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On stability and oscillation of fractional differential equations with a distributed delay

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Abstract: In this paper, we study the stability and oscillation of fractional differential equations

$$^{c}D^{\alpha}x(t)+ax(t)+\int_0^1 x(s+[t-1])dR(s)=0.$$

We discretize the fractional differential equation by variation of constant formula and semigroup property of Mittag–Leffler function, and get the difference equation corresponding to the integer points. From the equivalence analogy of qualitative properties between the difference equations and the original fractional differential equations, the necessary and sufficient conditions of oscillation, stability and exponential stability of the equations are obtained.

Key words: oscillation theory; stability; fractional differential equation

1. Introduction

In 2008, E. Braverman, S. Zhukovskiy [4] considered the stability and oscillation of equations with a distributed delay

$$x'(t)+ax(t)+\int_0^1 x(s+[t-1])dR(s)=0.$$

In 2015, M. Veselinova et al. [13] dealt with stability analysis of linear fractional differential system with distributed delay

$$_{RL}D^{a}_{0+}X(t)=\int_{-\sigma}^{0}[d_\theta U(\theta)]X(t+\theta),\ k=1,2,...,n, \tag{1.1}$$

where $_{RL}D^{a_k}_{0+}$ denotes Riemann–Liouville fractional derivative, $\alpha\in(0,1)$, $\sigma\in(0,\infty)$, $U:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

They proved the classical result that if all roots of the characteristic equation

$$\det(G(p))=0,$$

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where

$$G(p)=\begin{pmatrix} p^{\alpha_1}-\int_{-\sigma}^{0}e^{p\beta}d_1^1(\theta) & -\int_{-\sigma}^{0}e^{p\beta}d_1^2(\theta) & \dots & -\int_{-\sigma}^{0}e^{p\beta}d_1^n(\theta) \\ -\int_{-\sigma}^{0}e^{p\beta}d_2^1(\theta) & p^{\alpha_2}-\int_{-\sigma}^{0}e^{p\beta}d_2^2(\theta) & \dots & -\int_{-\sigma}^{0}e^{p\beta}d_2^n(\theta) \\ \dots & & & \\ -\int_{-\sigma}^{0}e^{p\beta}d_n^1(\theta) & -\int_{-\sigma}^{0}e^{p\beta}d_n^2(\theta) & \dots & p^{\alpha_n}-\int_{-\sigma}^{0}e^{p\beta}d_n^n(\theta) \end{pmatrix},$$

have negative real parts, then the zero solution of the considered homogeneous linear fractional differential system with distributed delays is globally asymptotic stable.

Motivated by the above papers, we consider Caputo fractional differential equations with a distributed delay

$$^{c}D^{\alpha}x(t)+ax(t)+\int_{0}^{1}x(s+[t-1])dR(s)=0,\quad t>0, \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo derivative of order $\alpha\in(0,1]$, $a<0$ is a constant, $\int_0^1 *dR(s)$ is Lebesgue–Stieltjes integral where $R(*):[0,1]\to\mathbb{R}$ is left–continuous function of bounded variation, $[t]=\max\{n\mid n\leqslant t,n\in\mathbb{N}\}$.

Let $R(s)=b\chi_{(\alpha,1]}$, where $\chi_{(b,c]}$ is characteristic function of interval $(b,c]$, i.e. $\chi_{(b,c]}=1$, if $x\in(b,c]$ and $\chi_{(b,c]}=0$, if $x\notin(b,c]$. Then (1.2) becomes fractional differential equations with piecewise continuous arguments

$$^{c}D^{\alpha}x(t)+ax(t)+x(\alpha+[t-1])=0,\quad t>0.$$

In recent years, there has been much research activity in the study of oscillation and stability of various classes of differential equations (see, e.g., the papers [5, 6, 10], dynamic equations on time scales (see, e.g., the papers [2, 3]), partial differential equations (see, e.g., the papers [7, 11, 12], where stability in the sense of globality of solutions to problems arising in mathematical biology and physics is addressed). However, to the best of our knowledge, a little literatures have discussed the stability and oscillation of fractional differential equations with distributed delay, see [13].

We introduce a new technique to solve this question by semigroup property of Mittag–Leffler function and variation of constant formula. We discretize the fractional differential equation into a second order difference equation with constant coefficients, which is equivalent to the original equation in terms of oscillation and stability, as well as use characteristic equations of second order difference equations to analyze oscillation, stability and exponential stability of fractional differential equations with distributed delays. In particular, fractional differential equations with piecewise continuous arguments is a special case of Equation (1.2). Our results improve and generalize the results in reference [4].

This paper is structured as follows. In Section 2, we present necessary notations, lemmas and definitions. In Section 3, we state and prove our main results. At last, an illustrative special case is proposed.

2. Preliminaries

As is known, there are many different definitions of the fractional derivative, all of which generalize the usual integer order derivative. Below we recall the definitions of Caputo fractional derivatives as well as some of their basic properties.

Definition 2.1 ([9]) The Caputo derivative of fractional order α of function $x(t)$ is defined as

$$^{c}D_{t_0}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha+n-1}x^{(n)}(\tau)d\tau,\quad t>t_0,$$

with $n = [\alpha] + 1$, where $[\alpha]$ means the integer part of α.

Note a variation of constant formula for Caputo fractional differential equations:

Lemma 2.1 ([1]) *The unique solution of*

$$^{c}D^{\alpha}_{a+}x(t) = Ax(t) + f(t), \quad x(0) = \eta,$$

where $A \in \mathbb{R}^{n,n}$ *and* $f : [0, \infty] \to \mathbb{R}^n$ *is measurable and bounded, is given by*

$$x(t) = E_\alpha(t^\alpha A)\eta + \int_a^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) f(\tau) d\tau, \tag{2.1}$$

where $E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $E_{\alpha,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\alpha)}$, $\alpha \in (0, 1]$.

Remark 2.1. Equation (2.1) is suitable for constant coefficient fractional differential equations with forcing terms.

Lemma 2.2 ([8]) *A difference equation with constant coefficients*

$$x_{n+2} - p_1 x_{n+1} + p_2 x_n = 0, \quad n = -1, 0, 1, ... \tag{2.2}$$

is stable if both roots of the characteristic equation $\lambda^2 - p_1\lambda + p_2 = 0$ *are on the unit circle and is exponentially stable if the roots are inside the unit circle. The latter condition is satisfied if* $|p_1| < p_2 + 1 < 2$ *and is also equivalent to the asymptotic stability of* (2.2)*. The former condition is satisfied if* $|p_1| \leqslant p_2 + 1 \leqslant 2$*. Equation* (2.2) *is oscillatory if and only if its characteristic equation has no positive roots, which is valid if either the discriminant is negative* $(p_1^2 < 4p_2)$ *or all coefficients are nonnegative* $(p_1 \leqslant 0,\ p_2 \geqslant 0)$.

3. Main results

We consider (1.2) with the initial condition

$$x(t) = \psi(t), \quad t \in [-1, 0], \tag{3.1}$$

under the following assumptions:

$(A1)$ $R(s) : [0, 1] \to \mathbb{R}$ is a left–continuous function of bounded variation which has a nonzero variation in $[0, 1]$;

$(A2)$ $\psi : [0, 1] \to \mathbb{R}$ is a Borel measurable function such that the Lebesgue–Stieltjes integral $\int_0^1 \psi(s-1)dR(s)$ exists (and is finite).

Definition 3.1 Continuous function $x(t)$ is a solution of (1.2), (3.1) if it satisfies (1.2) almost everywhere for $t \geqslant 0$ and (3.1) for $t \in [-1, 0]$.

Denote

$$x_n = x(n), \quad x_{-1} = \int_0^1 \psi(s-1)dR(s), \tag{3.2}$$

$$K_n = \int_0^1 x(s+n)dR(s), \quad K_{-1} = \int_0^1 \psi(s-1)dR(s), \tag{3.3}$$

$$P(a)=\int_0^1 E_\alpha(-as^\alpha)dR(s),\quad Q(a)=\int_0^1 \frac{1-E_\alpha(-as^\alpha)}{a}dR(s). \tag{3.4}$$

Next, we reduce the solution of (1.2) at the integer point to a solution of the second–order difference equation.

Lemma 3.1 (1) *The solution of* (1.2) *and* (3.1) *between integer points is*

$$x(t)=\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)}x_n+\frac{\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)}-1}{a}K_{n-1},\quad t\in[n,n+1), \tag{3.5}$$

where K_n, x_n *are defined by* (3.2), (3.3), $n=0,1,2,\dots$.

(2) *The solution of* (1.2), (3.1) *at integer points satisfies the second order difference equation*

$$x_{n+2}-(E(-a)-Q(a))x_{n+1}+(\frac{1-E_\alpha(-a)}{a}P(a)-E_\alpha(-a)Q(a))x_n=0,\quad n\geqslant -1. \tag{3.6}$$

Proof. (1) (i) When $t\in[0,1)$, from (1.2), (3.1), we have

$$^cD^\alpha x(t)+ax(t)+\int_0^1\psi(s-1)dR(s)=0. \tag{3.7}$$

From definition of K_{-1}, (3.7) can be written as $^cD^\alpha x(t)+ax(t)+K_{-1}=0$.

We can conclude from Lemma 2.1 that

$$\begin{aligned}
x(t)&=E_\alpha(-at^\alpha)x_0+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-a(t-\tau)^\alpha)(-K_{-1})d\tau\\
&=E_\alpha(-at^\alpha)x_0-K_{-1}\sum_{k=0}^\infty\int_0^t\frac{(-a)^k}{\Gamma(k\alpha+\alpha)}(t-\tau)^{k\alpha+\alpha-1}d\tau\\
&=E_\alpha(-at^\alpha)x_0+\frac{K_{-1}}{a}\sum_{k=0}^\infty\frac{(-at^\alpha)^{k+1}}{\Gamma((k+1)\alpha+1)}\\
&=E_\alpha(-at^\alpha)x_0+\frac{E_\alpha(-at^\alpha)-1}{a}K_{-1};
\end{aligned}$$

(ii) When $t\in[n,n+1)$, similar to the step (i), we can obtain

$$x(t)=\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)}x_n+\frac{\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)}-1}{a}K_{n-1},\quad t\in[n,n+1),\quad n\geqslant 1.$$

(2) Because $x(t)$ is a continuous function, we have

$$x_{n+1}=\frac{E_\alpha(-a(n+1)^\alpha)}{E_\alpha(-an^\alpha)}x_n+\frac{\frac{E_\alpha(-a(n+1)^\alpha)}{E_\alpha(-an^\alpha)}-1}{a}K_{n-1}.$$

From semigroup properties of Mittag–Leffler function, then $x_{n+1}=E_\alpha(-a)x_n+\frac{E_\alpha(-a)-1}{a}K_{n-1}$.

We can also see that

$$K_n = \int_0^1 x(s+n)dR(s) = \int_0^1 \left(E_\alpha(-as^\alpha)x_n + \frac{E_\alpha(-as^\alpha)-1}{a}K_{n-1} \right) dR(s)$$
$$= x_n \int_0^1 E_\alpha(-as^\alpha)dR(s) + K_{n-1}\int_0^1 \frac{E_\alpha(-as^\alpha)-1}{a}dR(s) = P(a)x_n - Q(a)K_{n-1}.$$

Letting $Y_n = (x_n, K_{n-1})^T$, then $Y_{n+1} = AY_n$, where

$$A = \begin{pmatrix} E_\alpha(-a) & \frac{E(-a)-1}{a} \\ P(a) & -Q(a) \end{pmatrix}.$$

Thus, x_n satisfies the second order difference equation

$$x_{n+2} - (E(-a) - Q(a))x_{n+1} + \left(\frac{1-E_\alpha(-a)}{a}P(a) - E_\alpha(-a)Q(a) \right) x_n = 0.$$

The proof is complete.□

Remark 3.1. By Lemma 3.1, the solution of (1.2), (3.1) at integer points satisfies the difference equation (3.6) with x_0, x_{-1} defined in (3.2).

Definition 3.2 A solution of (1.2) oscillates if it is neither eventually positive nor eventually negative. Equation (1.2) is oscillatory if all its solutions oscillate.

A solution of (3.6) oscillates if the solution $\{x_n\}$ is neither eventually positive nor eventually negative. Equation (3.6) is oscillatory if all its solutions oscillate.

Lemma 3.2 *Equation* (1.2) *is oscillatory if and only if* (3.6) *is oscillatory.*

Proof. “ $\Leftarrow$”: We can easily see that if a solution of (3.6) oscillates then the relevant solution of (1.2) (with an appropriate initial function, see Remark 3.1) cannot be eventually positive or negative.

“ $\Rightarrow$”: Because of

$$(E_\alpha(-at^\alpha))' = (\sum_{k=0}^{\infty} \frac{(-at^\alpha)^k}{\Gamma(k\alpha+1)})' = \sum_{k=0}^{\infty} \frac{(-a)^k(t^{k\alpha})'}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-a)^k k\alpha t^{k\alpha-1}}{\Gamma(k\alpha+1)}$$
$$= \sum_{k=0}^{\infty} \frac{(-at^\alpha)^k t^{-1}}{\Gamma(k\alpha)} = t^{-1}E_{\alpha,0}(-at^\alpha),$$

then

$$x'(t) = \left(\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)} \right)' x_n + \left(\frac{\frac{E_\alpha(-at^\alpha)}{E_\alpha(-an^\alpha)} - 1}{a} \right)' K_{n-1}$$
$$= \frac{t^{-1}E_{\alpha,0}(-at^\alpha)}{E_\alpha(-an^\alpha)} x_n + \frac{\frac{t^{-1}E_{\alpha,0}(-at^\alpha)}{E_\alpha(-an^\alpha)}}{a} K_{n-1} \tag{3.8}$$
$$= (x_n + \frac{K_{n-1}}{a}) \frac{t^{-1}E_{\alpha,0}(-at^\alpha)}{E_\alpha(-an^\alpha)}, \quad t \in [n, n+1).$$

Note that by (3.8) a solution of (1.2) increases in $[n, n+1)$ if $ax_n + K_{n-1} < 0$ and decreases if $ax_n + K_{n-1} > 0$. Thus, if $x(n)$, $x(n+1)$ have the same sign, so are all the points between n and $n+1$, hence oscillation of (1.2) implies that (3.6) is also oscillating.□

In the following, the equivalent analysis of the stability of Equations (1.2) and (3.6) is considered.

According to $(A2)$, the initial function is bounded, so we can define the supnorm:

$$\|\psi\| = \sup_{t\in[-1,0]} |\psi(t)|.$$

Definition 3.3 Equation (1.2) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any ψ satisfying $(A2)$ inequality $\|\psi\| < \delta$ implies $|x(t)| < \varepsilon$ for $t \geqslant 0$. Equation (1.2) is asymptotically stable if it is stable and $\lim_{t\to\infty} x(t) = 0$ for any initial conditions. Equation (1.2) is exponentially stable if there exist positive numbers N, γ such that any solution satisfies

$$|x(t)| < Ne^{-\gamma t}\|\psi\|.$$

Equation (3.6) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\max\{|x_0|, |x_{-1}|\} < \delta$ implies $|x_n| < \varepsilon$ for any $n \geqslant 0$. Equation (3.6) is asymptotically stable if it is stable and $\lim_{t\to\infty} x_n = 0$ for any initial conditions. Equation (3.6) is exponentially stable if there exist positive numbers N, γ such that any solution satisfies

$$|x_n| \leqslant Ne^{-\gamma n} \max\{|x_0|, |x_{-1}|\}.$$

Lemma 3.3 *Equation* (1.2) *is stable (asymptotically stable, exponentially stable) if and only if* (3.6) *is stable (asymptotically stable, exponentially stable).*

Proof. Since the proofs of three kinds of stability is similar, we only consider the case of exponential stability. Necessity is obvious. Next, we prove the sufficiency.

From the exponential stability of Equation (3.6), there exist positive numbers N, γ such that any solution satisfies

$$|x_n| < Ne^{-\gamma n} \max\{|x_0|, |x_{-1}|\}.$$

As in the previous corollary, for any solution of (1.2), $\max_{t\in[n,n+1]} |x(t)|$ is attained at the ends and equals either $|x(n)| = |x_n|$ or $|x(n+1)| = |x_{n+1}|$, so

$$\begin{aligned} |x(t)| &\leqslant |x_n| < Ne^{-\gamma n} \max\{|x_0|, |x_{-1}|\} \\ &= Ne^{-\gamma t} e^{-\gamma(n-t)} \max\{|x_0|, |x_{-1}|\} \\ &\leqslant Ne^{-\gamma t} \max\{1, e^{\gamma}\} \|\psi(t)\|. \end{aligned}$$

Therefore, (1.2) is exponentially stable.□

Next, we will obtain the necessary and sufficient conditions for the oscillation, stability and exponential stability of equation (1.2).

Theorem 3.4 *Suppose* $(A1)$–$(A2)$ *are satisfied. Equation* (1.2) *is oscillatory if and only if at least one of the two following conditions holds:*

$$\frac{1}{4}\left(E_\alpha(-a) + Q(a)\right)^2 < \frac{1 - E_\alpha(-a)}{a} P(a), \tag{3.9}$$

$$E_\alpha(-a) \leqslant Q(a) \leqslant \frac{\frac{1}{E_\alpha(-a)} - 1}{a} P(a). \tag{3.10}$$

Proof. By Lemma 3.2 Equation (3.6) is oscillatory if and only if either

$$\frac{1}{4}\left(E_\alpha(-a) - Q(a)\right)^2 < \frac{1 - E_\alpha(-a)}{a} P(a) - E_\alpha(-a) Q(a), \tag{3.11}$$

or

$$E_\alpha(-a) \leq Q(a) \leq \frac{1 - E_\alpha(-a)}{a E_\alpha(-a)} P(a), \tag{3.12}$$

where the former inequality is equivalent to (3.9) and the latter to (3.10).

Theorem 3.5 *Suppose* $(A1)$–$(A2)$ *are satisfied. Equation* (1.2) *is stable if and only if*

$$|Q(a) - E_\alpha(-a)| \leqslant \frac{1 - E_\alpha(-a)}{a} P(a) - E_\alpha(-a) Q(a) + 1 \leqslant 2$$

and is exponentially stable if and only if

$$|Q(a) - E_\alpha(-a)| < \frac{1 - E_\alpha(-a)}{a} P(a) - E_\alpha(-a) Q(a) + 1 < 2.$$

Proof. Combining Lemma 2.2 and Equation (3.6), we can get the result directly.

4. Particular case

In this section, we will consider a particular cases of (1.2).

Let $R(s)$ be a step function $R(s) = b\chi_{(r,1]}(t)$, $0 \leqslant r < 1$. Then (1.2) has the form

$${}^cD^\alpha x(t) + a x(t) + b x(r + [t-1]) = 0. \tag{4.1}$$

Then

$$P(a) = E_\alpha(-ar^\alpha) b, \quad Q(a) = \frac{1 - E_\alpha(-ar^\alpha)}{a} b.$$

So we can draw the following conclusions directly from Theorems 3.4 and 3.5. Equation (4.1) is oscillatory if and only if one of the following two inequalities holds

$$\frac{1}{4}\left(E_\alpha(-a) + \frac{1 - E_\alpha(-ar^\alpha)}{a} b\right)^2 < \frac{1 - E_\alpha(-a)}{a} E_\alpha(-ar^\alpha) b, \tag{4.2}$$

$$E_\alpha(-a) \leqslant \frac{1 - E_\alpha(-ar^\alpha)}{a} b \leqslant \frac{\frac{1}{E_\alpha(-a)} - 1}{a} E_\alpha(-ar^\alpha) b. \tag{4.3}$$

Equation (4.1) is stable if and only if

$$\left|\frac{1 - E_\alpha(-ar^\alpha)}{a} b - E_\alpha(-a)\right| \leqslant \frac{E_\alpha(-ar^\alpha) - E_\alpha(-a)}{a} b + 1 \leqslant 2,$$

and is exponentially stable if and only if

$$\left|\frac{1-E_\alpha(-ar^\alpha)}{a}b-E_\alpha(-a)\right| < \frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}b+1<2. \tag{4.4}$$

Theorem 4.1 *Let $0<r<1$. Equation (4.1) is oscillatory if and only if*

$$b > \left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\left[\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}-\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right]^2. \tag{4.5}$$

Proof. Using (4.2), (4.3), we will obtain explicit conditions for b, if a is given. Since $\frac{1-E_\alpha(-ar^\alpha)}{a}\geqslant 0$, then the left inequality in (4.3) becomes $b\geqslant \frac{aE_\alpha(-a)}{1-E_\alpha(-ar^\alpha)}$, while the right inequality $\frac{1-\frac{E_\alpha(-ar^\alpha)}{E_\alpha(-a)}}{a}b\leqslant 0$ is $b\geqslant 0$. Thus, (4.3) has the form

$$b\geqslant \{0,\frac{aE(-a)}{1-E(-ar^\alpha)}\}=\frac{aE(-a)}{1-E(-ar^\alpha)}.$$

Inequality (4.2) can be rewritten as a quadratic inequality in b:

$$\left(\frac{1-E_\alpha(-ar^\alpha)}{a}\right)^2 b^2+\left(2\frac{E_\alpha(-a)-2E_\alpha(-ar^\alpha)+E_\alpha(-a(r+1)^\alpha)}{a}\right)b+(E_\alpha(-a))^2<0. \tag{4.6}$$

The discriminant of the above quadratic inequality in b is

$$\begin{aligned}D&=\frac{4(E_\alpha(-a)-2E_\alpha(-ar^\alpha)+E_\alpha(-a(r+1)^\alpha))^2-4(E_\alpha(-a)-E_\alpha(-a(r+1)^\alpha))^2}{a^2}\\&=4\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a},\end{aligned}$$

which is positive as a product of two positive factors. A solution of inequality (4.6) is between the two roots $b_1<b_2$ of the relevant quadratic equation, the largest of them is

$$\begin{aligned}b_2&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\left(\frac{2E_\alpha(-ar^\alpha)-E_\alpha(-a)-E_\alpha(-a(r+1)^\alpha)}{a}\right)\\&+\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\left(2\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right)\\&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}+\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right)^2,\end{aligned}$$

similarly

$$b_1=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}-\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right)^2. \tag{4.7}$$

Thus, when $b_1 < b < b_2$, (4.2) is satisfied. Let us demonstrate the relationship between b_1, b_2 and $\frac{aE(-a)}{1-E(-ar^\alpha)}$.

$$\begin{aligned}
&b_1 - \frac{aE(-a)}{1-E(-ar^\alpha)}\\
&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\\
&\left[\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}-\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right)^2-\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}\right]\\
&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\\
&\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}-\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}+\sqrt{\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}}\right)\\
&\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}-\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}-\sqrt{\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}}\right),
\end{aligned}$$

the second term of the last equality is nonnegative since $\sqrt{x} \leqslant \sqrt{x+y}$ and the third term of the last equality is nonpositive since $\sqrt{x+y} \leqslant \sqrt{x}+\sqrt{y}$ for any nonnegative x, y, thus $b_1 - \frac{aE(-a)}{1-E(-ar^\alpha)} \leqslant 0$, and

$$\begin{aligned}
&b_2 - \frac{aE(-a)}{1-E(-ar^\alpha)}\\
&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\\
&\left[\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}+\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}\right)^2-\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}\right]\\
&=\left(\frac{a}{1-E_\alpha(-ar^\alpha)}\right)^2\\
&\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}+\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}+\sqrt{\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}}\right)\\
&\left(\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a(r+1)^\alpha)}{a}}+\sqrt{\frac{E_\alpha(-ar^\alpha)-E_\alpha(-a)}{a}}-\sqrt{\frac{E(-a)-E_\alpha(-a(r+1)^\alpha)}{a}}\right)\\
&\geqslant 0
\end{aligned}$$

as a product of three nonnegative terms, the last term is nonnegative since $\sqrt{x+y} \leqslant \sqrt{x}+\sqrt{y}$ for any nonnegative x, y. From the above two formulas, we get $b_1 \leqslant \frac{aE(-a)}{1-E(-ar^\alpha)} \leqslant b_2$. Hence $b > b_1$ is equivalent to $\{b \mid b_1 < b < b_2 \text{ or } b \geqslant \frac{aE_\alpha(-a)}{1-E_\alpha(-ar^\alpha)}\}$. Namely, (4.5) is necessary and sufficient for oscillation, which completes the proof.□

Theorem 4.2 *Let $0 < r < 1$. Equation (4.1) is stable if and only if*

$$-a \leqslant b \leqslant C,$$

where

$$C = \begin{cases} \min\left\{\frac{a(1+E_\alpha(-a))}{1+E_\alpha(-a)-2E_\alpha(-ar^\alpha)}, \frac{a}{E_\alpha(-ar^\alpha)-E_\alpha(-a)}\right\}, & \text{if } \frac{1+E_\alpha(-a)-2E_\alpha(-ar^\alpha)}{a} > 0 \\ \frac{a}{E_\alpha(-ar^\alpha)-E_\alpha(-a)}, & \text{if } \frac{1+E_\alpha(-a)-2E_\alpha(-ar^\alpha)}{a} \leqslant 0 \end{cases}$$

and is exponentially stable if and only if

$$-a < b < C. \tag{4.8}$$

Proof. By (4.4), we can obtain that (4.1) is exponentially stable if and only if

$$-\frac{E_\alpha(-ar^\alpha) - E_\alpha(-a)}{a}b - 1 < \frac{1 - E_\alpha(-ar^\alpha)}{a}b - E_\alpha(-a) < \frac{E_\alpha(-ar^\alpha) - E_\alpha(-a)}{a}b + 1, \tag{4.9}$$

$$\frac{E_\alpha(-ar^\alpha) - E_\alpha(-a)}{a}b < 1. \tag{4.10}$$

Inequality (4.10) can be written as $b > \frac{a}{E_\alpha(-ar^\alpha)-E_\alpha(-a)}$, while the left inequality of (4.9) is

$$\frac{1 - E_\alpha(-a)}{a}b > E_\alpha(-a) - 1, \quad \text{i.e.,} \quad b > \frac{E_\alpha(-a) - 1}{1 - E_\alpha(-a)}a = -a.$$

Further, consider the right inequality in (4.9) which is equivalent to

$$\frac{1 + E_\alpha(-a) - 2E_\alpha(-ar^\alpha)}{a}b < 1 + E_\alpha(-a). \tag{4.11}$$

When $\frac{1+E_\alpha(-a)-2E_\alpha(-ar^\alpha)}{a} < 0$, then

$$b < \frac{a}{E_\alpha(-ar^\alpha) - E_\alpha(-a)};$$

when $\frac{1+E_\alpha(-a)-2E_\alpha(-ar^\alpha)}{a} > 0$, then

$$b < \frac{a(1 + E_\alpha(-a))}{1 + E_\alpha(-a) - 2E_\alpha(-ar^\alpha)},$$

combining with (4.10), we have

$$b < \min\left\{\frac{a\,(1 + E_\alpha(-a))}{1 + E_\alpha(-a) - 2E_\alpha(-ar^\alpha)}, \frac{a}{E_\alpha(-ar^\alpha) - E_\alpha(-a)}\right\}.$$

So the necessity is proved, and vice versa. The proof of exponential stability is completed. Stability is considered similarly. □

5. Example

In this section, we will present an example to illustrate our main results.

Example 5.1. We consider differential equations with piecewise continuous argument

$$^{c}D^{\alpha}x(t)+ax(t)+x(r+[t-1])=0,\quad t>0, \tag{5.1}$$

where $\alpha=\frac{1}{2}$, $a=-1$, $r=\frac{5}{6}$.

When $\alpha=\frac{1}{2}$, Mittag–Leffler function $E_{\alpha}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}$ becomes $E_{\frac{1}{2}}(z)=\exp(z^2)(1-\operatorname{erfc}(-z))$. By the definition of error function, we have

$$E_{\frac{1}{2}}(z)=e^{z^2}(1+\frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}d\eta),\quad z>0.$$

Next, we calculate $\int_0^z e^{-\eta^2}d\eta$ by the method of polar transformation. Let $u=\int_0^z e^{-\eta^2}d\eta$. Thus

$$u^2=\int_0^z e^{-\eta^2}d\eta\int_0^z e^{-\eta^2}d\eta=\int_0^z e^{-x^2}dx\int_0^z e^{-y^2}dy=\int_0^z\int_0^z e^{-(x^2+y^2)}dxdy.$$

We introduce a polar transformation $x=R\cos\theta$, $y=R\sin\theta$, where $R\in[0,z]$, $\theta\in[0,\frac{\pi}{2}]$, then

$$u^2=\int_0^{\frac{\pi}{2}}d\theta\int_0^z Re^{-R^2}dR=\frac{\pi}{4}\int_0^z e^{-R^2}dR^2=\frac{\pi}{4}(1-e^{-z^2}).$$

Therefore $u=\frac{\sqrt{\pi}}{2}\sqrt{1-e^{-z^2}}$, and then $E_{\frac{1}{2}}(z)=e^{z^2}(1+\sqrt{1-e^{-z^2}})$. So

$$E_{\frac{1}{2}}(-a)=E_{\frac{1}{2}}(1)\approx 4.879,\ E_{\frac{1}{2}}(-ar^{\alpha})=E_{\frac{1}{2}}(\sqrt{\frac{5}{6}})\approx 4.031,\ E_{\frac{1}{2}}(-a(r+1)^{\alpha})=E_{\frac{1}{2}}(\sqrt{\frac{11}{6}})\approx 11.987.$$

Then

$$\left(\frac{a}{1-E_{\alpha}(-ar^{\alpha})}\right)^2\left[\sqrt{\frac{E_{\alpha}(-ar^{\alpha})-E_{\alpha}(-a(r+1)^{\alpha})}{a}}-\sqrt{\frac{E_{\alpha}(-ar^{\alpha})-E_{\alpha}(-a)}{a}}\right]^2\approx 0.393;$$

$$\frac{1+E_{\alpha}(-a)-2E_{\alpha}(-ar^{\alpha})}{a}\approx 2.171>0;$$

$$C=\min\left\{\frac{a\,(1+E_{\alpha}(-a))}{1+E_{\alpha}(-a)-2E_{\alpha}(-ar^{\alpha})}=2.708,\frac{a}{E_{\alpha}(-ar^{\alpha})-E_{\alpha}(-a)}=1.179\right\}=1.179.$$

From Theorems 4.1 and 4.2, we can get:

(i) when $b\in(0.393,1)\cup(1.179,\infty)$, Equation (5.1) is oscillatory;

(ii) when $b\in[1,1.79]$, Equation (5.1) is oscillatory and stable;

(iii) when $b\in(1,1.79)$, Equation (5.1) is oscillatory and exponential stable.

6. Conclusion

In this paper, we transform the study of the fractional differential equation into the study of second order difference equation with constant coefficients which is equivalent to the original equation in terms of oscillation and stability. In the same way, we can study the equation

$$^{c}D^{\alpha}x(t) + ax(t) + \sum_{j=0}^{n} \int_{0}^{\sigma} x_j(s + r[\frac{t-k}{r}])d_s R_k^j(s) = 0.$$

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