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Relation between matrices and the suborbital graphs by the special number sequences

Ümmügülsün AKBABA, Ali Hikmet DEĞER*
Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey,

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Abstract: Continued fractions and their matrix connections have been used in many studies to generate new identities. On the other hand, many examinations have been made in the suborbital graphs under circuit and forest conditions. Special number sequences and special vertex values of minimal length paths in suborbital graphs have been associated in our previous studies. In these associations, matrix connections of the special continued fractions $\mathcal{K}(-1/-k)$, where $k \in \mathbb{Z}^+,\ k \geq 2$ with the values of the special number sequences are used and new identities are obtained. In this study, by producing new matrices, new identities related to Fibonacci, Lucas, Pell, and Pell-Lucas number sequences are found by using both recurrence relations and matrix connections of the continued fractions. In addition, the farthest vertex values of the minimal length path in the suborbital graph $\mathbf{F}_{u,N}$ and these number sequences are associated.

Key words: Pell numbers, Pell-Lucas numbers, Fibonacci numbers, Lucas numbers, continued fractions, suborbital graphs

1. Introduction

Like Fibonacci numbers, which have many applications in literature and in daily life, Pell and Pell-Lucas numbers, which are their expanded versions, also have very interesting properties. It is interesting that they have algebraic applications in fields such as theories of combinatorics, graph, number, and continued fractions. Details on these number sequences can be found in [4, 14, 15]. The matrix relation of these sequences firstly was studied in [7, 10]. Also, the connections of these number sequences with matrices have been used in many application areas like information security, electrical network theory, the coding theory [3, 17, 20]. On the other hand, many studies have been conducted on circuits in suborbital graphs and on identities produced by special vertex values of paths of the minimal length on the suborbital graphs [2, 6]. In [12], Jones, Singerman, and Wicks have formed the suborbital graphs for subgroup $\Gamma_0(N)$ of the modular group Γ by using Sims's theory in [19]. It has been shown that the action of Γ on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ is transitive. Then, the congruence equations arising from the suborbital graphs have been studied in [8, 9]. Also, for the extended modular group $\hat{\mathbf{\Gamma}}$, the suborbital graphs have been examined in [13]. The authors in [16, 18] have studied new kinds of continued fractions which are consisted of suborbital graphs that are isomorphic to Farey graphs. In [6], Değer has defined the farthest vertices on the paths of the minimal length on the suborbital graphs and investigated the related continued fractions. The author has examined the relation between continued fractions and Fibonacci numbers.

*Correspondence: ahikmetd@ktu.edu.tr
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In this study, we examine the matrix which is obtained by recurrence relation of the continued fractions which are on the vertices of the paths of the minimal length on the suborbital graphs. In previous works, we have found the relation between Fibonacci numbers and the suborbital graphs by the relation in terms of matrices [5, 6]. By using these matrices, we get new matrices related to Fibonacci and Lucas numbers. Also, we see that we can write the vertices on the paths of the minimal length on the suborbital graphs by Lucas numbers.

Moreover, we examine the relation between Pell numbers and the suborbital graphs. For a special integer k, we write the vertices on the paths of the minimal length on suborbital graphs by Pell numbers. By the same motivation, we investigate the matrix which is obtained by the continued fractions for Pell and Pell-Lucas numbers. Also, by using the traces of these matrices, the vertices on the minimal length paths are characterized.

1.1. Fibonacci, Lucas, Pell, and Pell-Lucas numbers

For initial conditions $F_0 = 0$ and $F_1 = 1$ by recurrence relation $F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ n \in \mathbb{N}$, $\{F_n\}$ is called a Fibonacci sequence. Here, F_n is the n^{th} Fibonacci number. Similarly, L_n is the n^{th} Lucas number for recurrence relation $L_n = L_{n-1} + L_{n-2},\ n \geq 2,\ n \in \mathbb{N}$, where initial conditions are $L_0 = 2$ and $L_1 = 1$. For details, see [14]. From both of these number sequences, there are many identities that have been discovered. Let us give some of them:

$$F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n, n \geq 1,$$

$$F_{2n} = F_n L_n, n \geq 1,$$

$$L_n = F_{n-1} + F_{n+1} = F_{n+2} - F_{n-2}, n \geq 2, \tag{1.1}$$

$$5F_n = L_{n-1} + L_{n+1}, n \geq 2, \tag{1.2}$$

where $n \in \mathbb{Z}^+$.

Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The characteristic equation of the recurrence relation of Fibonacci numbers is $x^2 - x - 1 = 0$. Therefore, its solutions are φ and β, which are characteristic roots of this equation. Also, from the Binet formulas, $F_n = \frac{\varphi^n - \beta^n}{\sqrt{5}}$ and $L_n = \varphi^n + \beta^n$. Here, φ is known as the golden ratio.

P_n is the n^{th} Pell number which satisfies the recurrence relation $P_n = 2P_{n-1} + P_{n-2},\ n \geq 2,\ n \in \mathbb{Z}^+$ by initial conditions $P_0 = 0$ and $P_1 = 1$. Similarly, Q_n is the n^{th} Pell-Lucas number by $Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2,\ n \in \mathbb{Z}^+$ and initial conditions are $Q_0 = 1$ and $Q_1 = 1$. Binet-like formulas for P_n and Q_n are

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \ \texttt{and}\ Q_n = \frac{\gamma^n + \delta^n}{2}, \quad n \geq 2,\ n \in \mathbb{Z}^+,$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$.

Lemma 1.1 *[15] If P_n is the n^{th} Pell number and Q_n is the n^{th} Pell-Lucas number, then the following equations hold:*

$$P_n + P_{n-1} = Q_n,$$

$$Q_n + Q_{n-1} = 2P_n,$$

$$P_{n+1} + P_{n-1} = 2Q_n, \tag{1.3}$$

$$P_{2n} = 2P_nQ_n,$$

$$Q_{n+1} + Q_{n-1} = 4P_n, \tag{1.4}$$

where $n \geq 1,\ n \in \mathbb{Z}^+$.

1.2. Recurrence relations of continued fractions

For a continued fraction $b_0 + \mathcal{K}_{m=1}^{\infty}(a_m/b_m)$, let A_n is the n^{th} numerator and B_n is the n^{th} denominator. The continued fraction is defined by the recurrence relations

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}, \tag{1.5}$$

where $n = 1, 2, 3, \ldots$ with initial conditions $A_{-1} := 1, B_{-1} := 0, A_0 := b_0, B_0 := 1$. The modified approximant $T_n(z_n)$ can be written as $T_n(z_n) = \frac{A_n + A_{n-1}z_n}{B_n + B_{n-1}z_n}$, where $n = 0, 1, 2, 3, \ldots$. Hence, for the n^{th} approximant g_n, we have

$$g_n = T_n(0) = \frac{A_n}{B_n}, g_{n-1} = T_n(\infty) = \frac{A_{n-1}}{B_{n-1}}. \tag{1.6}$$

Let $\mathcal{K}(a_m/b_m)$ be a given continued fraction with n^{th} numerator A_n and n^{th}denominator B_n. Let $t_m(w) = \frac{a_m}{b_m + w}$, $x_m := \begin{pmatrix} 0 & a_m \\ 1 & b_m \end{pmatrix}$, $m = 1, 2, 3, \cdots$. Then the linear fractional transformation $T_n(w)$ and equation (1.5) lead to $X_n := x_1x_2x_3 \cdots x_n = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}$, $n = 1, 2, 3, \cdots$. Therefore, multiplication of 2x2 matrices can be used to construct the sequence $\{A_n\}, \{B_n\}$ and $\{g_n\}$, where g_n is given by (1.6). For details, see [4].

1.3. Suborbital graphs

By using the theory of Sims [19], in [12], the authors have defined the suborbital graphs $\mathbf{F}_{u,N}$, which is a subgraph of the graph $\mathbf{G}_{u,N}$. $\mathbf{F}_{u,N}$ is a generalization of the well-known Farey graph $F = \mathbf{G}_{1,1}$. For details, see [2] and [12]. By using Proposition 2.2. in [12], authors have determined the suborbital graphs $\mathbf{F}_{u,N}$ with imprimitive group action on a set which is

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \ c \equiv 0 (mod N),\ ad - bc = 1 \right\}$$

on $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. Here, $\Gamma_0(N)$ is the congruence subgroup of the modular group

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

$\Gamma_0(N)$ permutes the both vertices and the edges of $\mathbf{F}_{u,N}$ transitively. For reduced fractions $\frac{r}{s}, \frac{x}{y} \in \hat{\mathbb{Q}}$, $\frac{r}{s} \longrightarrow \frac{x}{y} \in \mathbf{F}_{u,N}$ iff $x \equiv \pm ur \ (modN), ry - sx = \pm N$. This is called the edge conditions for the suborbital graph $\mathbf{F}_{u,N}$, where $(u,N) = 1$ and $\frac{r}{s}, \frac{x}{y}$ are vertices.

Now let us give some definitions about suborbital graphs $\mathbf{F}_{u,N}$ that we use in our paper:

(a) Let sequence $v_0, v_1, \ldots, v_m$, where the elements of the sequence are vertices of the graph $\mathbf{F}_{u,N}$. When $m \geq 2$, we call the configuration $v_0 \to v_1 \to \ldots \to v_m \to v_0$ a *directed circuit* (*closed path*). If at least one arrow (not all) is reversed, the configuration is called an *undirected* (*antidirected*) *circuit*. When $m = 2$, the circuit is called a *triangle*, directed or not. When $m = 1$, the configuration $v_0 \to v_1 \to v_0$ is called a *self paired edge*.

(b) For visual convenience and because the elements of Γ send the hyperbolic lines to hyperbolic lines, the edges of graphs have been represented as hyperbolic geodesics in the upper half-plane

$$\mathcal{H} := \{z \in \mathbb{C} \,|\, Im(z) > 0\}$$

which is similar to euclidean semicircles or half-lines perpendicular to $\mathbb{R}$ as in [11].

(c) The configurations $v_0 \to v_1 \to \ldots \to v_m$ is called a *path* and $v_0 \to v_1 \to \ldots$ is called an *infinite path* on $\mathbf{F}_{u,N}$.

(d) Let the edge $\frac{r}{s} \overset{<}{\to} \frac{x}{y} \in \mathbf{F}_{u,N}$ $\left(\textbf{or } \frac{x}{y} \overset{<}{\leftarrow} \frac{r}{s} \in \mathbf{F}_{u,N}\right)$. The vertex $\frac{x}{y}$ is the *farthest vertex* for the vertex $\frac{r}{s}$, when there is no vertex which has greater (or smaller) value than $\frac{x}{y}$ joined with the vertex $\frac{r}{s}$ on the suborbital graph $\mathbf{F}_{u,N}$, by the edge conditions.

(e) The path $v_0 \to v_1 \to \ldots \to v_i \to v_{i+1} \to \ldots \to v_j \to \ldots \to v_m$ is called of *minimal length* if and only if $v_i \nleftrightarrow v_j$, where $i < j-1$, $i \in \{0,1,2,3,...,m-2\}, j \in \{2,3,...,m\}$ and v_{i+1} must be the farthest vertex which can be joined with the vertex v_i on $\mathbf{F}_{u,N}$.

(f) If A is a square matrix and n is positive integer, the n^{th} power of A is given by

$$A^n = A \times A \times \cdots \times A,$$

where there are n copies of matrix A.

From now on, in this work, instead of the vertex $\frac{r}{s} \in \hat{\mathbb{Q}}$, we use the column matrix notation as $\begin{pmatrix} r \\ s \end{pmatrix}$.

2. Main results

Let $\varphi_i = \begin{pmatrix} -u & \frac{u^2+(-1)^i k_i u+1}{N} \\ -N & u+(-1)^i k_i \end{pmatrix} \in \Gamma_0(N)$, $i = 1,2$. If $(u,N) = 1$, then there exists an integer k_i, $i = 1,2$ such that $u^2 + (-1)^i k_i u + 1 \equiv 0 \ (modN), i = 1,2$. On $\mathbf{F}_{u,N}$, φ_i, $i = 1,2$ is a transformation which joins the vertices to each other respectively on the infinite path of minimal length to both directions. Thus, if $\begin{pmatrix} u+(-1)^i \frac{x}{y} \\ N \end{pmatrix}$, $i = 1,2$ is a vertex on the path of minimal length on $\mathbf{F}_{u,N}$, then the farthest vertex which can be joined with this vertex is $\varphi_i \begin{pmatrix} u+(-1)^i \frac{y}{k_i y - x} \\ N \end{pmatrix}$, $i = 1,2$. Since $v_0 = \begin{pmatrix} u \\ N \end{pmatrix}$ is the initial vertex, which is joined with the vertex $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on the suborbital graph $\mathbf{F}_{u,N}$, then $v_n = {\varphi_i}^n(v_0),\ i = 1,2,\ n \in \mathbb{Z}^+$. Here, v_n is the n^{th} vertex on the path of minimal length on $\mathbf{F}_{u,N}$ for both directions. If $i = 1$, then for the left direction, we represent the n^{th} vertex as the farthest vertex with $\overleftarrow{v}_n$ and similarly, if $i = 2$, then for the right direction, we represent the n^{th} vertex as farthest vertex with $\overrightarrow{v}_n$.

Let $\mathcal{K}_{n=1}^{\infty}\left(\frac{-1}{-k_i}\right)$ be a continued fraction, where $a_n = -1,\ b_n = -k_i$ for $n \in \mathbb{Z}^+$ and $i = 1,2$. From recurrence relation (1.5), for this continued fraction, $B_n = -A_{n+1}$. Then, n^{th} vertex on the path of minimal length in the suborbital graph $\mathbf{F}_{u,N}$ can be given by

$$\frac{u + (-1)^i T_n(0)}{N} = \frac{u + (-1)^i \frac{A_n}{B_n}}{N} = \frac{A_{n+1}u - (-1)^i A_n}{A_{n+1}N},\ i = 1,2$$

for left and right directions, respectively. We can see the vertices of the minimal length path of suborbital graph $\mathbf{F}_{u,N}$ at Figure 1.

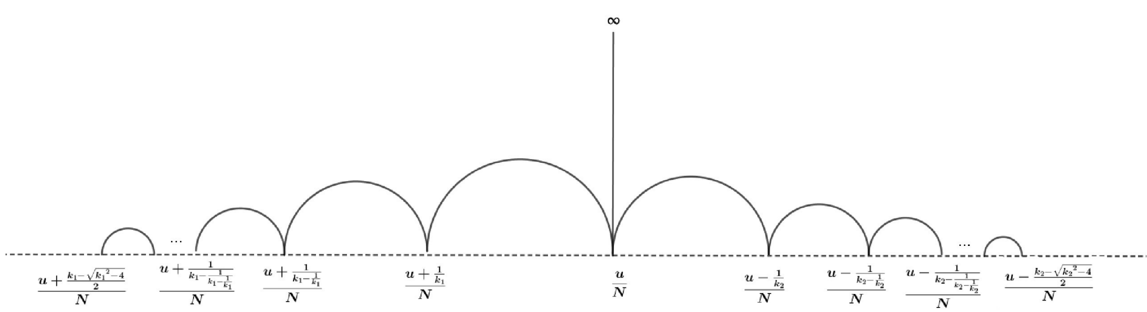

Figure 1. The vertices on the minimal length path on the suborbital graph $\mathbf{F}_{u,N}$.

Also, from matrix connections for continued fractions, we get

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k_i \end{pmatrix}^n,\ i = 1,2. \tag{2.1}$$

Thus, set of vertices of the path for both direction is

$$M_i := \bigcup_{n=0}^{\infty}\left\{\frac{u + (-1)^i T_n(0)}{N} : T_n = t_0 t_1 t_2 \dots t_n,\ t_0(z) = z, t_n(z) := t(z) = \frac{-1}{-k_i + z},\ i = 1,2\right\} \cup \{\infty\}.$$

From the set $M_i,\ i = 1,2$, if $k_i = 2$, then the Mobius transformation t is parabolic and if $k_i \geq 2$, then t is hyperbolic. Therefore, φ_i is an element of $\Gamma_0(N)$ and also if $k_i = 0$ and $k_i = 1$, then φ_i is an elliptic element of order 2 and 3, respectively. Moreover, if we solve the characteristic equation $A_n = -k_i A_{n-1} - A_{n-2},\ i = 1,2$, when $k_i = 2$, then $A_n = (-1)^n n$ and when $k_i > 2$, then $A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n}(k_i + \sqrt{{k_i}^2 - 4})^{n-t}(k_i - \sqrt{{k_i}^2 - 4})^{t-1}$. Hence, if $k_i = 2,\ i = 1,2$, then for left and right direction, the n^{th} vertices on the path of minimal length starting with the vertex $\frac{u}{N}$ in the suborbital graph $\mathbf{F}_{u,N}$ are

$$\overleftarrow{v_n} = \frac{(n+1)u - n}{(n+1)N} \ \text{ and } \ \overrightarrow{v_n} = \frac{(n+1)u + n}{(n+1)N},$$

respectively. Furthermore, if $k_i = 3,\ i = 1,2$, then for left and right direction, the n^{th} vertices on the path of minimal length starting with the vertex $\frac{u}{N}$ in the suborbital graph $\mathbf{F}_{u,N}$ are

$$\overleftarrow{v_n} = \frac{u - \frac{F_{2n}}{F_{2n+2}}}{N} \text{ and } \overrightarrow{v_n} = \frac{u + \frac{F_{2n}}{F_{2n+2}}}{N}, \tag{2.2}$$

where F_n is n^{th} Fibonacci number [6].

New results on relation between matrices and even indexed terms of Fibonacci and Lucas numbers are examined in [5]. By taking $k_i = 3$ on equation (2.1), the following lemma gives a special matrix which generates the even indexed Fibonacci numbers:

Lemma 2.1 *[5] If F_n is the n^{th} Fibonacci number, then*

$$\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n = \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} = S_n,$$

where $n \in \mathbb{Z}^+$.

2.1. Matrices with even indexed terms of Fibonacci and Lucas numbers

In this section, by the similar motivation, by using Lemma 2.1 new matrices related to Fibonacci and Lucas numbers and the new identities are examined.

Lemma 2.2 *If L_n is the n^{th} Lucas number, then*

$$L_{2n} = (-1)^n tr[S_n],\ n \in \mathbb{Z}^+, \tag{2.3}$$

where $tr[S_n]$ is the trace of the matrix S_n.

Proof From identity (1.1), we have $L_{2n} = F_{2n+2} - F_{2n-2}, n \geq 2$. By this equation, if we write L_{2n} as $L_{2n} = (-1)^n[(-1)^n F_{2n+2} - (-1)^n F_{2n-2}] = (-1)^n[(-1)^n F_{2n+2} + (-1)^{n-1} F_{2n-2}]$, then we have equation (2.3). □

By using matrix S_n, we obtained the following theorem and according to Theorem 2.3, we can generate even indexed terms of Lucas numbers by using matrix T_n.

Theorem 2.3 *If L_n is the n^{th} Lucas number, then*

$$\begin{pmatrix} -3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n = \begin{pmatrix} (-1)^{n-1}L_{2n-2} & (-1)^n L_{2n} \\ (-1)^{n+1}L_{2n} & (-1)^n L_{2n+2} \end{pmatrix} = T_n \tag{2.4}$$

and

$$det(T_n) = {L_{2n}}^2 - L_{2n-2}L_{2n+2} = -5, \tag{2.5}$$

where $n \in \mathbb{Z}^+$.

Proof For the proof we use the identity $5F_n = L_{n-1} + L_{n+1}$.
Firstly, when we write that identity for $2n-2$ and using $L_{2n-3} = L_{2n-1} - L_{2n-2}$, $L_{2n-1} = L_{2n} - L_{2n-2}$, we

get

$$\begin{aligned} 5F_{2n-2} &= L_{2n-3} + L_{2n-1} \\ &= -3L_{2n-2} + 2L_{2n} \\ -F_{2n-2} &= \frac{3}{5}L_{2n-2} - \frac{2}{5}L_{2n}. \end{aligned}$$

Secondly, we write the identity for $2n$ and using $L_{2n-1} = L_{2n+1} - L_{2n}$, $L_{2n+1} = L_{2n+2} - L_{2n}$, we get

$$\begin{aligned} 5F_{2n} &= L_{2n-1} + L_{2n+1} \\ &= -3L_{2n} + 2L_{2n+2} \\ F_{2n} &= -\frac{3}{5}L_{2n} + \frac{2}{5}L_{2n+2} \end{aligned}$$

and at the same time

$$-F_{2n} = \frac{2}{5}L_{2n-2} - \frac{3}{5}L_{2n}.$$

Lastly, we write the identity for $2n+2$ and using $L_{2n+1} = L_{2n+2} - L_{2n}$, $L_{2n+3} = L_{2n+1} - L_{2n+2}$, we get

$$\begin{aligned} 5F_{2n+2} &= L_{2n+1} + L_{2n+3} \\ &= -2L_{2n} + 3L_{2n+2} \\ F_{2n-2} &= -\frac{2}{5}L_{2n} + \frac{3}{5}L_{2n+2}. \end{aligned}$$

Therefore, we can write the matrix S_n in a new form as

$$\begin{aligned} S_n &= \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} \\ &= (-1)^n \begin{pmatrix} \frac{3}{5}L_{2n-2} - \frac{2}{5}L_{2n} & -\frac{3}{5}L_{2n} + \frac{2}{5}L_{2n+2} \\ \frac{2}{5}L_{2n-2} - \frac{3}{5}L_{2n} & -\frac{2}{5}L_{2n} + \frac{3}{5}L_{2n+2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} (-1)^{n-1}L_{2n-2} & (-1)^n L_{2n} \\ (-1)^{n+1}L_{2n} & (-1)^n L_{2n+2} \end{pmatrix}. \end{aligned}$$

Let $A = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}$. Hence, the inverse of A is $A^{-1} = \begin{pmatrix} -3 & 2 \\ -2 & 3 \end{pmatrix}$. If we multiply both sides of the last equation by the matrix A^{-1}, we can write the matrix T_n for even indexed Lucas numbers. When we examine determinant of matrix T_n for both side of the equation (2.4), we get $(-5)1 = (-1)^{n-1}L_{2n-2}(-1)^n L_{2n+2} - (-1)^n(-1)^{n+1}{L_{2n}}^2$. Therefore, the equation (2.5) is obtained. □

Lemma 2.4 *If F_n is the n^{th} Fibonacci number, then*

$$5F_{2n} = (-1)^n tr[T_n], \; n \in \mathbb{Z}^+. \tag{2.6}$$

Proof

Since the trace of the matrix T_n is $tr[T_n] = (-1)^n(-1)L_{2n-2} + (-1)^n L_{2n+2} = (-1)^n[L_{2n+2} - L_{2n-2}] = (-1)^n[L_{2n+1}+L_{2n-1}]$, then from the identity (1.2), we get $5F_{2n} = L_{2n-1}+L_{2n+1}$. So, $5F_{2n} = (-1)^n tr[T_n]$, $n \in \mathbb{Z}^+$. □

Therefore, we obtain the following corollary which gives the relation between the values of the vertices on the path of the minimal length on the suborbital graphs $\mathbf{F}_{u,N}$ and matrix T_n.

Corollary 2.5 *If $k_i = 3,\ i = 1, 2$, then for left and right direction, the n^{th} vertices on the path of minimal length starting with the vertex $\frac{u}{N}$ in the suborbital graph $\mathbf{F}_{u,N}$ are*

$$\overleftarrow{v_n} = \frac{u + \frac{tr[T_n]}{tr[T_{n+1}]}}{N} \text{ and } \overrightarrow{v_n} = \frac{u - \frac{tr[T_n]}{tr[T_{n+1}]}}{N}, \tag{2.7}$$

respectively.

Proof From the equation (2.6) we get $F_{2n} = \frac{(-1)^n}{5} tr[T_n]$. Then, if we use this in (2.2), then the equation (2.7) is obtained. □

Lemma 2.6 *[1] Let F_n is the n^{th} Fibonacci number, then we obtain the following equation as*

$$P^n = \begin{pmatrix} (-1)^n F_{n-1} & 0 & (-1)^{n+1} F_n \\ 3[(-1)^{n+1} F_{n-2} - 1] & 1 & 3[(-1)^n F_{n-1} - 1] \\ (-1)^{n+1} F_n & 0 & (-1)^n F_{n+1} \end{pmatrix},$$

where $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$ *and* $n \in \mathbb{Z}^+$.

By using the matrix which is given on equation (2.1), we can write the Fibonacci numbers with a new matrix as follows:

Theorem 2.7 *Let F_n is the n^{th} Fibonacci number, then we obtain the following equation as*

$$Z^n = \begin{pmatrix} (-1)^n F_{n-1} & 0 & (-1)^{n+1} F_n \\ k[(-1)^{n+1} F_{n-2} - 1] & 1 & k[(-1)^n F_{n-1} - 1] \\ (-1)^{n+1} F_n & 0 & (-1)^n F_{n+1} \end{pmatrix},$$

where $Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -k \\ 1 & 0 & -1 \end{pmatrix}$ *and* $n \in \mathbb{Z}^+$.

Proof For the proof we use the diagonal matrix of matrix Z. Because we want to write n^{th} power of matrix Z^n with the equation,

$$D^n = (P^{-1}ZP)^n = P^{-1}ZPP^{-1} \cdots P^{-1}ZP = P^{-1}Z^nP,$$

where D is diagonal matrix, P is the matrix whose column i is i^{th} eigenvector. Hence, we get $Z^n = PD^nP^{-1}$. Now let us find the eigenvalues and eigenvectors of matrix Z. Then by the equation

$$P_Z(\lambda) = det(\lambda I - Z) = \lambda^3 - 2\lambda + 1,$$

we can find the eigenvalues as $\lambda_{1,2,3} = 1, \frac{-1+\sqrt{5}}{2}, \frac{-\sqrt{5}-1}{2}$ and the eigenvectors as

$$V_{1,2,3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ \frac{2k}{3-\sqrt{5}} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} \\ \frac{2k}{3+\sqrt{5}} \\ 1 \end{pmatrix}.$$

When we write matrix P and its inverse P^{-1}. Then

$$P = \begin{pmatrix} 0 & \frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 & \frac{2k}{3-\sqrt{5}} & \frac{2k}{3+\sqrt{5}} \\ 0 & 1 & 1 \end{pmatrix} \quad and \quad P^{-1} = \begin{pmatrix} -k & 1 & -k \\ \frac{1}{\sqrt{5}} & 0 & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}.$$

Therefore, we can write the diagonal matrix as $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{5}-1}{2} & 0 \\ 0 & 0 & \frac{-\sqrt{5}-1}{2} \end{pmatrix}$. Now, we can write the matrix Z^n as:

$$Z^n = PD^nP^{-1} = \begin{pmatrix} \frac{(\sqrt{5}+5)(\sqrt{5}-1)^n - (\sqrt{5}-5)(-\sqrt{5}-1)^n}{10\ 2^n} & 0 & \frac{\sqrt{5}(\sqrt{5}-1)^n - \sqrt{5}(-\sqrt{5}-1)^n}{5\ 2^n} \\ \frac{(\frac{3\sqrt{5}+5}{10})k(\sqrt{5}-1)^n + (\frac{-3\sqrt{5}+5}{10})k(-\sqrt{5}-1)^n}{2^n} - k & 1 & \frac{(\frac{\sqrt{5}+5}{10})k(\sqrt{5}-1)^n + (\frac{-\sqrt{5}+5}{10})k(-\sqrt{5}-1)^n}{2^n} - k \\ \frac{\sqrt{5}(\sqrt{5}-1)^n - \sqrt{5}(-\sqrt{5}-1)^n}{5\ 2^n} & 0 & \frac{(\frac{\sqrt{5}+5}{10})(-\sqrt{5}-1)^n + (\frac{-\sqrt{5}+5}{10})(\sqrt{5}-1)^n}{2^n} \end{pmatrix}.$$

If we use Binet formula of Fibonacci numbers as $F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$, we consist the matrix Z^n with Fibonacci numbers. □

The following lemma provides us to find the even indexed terms of Fibonacci and Lucas numbers by a new matrix.

Theorem 2.8 *Let F_n is the n^{th} Fibonacci number and L_n is the n^{th} Lucas number. So, we have the following matrix equation as*

$$X^n = \begin{pmatrix} 2 & -3 \\ 3 & -7 \end{pmatrix}^n = \left\{ \begin{array}{ll} 5^{\frac{n-1}{2}} \begin{pmatrix} L_{2n-2} & -L_{2n} \\ L_{2n} & -L_{2n+2} \end{pmatrix}, & if\, n\ is\ odd \\ \\ 5^{\frac{n}{2}} \begin{pmatrix} -F_{2n-2} & F_{2n} \\ -F_{2n} & F_{2n+2} \end{pmatrix}, & if\, n\ is\ even \end{array} \right\}, \tag{2.8}$$

where $n \in \mathbb{Z}^+$.

Proof For the proof, we will use the mathematical induction method for odd n integers and even n integers, respectively. For odd n integers, the case for $n = 1$ is true. We suppose that for $n = k$, the case is true and we should show the case is true for odd $n = k+2$ integer. So,

$$\begin{aligned} X^{k+2} &= X^k X^2 \\ &= \begin{pmatrix} 2 & -3 \\ 3 & -7 \end{pmatrix}^k \begin{pmatrix} 2 & -3 \\ 3 & -7 \end{pmatrix}^2 \\ &= 5^{\frac{k-1}{2}} \begin{pmatrix} L_{2k-2} & -L_{2k} \\ L_{2k} & -L_{2k+2} \end{pmatrix} \begin{pmatrix} -5 & 15 \\ -15 & 40 \end{pmatrix} \\ &= 5^{\frac{k+1}{2}} \begin{pmatrix} L_{2k} & -L_{2k+2} \\ L_{2k+2} & -L_{2k+4} \end{pmatrix}, \end{aligned}$$

where we use recurrence relation for Lucas numbers. So, the allegation for all odd n integers is valid. For even n integers, the proof can be made similarly. □

2.2. Matrices with even indexed terms of Pell and Pell-Lucas numbers

In this section, firstly we introduce the relation between Pell numbers and the suborbital graphs. To consist the relation, we use the congruence equation $u^2 + (-1)^i k_i u + 1 \equiv 0 (mod N)$, $i = 1, 2$. Hereby, we can write the following theorem;

Theorem 2.9 *If $k_i = 6$, $i = 1, 2$, then for left and right direction, the n^{th} vertices on the path of minimal length starting with the vertex $\frac{u}{N}$ in the suborbital graph $\mathbf{F}_{u,N}$ are*

$$\overleftarrow{v}_n = \frac{u - \frac{P_{2n}}{P_{2n+2}}}{N} \text{ and } \overrightarrow{v}_n = \frac{u + \frac{P_{2n}}{P_{2n+2}}}{N}, \tag{2.9}$$

where P_n is n^{th} Pell number, $n \in \mathbb{Z}^+$ and v_n is a reduced fraction.

Proof Let $k_1 = 6$. So, we get

$$\begin{aligned} A_n &= (-1)^n \frac{1}{2^n} \frac{1}{2\sqrt{8}} [(6+2\sqrt{8})^n - (6-2\sqrt{8})^n] \\ &= (-1)^n \frac{1}{2\sqrt{8}} \left[\frac{(2+\sqrt{8})^{2n}}{2^{2n}} - \frac{(2-\sqrt{8})^{2n}}{2^{2n}} \right] \\ &= (-1)^n \frac{1}{2} \left[\frac{(1+\sqrt{2})^{2n}}{2\sqrt{2}} - \frac{(1-\sqrt{2})^{2n}}{2\sqrt{2}} \right], \end{aligned}$$

where $n \geq 0$. By using the Binet formula for Pell numbers as $P_n = \frac{(1+\sqrt{2})^n}{2\sqrt{2}} - \frac{(1-\sqrt{2})^n}{2\sqrt{2}}$, we get

$$A_n = \frac{(-1)^n P_{2n}}{2}. \tag{2.10}$$

So, for the right direction, the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ is

$$\frac{A_{n+1}u - A_n}{A_{n+1}N} = \frac{uP_{2n+2} + P_{2n}}{P_{2n+2}N}$$

and for the left direction the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ is

$$\frac{A_{n+1}u+A_n}{A_{n+1}N}=\frac{uP_{2n+2}-P_{2n}}{P_{2n+2}N}.$$

□

Similar to the Fibonacci numbers in previous sections, we can generate even indexed Pell numbers with a special matrix. Consequently, we obtain the following corollary.

Corollary 2.10 *If P_n is the n^{th} Pell number, then*

$$\begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n = \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} & \frac{(-1)^n}{2}P_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} & \frac{(-1)^n}{2}P_{2n+2} \end{pmatrix} = V_n.$$

Proof From the matrix connection of continued fractions,

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k \end{pmatrix}^n$$

is given. On the other hand, for $k=6$, from the equation (2.10), we get $A_{n-1}=\frac{(-1)^{n-1}P_{2n-2}}{2}$ and $A_{n+1}=\frac{(-1)^{n+1}P_{2n+2}}{2}$. So, we have

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} & \frac{(-1)^n}{2}P_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} & \frac{(-1)^n}{2}P_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n.$$

□

Example 2.11 *Let $k_2=6$ for the right direction. From the congruence equation $u^2+6u+1\equiv 0\ (modN)$, let $u=1$ and $N=8$. So, the vertices of minimal length path on the suborbital graph $\mathbf{F}_{1,8}$ are on Figure 2 and as follows:*

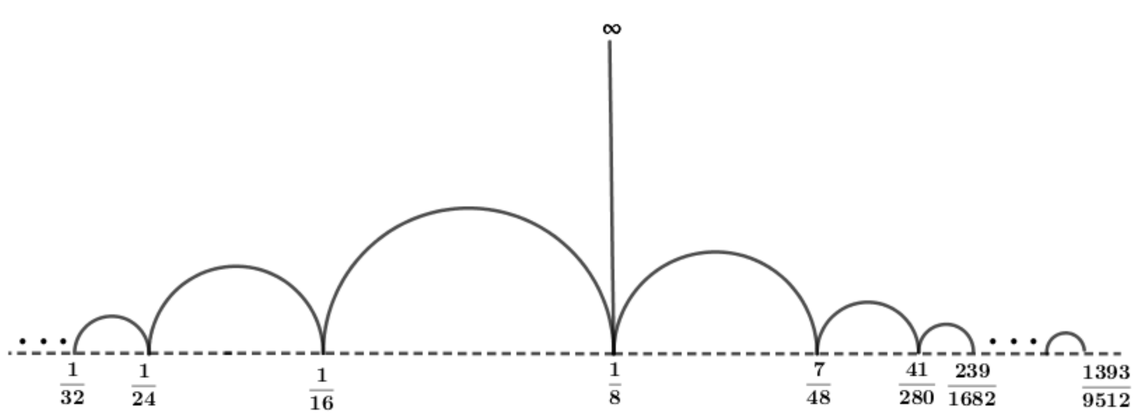

Figure 2. The vertices on the minimal length path on the suborbital graph $\mathbf{F}_{1,8}$.

$$\infty=\frac{1}{0}\rightarrow\underbrace{\frac{1}{8}}_{\overrightarrow{v}_0}\rightarrow\underbrace{\frac{1+\frac{1}{6}}{8}}_{\overrightarrow{v}_1}\rightarrow\underbrace{\frac{1+\frac{1}{6-\frac{1}{6}}}{8}}_{\overrightarrow{v}_2}\rightarrow\cdots\rightarrow\underbrace{\frac{1+\frac{1}{6-\frac{1}{6-\ddots-\frac{1}{6}}}}{8}}_{\overrightarrow{v}_{n-1}}\rightarrow\underbrace{\frac{1+\frac{1}{6-\frac{1}{6-\ddots-\frac{1}{6-\frac{1}{6}}}}}{8}}_{\overrightarrow{v}_n}\rightarrow\cdots.$$

Here, we can write n^{th} vertex as $\overrightarrow{v}_n=\frac{1+\frac{P_{2n}}{P_{2n+2}}}{8}$. For example; if we want to obtain the value of the 5^{th} vertex of the minimal length path on suborbital graph $\mathbf{F}_{1,8}$, we can use two methods. Firstly, we can write $\overrightarrow{v}_5=\frac{1+\frac{1}{6-\frac{1}{6-\frac{1}{6-\frac{1}{6-\frac{1}{6}}}}}}{8}$ by the general method. Secondly, the 5^{th} vertex can be found by more useful method which is given on Theorem 2.9 as $\overrightarrow{v}_5=\frac{1+\frac{P_{10}}{P_{12}}}{8}=\frac{1+\frac{2378}{13860}}{8}=\frac{8119}{55440}$. At the same time, we can find the matrix by using even indexed terms of Pell numbers from Corollary 2.10 as

$$\begin{pmatrix}\frac{P_{16}}{2} & -\frac{P_{18}}{2}\\ \frac{P_{18}}{2} & -\frac{P_{20}}{2}\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & -6\end{pmatrix}^9=\begin{pmatrix}235416 & -1372105\\ 1372105 & -7997214\end{pmatrix}.$$

2.3. Some applications on Pell and Pell-Lucas number matrices

In this section, special matrix V_n which gives even indexed Pell numbers are examined.

Lemma 2.12 *Q_n is the n^{th} Pell-Lucas number, then*

$$Q_{2n}=\frac{(-1)^n}{2}tr(V_n),\ n\in\mathbb{Z}^+.$$

Proof The trace of matrix V_n is $tr(V_n)=\frac{(-1)^n}{2}[P_{2n+2}-P_{2n-2}]$. From the recurrence relation of Pell numbers, we get $P_{2n+2}=2P_{2n+1}+P_{2n}$ and $P_{2n}=2P_{2n-1}+P_{2n-2}$. If we put them in the equation of $tr(V_n)$,

$$tr(V_n)=\frac{(-1)^n}{2}[2P_{2n+1}+P_{2n}-P_{2n}+2P_{2n-1}]=\frac{(-1)^n}{2}2\ [P_{2n+1}+P_{2n-1}]$$

and by using identity (1.3), we have $tr(V_n)=(-1)^n 2\ Q_{2n}$. □

Theorem 2.13 *If Q_n is the n^{th} Pell-Lucas number, then we get*

$$\begin{pmatrix}-6 & 2\\ -2 & 6\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & -6\end{pmatrix}^n=\begin{pmatrix}2(-1)^{n-1}Q_{2n-2} & 2(-1)^nQ_{2n}\\ 2(-1)^{n+1}Q_{2n} & 2(-1)^nQ_{2n+2}\end{pmatrix}=W_n,$$

where $n\in\mathbb{Z}^+$.

Proof For the proof we use the identity $4P_n=Q_{n+1}+Q_{n-1}$ basically.
Firstly, when we write that identity for $2n-2$ and using $Q_{2n}=2Q_{2n-1}+Q_{2n-2}$, $Q_{2n+1}=2Q_{2n}+Q_{2n-1}$,

$Q_{2n-1} = 2Q_{2n-2} + Q_{2n-3}$, we get

$$\begin{aligned} 4P_{2n-2} &= Q_{2n-1} + Q_{2n-3} \\ &= -3Q_{2n-2} + Q_{2n} \\ -Q_{2n-2} &= \frac{3}{4}Q_{2n-2} - \frac{1}{4}Q_{2n}. \end{aligned}$$

Secondly, if we write the identity for $2n$ and using $Q_{2n+2} = 2Q_{2n+1} + Q_{2n}$, $Q_{2n+3} = 2Q_{2n+2} + Q_{2n+1}$, then

$$\begin{aligned} 4P_{2n} &= Q_{2n+1} + Q_{2n-1} \\ &= Q_{2n+2} - 3Q_{2n} \\ P_{2n} &= \frac{3}{4}Q_{2n} + \frac{1}{4}Q_{2n+2} \end{aligned}$$

and at the same time

$$-P_{2n} = \frac{1}{4}Q_{2n-2} - \frac{3}{4}Q_{2n}$$

is obtained. Lastly, if we write the identity for $2n+2$ and using $Q_{2n+3} = 2Q_{2n+2} + Q_{2n+1}$, $Q_{2n+2} = 2Q_{2n+1} + Q_{2n}$,

$$\begin{aligned} 4P_{2n+2} &= Q_{2n+3} + Q_{2n+1} \\ &= 3Q_{2n+2} - 2Q_{2n} \\ P_{2n-2} &= -\frac{1}{4}Q_{2n} + \frac{3}{4}Q_{2n+2} \end{aligned}$$

is obtained. Therefore, we can write the matrix V_n in a new form as

$$\begin{aligned} V_n &= \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} & \frac{(-1)^n}{2}P_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} & \frac{(-1)^n}{2}P_{2n+2} \end{pmatrix} \\ &= \frac{(-1)^n}{2}\begin{pmatrix} \frac{3}{4}Q_{2n-2} - \frac{1}{4}Q_{2n} & \frac{3}{4}Q_{2n} + \frac{1}{4}Q_{2n+2} \\ \frac{1}{4}Q_{2n-2} - \frac{3}{4}Q_{2n} & -\frac{1}{4}Q_{2n} + \frac{3}{4}Q_{2n+2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{16} & \frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} \end{pmatrix}\begin{pmatrix} 2(-1)^{n-1}Q_{2n-2} & 2(-1)^n Q_{2n} \\ 2(-1)^{n+1}Q_{2n} & 2(-1)^n Q_{2n+2} \end{pmatrix}. \end{aligned}$$

When we take $A = \begin{pmatrix} -\frac{3}{16} & \frac{1}{16} \\ -\frac{1}{16} & \frac{3}{16} \end{pmatrix}$, then the inverse of A is $A^{-1} = \begin{pmatrix} -6 & 2 \\ -2 & 6 \end{pmatrix}$. If we multiply both sides of the last equation with the matrix A^{-1}, we get the matrix W_n for even indexed Pell-Lucas numbers. $\square$

Lemma 2.14 *P_n is the n^{th} Pell number, then*

$$P_{2n} = \frac{(-1)^n}{16}tr(W_n), \ n \in \mathbb{Z}^+. \tag{2.11}$$

Proof The trace of the matrix W_n is $tr(W_n) = 2(-1)^{n-1}Q_{2n-2} + 2(-1)^n Q_{2n+2}$. From the recurrence relation of Pell-Lucas numbers,when we write $Q_{2n+2} = 2Q_{2n+1} + Q_{2n}$, $Q_{2n-2} = Q_{2n} - 2Q_{2n-1}$ on the equation, we get $tr(W_n) = 2(-1)^{n-1}(-2)[Q_{2n+1} + Q_{2n-1}]$. By using the identity $Q_{n+1} + Q_{n-1} = 4P_n$, we have equation (2.11). □

Consequently, we can write the vertices on the path of minimal length suborbital graphs where $k_i = 6$ by the trace of a special matrix.

Corollary 2.15 *If $k_i = 6,\ i = 1, 2$, then for left and right direction, the n^{th} vertices v_n, which are reduced fractions on the path of minimal length starting with the vertex $\frac{u}{N}$ in the suborbital graph $\mathbf{F}_{u,N}$ are*

$$\overleftarrow{v_n} = \frac{u + \frac{tr[W_n]}{tr[W_{n+1}]}}{N} \text{ and } \overrightarrow{v_n} = \frac{u - \frac{tr[W_n]}{tr[W_{n+1}]}}{N}, \tag{2.12}$$

respectively.

Proof According to Lemma 2.14, $P_{2n} = \frac{(-1)^n}{16} tr(W_n)$ is written. Hence, when we use that equation on equation (2.9), we get equation (2.12). □

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