Fixed points of nonexpansive maps on closed, bounded, convex sets in ℓ^1. PhD, University of Pittsburgh, Pittsburgh, PA, USA, 2013.

[15] Falset JG. The fixed point property in Banach spaces with the NUS-property. Journal of Mathematical Analysis and Applications 1997; 215 (2): 532-542.

[16] Goebel K, Kuczumow T. Irregular convex sets with fixed-point property for nonexpansive mappings. Colloquium Mathematicum 1979; 2 (40): 259-264.

[17] Hardy GH, Littlewood JE, Polya G. (1952). Inequalities. Cambridge: Cambridge university press, 1952.

[18] Kızmaz H. On certain sequence spaces. Canadian Mathematical Bulletin 1981; 24 (2): 169-176.

[19] Kirk WA. A fixed point theorem for mappings which do not increase distances. The American Mathematical Monthly 1965; 72 (9): 1004-1006.

[20] Lindenstraus J, Tzafriri L. Classical Banach spaces I: Sequence spaces. Berlin and New York: Springer-Verlag, 1977.

[21] Lindenstraus J, Tzafriri L. Classical Banach spaces II: function spaces. (Vol. 97), Springer Science & Business Media, 2013.

[22] Llorens–Fuster E, Sims B. The fixed point property in c_0. Canadian Mathematical Bulletin 1998; 41 (4): 413-422.

[23] Lorentz GG. Some new functional spaces. Annals of Mathematics 1950; 37-55.

[24] Maurey B. Points fixes des contractions de certains faiblement compacts de L^1. Seminaire d'Analyse Fonctionelle, 1980-1981, Centre de Mathématiques, École Polytech., Palaiseau, Exp. No. VIII, 19 pp., 1981.

[25] NgPeng-Nung NN, LeePeng-Yee LY. Cesàro sequence spaces of nonabsolute type. Commentationes mathematicae 1978; 20 (2): 429-433.

[26] Orhan C. Casaro Differance Sequence Spaces and Related Matrix Transformations. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 1983; 32: 55-63.

[27] Rezapour S, Sakar FM, Aydoğan SM, Ravash E. Some results on a system of multiterm fractional integro-differential equations. Turkish Journal of Mathematics 2020; 44 (6) : 2004-2020.

[28] Schauder J. Der Fixpunktsatz in Funktional-raumen. Studia Studia Mathematica 1930; 2 (1): 171-180

[29] Shiue JS. (1970). On the Cesaro sequence spaces. Tamkang Journal of Mathematics 1970; 1 (1): 19-25.

[30] Sine R. Remarks on the example of Alspach. Nonlinear Analysis and Applications, Marcel Dekker 1981; 237-241.

[31] Sivek J. Differentiability, summability, and fixed points in Banach spaces. PhD, University of Pittsburgh, Pittsburgh, PA, USA, 2014.

[32] Steiner J. Gesammelte Werke I, II, Reimer. 1881.

[33] Tripathy BC, Esi A, Tripathy B. On new types of generalized difference Cesaro sequence spaces. Soochow Journal of Mathematics 2005; 31 (3): 333-340.