[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

[Volume 46](https://journals.tubitak.gov.tr/math/vol46)
[Number 2 *SI-1*](https://journals.tubitak.gov.tr/math/vol46/iss2)
[Article 2](https://journals.tubitak.gov.tr/math/vol46/iss2/2)

1-1-2022

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2022) 46: 360 – 376
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doi:10.3906/mat-2104-56

Uniformly convergent finite difference method for reaction-diffusion type third order singularly perturbed delay differential equation

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Received: 15.04.2021 • **Accepted/Published Online:** 06.06.2021 • **Final Version:** 21.01.2022

Abstract: A class of third order reaction-diffusion type singularly perturbed ordinary delay differential equations is considered in this article. A fitted finite difference method on Shishkin mesh is suggested to solve the problem. Moreover, we present a class of nonlinear problems. An error estimation is obtained based on the maximum norm and it is of almost first order convergence. Numerical results are given to support theoretical claims.

Key words: Third order differential equations, singularly perturbed problem, reaction-diffusion, Shishkin mesh

1. Introduction

Singularly perturbed delay differential equations (SPDDEs) have received a lot of attention in recent years since they have been shown to be useful tools in a variety of fields of science and mathematical modeling, such as the variational problem in control theory [7], the predator-prey model [8], and the description of the human pupil-light reflex [12]. In recent years, many authors have developed numerical techniques for the solutions of singularly perturbed differential equations, in particular for the third order convection-diffusion and reaction-diffusion type problems without delay arguments in [3, 6, 22, 23] and the references therein to cite a few. Higher order delay differential equations (DDEs) are used to model some of the scientific and engineering problems. For example, a system of DDEs governs a mechanical system, which interestingly transformed into the third order delay differential equation [1]. The stability results for the third order DDEs are discussed in [2, 4] and the references therein. In [15], Nouioua et al. discussed the existence of positive periodic solutions for the third order delay differential equation by employing Green's function and Krasnoselskii's fixed point theorem. Moreover, for the third order delay differential equation with discontinuous data functions, the existence results are given in [18]. The analysis of the third order SPDDEs has received far less attention in the literature than that of the third order singularly perturbed differential equations. To cite a very few, the authors in [13, 18] suggested fitted finite difference method (FFDM) for the third order SPDDEs of convection-diffusion type. The authors in [19, 20] developed asymptotic numerical methods for both convection-diffusion and reaction-diffusion type SPDDEs on piecewise uniform Shishkin meshes. The proposed methods are of almost first order convergence. Sekar and Tamilselvan [16, 17] suggested FFDM for the third order SPDDEs with integral boundary conditions and obtained almost first order convergence. In this paper, the third order equation is converted into a weakly coupled system of equations. For a weakly coupled system, we proved the maximum principle, stability result,

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2010 *AMS Mathematics Subject Classification:* 34K10, 34K26, 34K28

and derivative estimates, etc. Using the maximum principle, one can prove that the solution of the original problem (2.1) is stable and it is unique, if it exists. In this paper, we proposed a FFDM for solving reaction-diffusion type third order SPDDEs using piecewise uniform Shishkin mesh. The proposed numerical method is uniformly convergent with order of convergence one. This method can be applied to the nonlinear problem (Section 5).

The rest of the paper is organized as follows: In Section 2, continuous problem, maximum principle, stability result, and the derivative estimates of the solution are presented. In Section 3, the problem is discretized using the standard finite difference method (FDM) on piecewise uniform Shishkin mesh. An error estimate is presented in Section 4. The quasilinearization technique is applied for nonlinear problem and then the fitted numerical method is also applied (Section 5). In Section 6, numerical results are presented to validate the theoretical findings. The proposed method is $\varepsilon-$ uniform convergent method. Furthermore, the method is of first order accurate. Finally, we conclude the paper with some discussion (Section 7).

Let ε be a small positive parameter such that $0 < \varepsilon \ll 1$ and C, C_1 denote generic positive constants independent of ε and N. Furthermore, let $\Omega = (0,2)$ be a set, its closure $\bar{\Omega} = [0,2]$ and $\Omega^* = \Omega^- \cup \Omega^+$, $\Omega^- = (0,1)$, $\Omega^+ = (1,2)$. The set $\bar{\Omega}^N$ denotes the set of grid points $\{x_0,\ x_1,\ \ldots, x_N\}$. The collections $Y,\ Y_1$, and Y_2, respectively denote $C^1(\bar{\Omega}) \cap C^3(\Omega)$, $C^0(\bar{\Omega}) \cap C^1(\Omega \cup \{2\})$, and $C^0(\bar{\Omega}) \cap C^2(\Omega)$. The norms $\|w\|_\Omega = \sup_{x\in\Omega} \mid w(x) \mid$ and $\|\bar{w}\|_\Omega = \max\{\|w_1\|_\Omega,\ \|w_2\|_\Omega\}$ are used in the following.

2. Problem statement and analytical results

2.1. Statement of the problem

Motivated by the work of [9–11, 24], we consider the following SPDDE:

Find $u \in Y$ such that

$$\begin{cases} -\varepsilon u'''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x),\ x \in \Omega, \\ u(x) = \varphi(x),\ x \in [-1,0],\ u'(0) = \varphi'(0),\ u'(2) = l,\ \varphi \in C^1([-1,0]), \end{cases} \tag{2.1}$$

where $b(x) \geq \beta > 0$, $c(x) \leq \gamma < 0$, $d(x) \leq \eta < 0$, $\beta + 24\gamma + 5\eta > 0$. Furthermore, b, c, d, and f are sufficiently differentiable, bounded on $\bar{\Omega}$ and φ is sufficiently differentiable on $[-1,0]$.

Let $u = u_1$ and $u' = u_2$, then the above problem (2.1) can be written as follows:

Find $\bar{u} = (u_1,\ u_2)$, $u_1 \in Y_1$ and $u_2 \in Y_2$ such that

$$P_1\bar{u}(x) := u_1'(x) - u_2(x) = 0,\ x \in \Omega \cup \{2\}, \tag{2.2}$$

$$P_2\bar{u}(x) := \begin{cases} -\varepsilon u_2''(x) + b(x)u_2(x) + c(x)u_1(x) = f(x) - d(x)\varphi'(x-1),\ x \in \Omega^- \cup \{1\}, \\ -\varepsilon u_2''(x) + b(x)u_2(x) + c(x)u_1(x) + d(x)u_2(x-1) = f(x),\ x \in \Omega^+, \end{cases} \tag{2.3}$$

$$u_1(0) = \varphi(0),\ u_2(0) = \varphi'(0),\ u_2(2) = l.$$

In the rest of the article, we consider the above problem (2.2)-(2.3).

2.2. Stability result

In this section, a maximum principle for the above problem (2.2)-(2.3) is presented.

Theorem 2.1 *[13]* Let $\bar{w} = (w_1,\ w_2)$ be any function satisfying $w_1(0) \geq\ 0$ and $w_2(0) \geq\ 0,\ w_2(2) \geq\ 0$, $P_1\bar{w}(x) \geq\ 0,\ \forall x \in \Omega \cup \{2\},\ P_2\bar{w}(x) \geq\ 0,\ \forall x \in \Omega^*$, and $w_2'(1+) - w_2'(1-) = [w_2'](1) \leq\ 0$. Then $w_i(x) \geq\ 0,\ \forall x \in \bar{\Omega},\ i\ =\ 1, 2$, where $w_1 \in C^1(\Omega)$ and $w_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.

Corollary 2.2 *[13]* If $\bar{\psi} = (\psi_1, \psi_2),\ \psi_1 \in Y_1,\ \psi_2 \in Y_2$ is any function that satisfies the system (2.2)-(2.3), then we have $|\ \psi_i(x)\ | \leq C \max\left\{\ |\ \psi_1(0)\ |,\ |\ \psi_2(0)\ |,\ |\ \psi_2(2)\ |,\ \sup_{\zeta_1 \in \Omega \cup \{2\}} |\ P_1\bar{\psi}(\zeta_1)\ |,\ \sup_{\zeta_2 \in \Omega^*} |\ P_2\bar{\psi}(\zeta_2)\ |\ \right\},\ \forall\ x \in \bar{\Omega},\ i\ =\ 1, 2$.

Note: The condition $\beta +\ 24\gamma +\ 5\eta >\ 0$ among the coefficient functions is used to prove the stability of the solution. The consequence of the above corollary is that the solution of the above problem (2.2)-(2.3) is unique, if it exists.

2.3. Derivative estimates

We estimate upper bounds for derivatives of the solution of the problem (2.2)-(2.3) in this section. We also determine the sharp bounds for the derivatives.

Theorem 2.3 *If $\bar{u}$ is the solution of (2.2)-(2.3), then for $k\ =\ 0, 1, 2, 3,$ we have*

$$\|\ u_1^{(k)}\ \|_{\Omega^*} \leq C\varepsilon^{\frac{1\ -\ k}{2}},\ \|\ u_2^{(k)}\ \|_{\Omega^*} \leq C\varepsilon^{-\ \frac{k}{2}}.$$

Proof The proof of the theorem is similiar to that of [13, Lemma 4.1]. □

The sharp bounds on the derivatives of the solution $\bar{u}$ is obtained by taking $\bar{u}$ as equal to sum of the regular component $\bar{v}$ and the singular component $\bar{w}$ as described in [14]. That is,

$$\bar{u}(x) = \bar{v}(x) + \bar{w}(x),$$

where $\bar{v} = \bar{v}_0 + \sqrt{\varepsilon}\bar{v}_1 + \varepsilon\bar{v}_2$ and $\bar{v}_0,\ \bar{v}_1$, and $\bar{v}_2$ are the solutions of the following problems:

Find $\bar{v}_0 = (v_{0,1}, v_{0,2})$ such that

$$\begin{cases} v_{0,1}'(x) - v_{0,2}(x) =\ 0,\ x \in \Omega^* \cup \{2\}, \\ b(x)v_{0,2}(x) + c(x)v_{0,1}(x) + d(x)v_{0,2}(x\ -\ 1) = f(x),\ x \in \Omega^* \cup \{0, 2\}, \\ v_{0,1}(0) = \varphi(0),\ v_{0,2}(x) = \varphi'(x),\ x \in [-\ 1, 0), \end{cases} \tag{2.4}$$

$\bar{v}_1 = (v_{1,1}, v_{1,2})$, such that

$$\begin{cases} v_{1,1}'(x) - v_{1,2}(x) =\ 0,\ x \in \Omega^* \cup \{2\}, \\ b(x)v_{1,2}(x) + c(x)v_{1,1}(x) + d(x)v_{1,2}(x\ -\ 1) = \sqrt{\varepsilon}v_{0,2}''(x),\ x \in \Omega^* \cup \{2\}, \\ v_{1,1}(0) =\ 0,\ v_{1,2}(x) =\ 0,\ x \in [-\ 1, 0), \end{cases} \tag{2.5}$$

and $\bar{v}_2 = (v_{2,1}, v_{2,2})$, such that

$$\begin{cases} P_1\bar{v}_2 := v_{2,1}'(x) - v_{2,2}(x) =\ 0,\ x \in \Omega^* \cup \{2\}, \\ P_2\bar{v}_2 := -\varepsilon v_{2,2}''(x) + b(x)v_{2,2}(x) + c(x)v_{2,1}(x) + d(x)v_{2,2}(x\ -\ 1) = \sqrt{\varepsilon}v_{1,2}''(x),\ x \in \Omega^*, \\ v_{2,1}(0) =\ 0,\ v_{2,2}(x) =\ 0,\ x \in [-\ 1, 0],\ v_{2,2}(2) =\ 0. \end{cases} \tag{2.6}$$

Thus, the regular component $\bar{v}$ satisfies the following: Find $\bar{v} = (v_1, v_2)$, such that

$$\begin{cases} P_1\bar{v}(x) := v_1'(x) - v_2(x) = 0, \; x \in \Omega^* \cup \{2\}, \\ P_2\bar{v}(x) := -\varepsilon v_2''(x) + b(x)v_2(x) + c(x)v_1(x) + d(x)v_2(x-1) = f(x), \; x \in \Omega^*, \\ v_1(0) = \varphi(0), \; v_2(x) = \begin{cases} \varphi'(x), & x \in [-1, 0), \\ v_{0,2}(0) + \sqrt{\varepsilon}v_{1,2}(0) + \sqrt{\varepsilon}v_{2,2}(0), & x = 0, \end{cases} \\ v_2(2) = v_{0,2}(2) + \sqrt{\varepsilon}v_{1,2}(2). \end{cases} \tag{2.7}$$

Furthermore, $\bar{w}$ satisfies the following problem: Find $\bar{w} = (w_1, w_2)$ such that

$$\begin{cases} P_1\bar{w}(x) := w_1'(x) - w_2(x) = 0, \; x \in \Omega^* \cup \{2\}, \\ P_2\bar{w}(x) := -\varepsilon w_2''(x) + b(x)w_2(x) + c(x)w_1(x) + d(x)w_2(x-1) = 0, \; x \in \Omega^*, \\ w_1(0) = 0, \; w_2(x) = \begin{cases} 0, & x \in [-1, 0), \\ \varphi'(0) - v_2(0), & x = 0, \end{cases} \\ [w_2](1) = -[v_2](1), \; [w_2'](1) = -[v_2'](1), w_2(2) = l - [v_{0,2}(2) + \sqrt{\varepsilon}v_{1,2}(2)]. \end{cases} \tag{2.8}$$

Theorem 2.4 *If $\bar{u} = \bar{v} + \bar{w}$. Then for $r = 0, 1, 2, 3$ and $k = 1, 2$, we have*

$$\| v_1^{(r)} \|_{\Omega^*} \leq C(1 + \varepsilon^{\frac{3-r}{2}}), \tag{2.9}$$

$$\| v_2^{(r)} \|_{\Omega^*} \leq C(1 + \varepsilon^{\frac{2-r}{2}}), \tag{2.10}$$

$$| w_k^{(r)}(x) | \leq C\varepsilon^{\frac{2-k-r}{2}} \begin{cases} e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, & x \in (0,1), \\ e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}, & x \in (1,2). \end{cases} \tag{2.11}$$

Proof The inequalities (2.9)-(2.10) can be proved by integrating (2.4)-(2.6) and using Corollary 2.2. The following barrier functions $\bar{\varphi}^{\pm} = (\varphi_1^{\pm}, \varphi_2^{\pm})$ and $\bar{\psi}^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$ are used to prove (2.11) in [0,1] and [1,2], respectively.

Let $x \in [0,1]$, then

$$\varphi_1^{\pm}(x) = C_1\varepsilon\{\sqrt{\beta} - \sqrt{\beta}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + \sqrt{\beta}e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}\} \pm w_1(x), \; x \in \Omega^-,$$

$$\varphi_2^{\pm}(x) = C_1\{\beta e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + \beta e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}\} \pm w_2(x), \; x \in \Omega^-.$$

Note that, $\varphi_1^{\pm}(0) \geq 0$, $\varphi_2^{\pm}(0) \geq 0$, $\varphi_2^{\pm}(1) \geq 0$ and

$$P_1\bar{\varphi}^{\pm}(x) = C\{\sqrt{\varepsilon}(\frac{\beta}{\sqrt{\varepsilon}}e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + \frac{\beta}{\sqrt{\varepsilon}}e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}) - (\beta e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + \beta e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}})\} \pm 0 \geq 0,$$

$$P_2\bar{\varphi}^{\pm}(x) = C\{[\beta(b(x) - \beta) - \sqrt{\varepsilon}c(x)\sqrt{\beta}]e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + [\beta(b(x) - \beta) - \sqrt{\varepsilon}c(x)\sqrt{\beta}]e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}$$
$$+ \sqrt{\varepsilon}c(x)\sqrt{\beta}\} \pm 0 \geq 0.$$

By [21, Theorem 2.1], we have

$$|w_k^{(r)}(x)| \leq C\varepsilon^{\frac{2-k-r}{2}}\{e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}\}, \; x \in [0,1], \; r = 0, 1, 2, 3.$$

Let $x \in [1,2]$, then $\bar{\psi}^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$, where

$$\psi_1^{\pm}(x) = C_1\varepsilon\{\sqrt{\beta} - \sqrt{\beta}e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + \sqrt{\beta}e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}\} \pm w_1(x),\ x \in \Omega^+,$$

$$\psi_2^{\pm}(x) = C_1\{\beta e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + \beta e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}\} \pm w_2(x),\ x \in \Omega^+.$$

Also note that $\psi_1^{\pm}(1) \geq 0$, $\psi_2^{\pm}(1) \geq 0$, $\psi_2^{\pm}(2) \geq 0$ and

$$P_1\bar{\psi}^{\pm}(x) = C\{\sqrt{\varepsilon}(\frac{\beta}{\sqrt{\varepsilon}}e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + \frac{\beta}{\sqrt{\varepsilon}}e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}) - (\beta e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + \beta e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}})\} \pm 0 \geq 0,$$

$$P_2\bar{\psi}^{\pm}(x) = C\{[\beta(b(x)-\beta) - \sqrt{\varepsilon}c(x)\sqrt{\beta}]e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + [\beta(b(x)-\beta) - \sqrt{\varepsilon}c(x)\sqrt{\beta}]e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}$$
$$+ \sqrt{\varepsilon}c(x)\sqrt{\beta}\} \pm 0 \geq 0.$$

Again by using [21, Theorem 2.1], we have

$$|\, w_k^{(r)}(x) \,| \leq C\varepsilon^{\frac{2-k-r}{2}}\{e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}\},\ x \in [1,2],\ r = 0,1,2,3.$$

□

Note: It is easy to see from the above theorem that, for $k = 1,2$

$$|\, u_k(x) - v_k(x) \,| \leq C\varepsilon^{\frac{2-k}{2}} \begin{cases} e^{-x\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, & x \in \Omega^-, \\ e^{-(x-1)\sqrt{\frac{\beta}{\varepsilon}}} + e^{-(2-x)\sqrt{\frac{\beta}{\varepsilon}}}, & x \in \Omega^+. \end{cases}$$

3. Finite difference method

In this section, a standard FDM is presented for (2.2)-(2.3). The index set I_N is defined as, $I_N = \{0,1,2,\ldots,N\}$.

3.1. Shishkin mesh

For the discretization of the problem (2.2)-(2.3) described in Section 3.2, we shall use the mesh that is adapted for the boundary layers at $x = 0$, $x = 2$ and interior twin layers at $x = 1$. Therefore, the domain $\bar{\Omega}$ is splitted into six subdomains namely, $[0,\tau]$, $[\tau, 1-\tau]$, $[1-\tau, 1]$, $[1, 1+\tau]$, $[1+\tau, 2-\tau]$, $[2-\tau, 2]$, where $\tau = \min\{0.25, \frac{2\sqrt{\varepsilon}\ln N}{\sqrt{\beta}}\}$. The mesh $\bar{\Omega}^N = \{x_0, x_1, \ldots, x_N\}$ is defined by $x_0 = 0.0$, $x_i = x_0 + ih$, $i = 1,\ldots,\frac{N}{8}$, $x_{i+\frac{N}{4}} = x_{\frac{N}{4}} + iH$, $i = 1,\ldots,\frac{N}{4}$, $x_{i+\frac{3N}{8}} = x_{\frac{3N}{8}} + ih$, $i = 1,\ldots,\frac{N}{8}$, $x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih$, $i = 1,\ldots,\frac{N}{8}$, $x_{i+\frac{5N}{8}} = x_{\frac{5N}{8}} + iH$, $i = 1,\ldots,\frac{N}{4}$, $x_{i+\frac{7N}{8}} = x_{\frac{7N}{8}} + ih$, $i = 1,\ldots,\frac{N}{8}$, where $h = 8N^{-1}\tau$ and $H = 4N^{-1}(1-2\tau)$.

3.2. A finite difference scheme

On the mesh $\bar{\Omega}^N$, the standard FDM is applied.

$$P_1^N\bar{U}(x_i) := D^-U_1(x_i) - U_2(x_i) = 0,\ i \in I_N \setminus \{0\}, \tag{3.1}$$

$$P_2^N\bar{U}(x_i) := -\varepsilon\delta^2U_2(x_i) + b(x_i)U_2(x_i) + c(x_i)U_1(x_i) + d(x_i)U_2^I(x_i) = f^*(x_i),\ i \in I_N \setminus \{0, \frac{N}{2}, N\}, \tag{3.2}$$

$$U_1(x_0) = \varphi(0),\ U_2(x_0) = \varphi'(0),\ D^-U_2(x_{\frac{N}{2}}) = D^+U_2(x_{\frac{N}{2}}),\ U_2(x_N) = l, \tag{3.3}$$

where $\delta^2,\ D^+,\ D^-$ are central, forward, backward difference operators, respectively [13] and

$$f^*(x_i)=\begin{cases} f(x_i)-d(x_i)\varphi'(x_i-1), & i<\frac{N}{2},\\ f(x_i), & i>\frac{N}{2},\end{cases}\quad U_2^I(x_i)=\begin{cases}0, & i<\frac{N}{2},\\ U_2(x_{i-\frac{N}{2}}), & i>\frac{N}{2}.\end{cases}$$

3.3. Discrete maximum principle and stability result

Lemma 3.1 *[13] Let $\bar{Z}(x_i)=(Z_1(x_i),Z_2(x_i))$ be mesh function satisfying $Z_1(x_0)\geq 0$, $Z_2(x_0)\geq 0$, $Z_2(x_N)\geq 0$, $P_1^N\bar{Z}(x_i)\geq 0$, $P_2^N\bar{Z}(x_i)\geq 0$, and $[D]Z_2(x_{N/2})\leq 0$. Then, $Z_1(x_i)\geq 0$ and $Z_2(x_i)\geq 0$, $x_i\in\bar{\Omega}^N$.*

Lemma 3.2 *If $\bar{U}(x_i)=(U_1(x_i),U_2(x_i))$ is the discrete problem solution. Then,*

$$\mid U_k(x_i)\mid\leq C\max\Big\{\mid U_1(x_0)\mid,\ \mid U_2(x_0)\mid,\ \mid U_2(x_N)\mid,\ \max_{j\in I_N}\mid P_1^N\bar{U}(x_j)\mid,$$
$$\max_{j\in I_N\setminus\{0,\frac{N}{2},N\}}\mid P_2^N\bar{U}(x_j)\mid\Big\},\ i\in I_N,\ k=1,2.$$

The numerical solution $\bar{U}(x_i)$ described by (3.1)-(3.3) can be written as

$$\bar{U}(x_i)=\bar{V}(x_i)+\bar{W}(x_i),\tag{3.4}$$

where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ satisfy the following:

$$\begin{cases} P_1^N\bar{V}(x_i):=D^-V_1(x_i)-V_2(x_i)=0,\ i\in I_N\setminus\{0\},\\ P_2^N\bar{V}(x_i):=-\varepsilon\,\delta^2V_2(x_i)+b(x_i)V_2(x_i)+c(x_i)V_1(x_i)+d(x_i)V_2^I(x_i)=f^*(x_i),\ i\in I_N\setminus\{0,\frac{N}{2},N\},\\ V_j(x_0)=v_j(0),\ [D]V_2(x_{\frac{N}{2}})=[v_2'](1),\ V_2(x_N)=v_2(2),\ j=1,2\end{cases}\tag{3.5}$$

and

$$\begin{cases} P_1^N\bar{W}(x_i):=D^-W_1(x_i)-W_2(x_i)=0,\ i\in I_N\setminus\{0\},\\ P_2^N\bar{W}(x_i):=-\varepsilon\,\delta^2W_2(x_i)+b(x_i)W_2(x_i)+c(x_i)W_1(x_i)+d(x_i)W_2^I(x_i)=0,\ i\in I_N\setminus\{0,\frac{N}{2},N\},\\ W_j(x_0)=w_j(0),\ [D]W_2(x_{\frac{N}{2}})=-[D]V_2(x_{\frac{N}{2}}),\ W_2(x_N)=w_2(2),\ j=1,2.\end{cases}\tag{3.6}$$

The estimate of the difference between the solutions of (3.1)-(3.3) and (3.5) can be determined using the following theorem.

Theorem 3.3 *If $\bar{U}(x_i)$ is a discrete problem solution of (2.2)-(2.3) defined by (3.1)-(3.3) and $\bar{V}(x_i)$ is a solution of (2.7) defined by (3.5); furthermore, if $\delta=\max\Big\{\mid\varphi'(0)-V_2(x_0)\mid,\mid U_1(x_{\frac{N}{2}})-V_1(x_{\frac{N}{2}})\mid,\mid U_2(x_{\frac{N}{2}})-V_2(x_{\frac{N}{2}})\mid,\mid l-V_2(x_N)\mid\Big\}$. Then,*

$$\mid U_k(x_i)-V_k(x_i)\mid\leq C\begin{cases} N^{-1}, & i\in\{\frac{N}{8},\dots,\frac{3N}{8}\}\cup\{\frac{5N}{8},\dots,\frac{7N}{8}\},\\ N^{-1}+\delta, & \textit{otherwise},\end{cases}\qquad k=1,2.$$

Proof Consider a mesh function

$$\varphi_k^{\pm}(x_i)=C_1(N^{-1}s_k(x_i)+\psi_k(x_i))\pm(U_k(x_i)-V_k(x_i)),\ k=1,2,$$

where

$$\psi_1(x_i)=\begin{cases}0, & i\in\{\frac{N}{8},\dots,\frac{3N}{8}\}\cup\{\frac{5N}{8},\dots,\frac{7N}{8}\},\\ 3x_i\delta, & i\in\{1,\dots,\frac{N}{8}-1\}\cup\{\frac{3N}{8}+1,\dots,\frac{N}{2}\},\\ 2x_i\delta, & i\in\{\frac{N}{2}+1,\dots,\frac{5N}{8}-1\},\\ \frac{3x_i}{2}\delta, & i\in\{\frac{7N}{8}+1,\dots,N\},\end{cases}$$

$$\psi_2(x_i)=\begin{cases}0, & i\in\{\frac{N}{8},\dots,\frac{3N}{8}\}\cup\{\frac{5N}{8},\dots,\frac{7N}{8}\},\\ (\frac{1}{8}+\frac{x_i}{2})\delta, & i\in\{1,\dots,\frac{N}{8}-1\}\cup\{\frac{3N}{8}+1,\dots,\frac{N}{2}\},\\ (\frac{3}{8}+\frac{x_i}{4})\delta, & i\in\{\frac{N}{2}+1,\dots,\frac{5N}{8}-1\}\cup\{\frac{7N}{8}+1,\dots,N\}\end{cases}$$

and $s_1(x_i) = 1 + x_i,\ x_i \in \bar{\Omega}^N,\ s_2(x_i) = \begin{cases}\frac{1}{8}+\frac{x_i}{2}, & x_i\in\Omega^-\cap\bar{\Omega}^N,\\ \frac{3}{8}+\frac{x_i}{4}, & x_i\in\Omega^+\cap\bar{\Omega}^N.\end{cases}$ It is verified that $\varphi_k^{\pm}(x_0) \geq 0$, $k = 1,2$ and $\varphi_2^{\pm}(x_N) \geq 0$, for a suitable choice of $C_1 > 0$.

When $x_i \in (0,\tau)\cap\bar{\Omega}^N$,

$$P_1^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}(\frac{7}{8}-\frac{\tau}{2})+C_1\delta(\frac{23}{8}-\frac{\tau}{2}) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}[\frac{\beta}{8}+(1+\tau)\gamma]+C_1\delta[\frac{\beta}{8}+3\gamma\tau] \pm 0 \geq 0.$$

When $x_i \in [\tau, 1-\tau]\cap\bar{\Omega}^N$,

$$P_1^N\bar{\varphi}^{\pm}(x_i) = C_1N^{-1}[1-(\frac{1}{8}+\frac{x_i}{2})] \pm 0 \geq C_1N^{-1}(\frac{7}{8}-\frac{1-\tau}{2}) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}[\beta(\frac{1}{8}+\frac{\tau}{2})+(2-\tau)\gamma] \pm 0 \geq 0.$$

When $x_i \in (1-\tau, 1)\cap\bar{\Omega}^N$,

$$P_1^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}(\frac{3}{8})+C_1\delta(\frac{19}{8}) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}[\beta(\frac{1}{8}+\frac{1-\tau}{2})+2\gamma]+C_1\delta[\beta(\frac{1}{8}+\frac{1-\tau}{2})+3\gamma] \pm 0 \geq 0.$$

When $x_i \in (1, 1+\tau)\cap\bar{\Omega}^N$,

$$P_1^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}(\frac{5}{8}-\frac{1+\tau}{4})+C_1\delta(\frac{21}{8}-\frac{1+\tau}{4}) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}[\frac{5\beta}{8}+(2+\tau)\gamma+(\frac{3}{8}+\frac{\tau}{4})\eta]+C_1\delta[\frac{5\beta}{8}+2\gamma(1+\tau)+(\frac{3}{8}+\frac{\tau}{4})\eta] \pm 0 \geq 0.$$

When $x_i \in [1+\tau, 2-\tau]\cap\bar{\Omega}^N$,

$$P_1^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}(\frac{1}{8}+\frac{\tau}{4}) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) \geq C_1N^{-1}[\beta(\frac{3}{8}+\frac{1+\tau}{4})+(3-\tau)\gamma+(\frac{1}{8}+\frac{2-\tau}{4})\eta] \pm 0 \geq 0.$$

When $x_i \in (2 - \tau, 2) \cap \bar{\Omega}^N$,

$$P_1^N \bar{\varphi}^{\pm}(x_i) \geq C_1 N^{-1}(\frac{1}{8}) + C_1\delta(\frac{1}{8}) \geq 0,$$

$$P_2^N \bar{\varphi}^{\pm}(x_i) \geq C_1 N^{-1}[\frac{\beta + 24\gamma + 5\eta}{8}] + C_1\delta[\frac{5\beta + 24\gamma + 5\eta}{8}] \pm 0 \geq 0.$$

When $x_i = x_{\frac{N}{2}}$, we have $[D]\varphi_2^{\pm}(x_{\frac{N}{2}}) = C_1 N^{-1}(-\frac{1}{4}) + C_1\delta(-\frac{1}{4}) \mp [v_2'](1) < 0$, for a suitable choice of $C_1 > 0$. Hence, the proof. □

4. Error estimate

Theorem 4.1 *If $\bar{V}(x_i)$ is the discrete problem solution of (2.7) defined by (3.5), then for $k = 1, 2$, we have*

$$| v_k(x_i) - V_k(x_i) | \leq CN^{-1}, \ i \in I_N.$$

Proof Now,

$$P_1^N(\bar{v}(x_i) - \bar{V}(x_i)) = P_1^N \bar{v}(x_i) - P_1^N \bar{V}(x_i) = (D^- - \frac{d}{dx})v_1(x_i),$$

$$P_2^N(\bar{v}(x_i) - \bar{V}(x_i)) = -\varepsilon\Big(\delta^2 - \frac{d^2}{dx^2}\Big)v_2(x_i) + d(x_i)\begin{cases} 0, & i \leq \frac{N}{2} - 1, \\ v_2^I(x_i) - v_2(x_i - 1), & i \geq \frac{N}{2} + 1. \end{cases}$$

$$| P_k^N(\bar{v}(x_i) - \bar{V}(x_i)) | \leq CN^{-1}, \ i \in I_N \setminus \{0, \frac{N}{2}, N\}, \ k = 1, 2.$$

By the Lemma 3.2, we have

$$| v_k(x_i) - V_k(x_i) | \leq CN^{-1}, \ i \in I_N, \ k = 1, 2.$$

Hence, the proof. □

Theorem 4.2 *If $\bar{W}(x_i)$ is the discrete problem solution of (2.8) defined by (3.6). Then for $k = 1, 2$, we have $| w_k(x_i) - W_k(x_i) | \leq CN^{-1}\ln^2 N, \ i \in I_N$.*

Proof Note that

$$| w_k(x_i) - W_k(x_i) | \leq | u_k(x_i) - U_k(x_i) | + | v_k(x_i) - V_k(x_i) |, \ k = 1, 2.$$

Then by (2.3), Theorem 3.3, and Theorem 4.1, we have

$$| u_k(x_i) - U_k(x_i) | \leq | U_k(x_i) - V_k(x_i) | + | v_k(x_i) - V_k(x_i) | + | u_k(x_i) - v_k(x_i) |$$

$$\leq CN^{-1} + CN^{-1} + CN^{-1} \leq CN^{-1}, \ i \in \{\frac{N}{8}, \ldots, \frac{3N}{8}\} \cup \{\frac{5N}{8}, \ldots, \frac{7N}{8}\}.$$

Therefore,

$$| w_k(x_i) - W_k(x_i) | \leq | u_k(x_i) - U_k(x_i) | + | v_k(x_i) - V_k(x_i) |, \ k = 1, 2$$

$$| w(x_i) - W(x_i) | \leq CN^{-1}, \ i \in \{\frac{N}{8}, \ldots, \frac{3N}{8}\} \cup \{\frac{5N}{8}, \ldots, \frac{7N}{8}\}. \tag{4.1}$$

Now, consider a mesh function $\bar{\varphi}^{\pm}(x_i) = (\varphi_1^{\pm}(x_i), \varphi_2^{\pm}(x_i))$, where

$$\varphi_1^{\pm}(x_i) = C_1N^{-1}\{s_1(x_i) - \frac{1}{\sqrt{\varepsilon}}(\tau - x_i)\} \pm (w_1(x_i) - W_1(x_i)),\ x_i \in [0, \tau] \cap \bar{\Omega}^N,$$

$$\varphi_2^{\pm}(x_i) = C_1N^{-1}\{s_2(x_i) + \frac{1}{\sqrt{\varepsilon}}(\tau - x_i)\} \pm (w_2(x_i) - W_2(x_i)),\ x_i \in [0, \tau] \cap \bar{\Omega}^N.$$

From (4.1), it is easy to verify that $\varphi_k^{\pm}(x_0) \geq 0,\ k = 1, 2$ and $\varphi_2^{\pm}(x_{\frac{N}{8}}) \geq 0$, for a suitable choice of $C_1 > 0$.

$$P_1^N\bar{\varphi}^{\pm}(x_i) = C_1N^{-1}[1 - s_2(x_i)] + \frac{1}{\sqrt{\varepsilon}}[1 - \tau + x_i] \pm (P_1^N - P_1)\bar{w}(x_i) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) = C_1N^{-1}\{b(x_i)[s_2(x_i) + \frac{1}{\sqrt{\varepsilon}}(\tau - x_i)] + c(x_i)[s_1(x_i) + \frac{1}{\sqrt{\varepsilon}}(\tau - x_i)]\} \pm (P_2^N - P_2)\bar{w}(x_i)$$

$$\geq C_1N^{-1}\{\frac{\beta}{8} + \gamma(1 + \tau) + \frac{\beta_0}{\sqrt{\varepsilon}} + \frac{\tau\gamma}{\sqrt{\varepsilon}}\} \mp CN^{-1}\varepsilon^{-\frac{1}{2}} \geq 0.$$

Consider a mesh function $\bar{\varphi}^{\pm}(x_i) = (\varphi_1^{\pm}(x_i), \varphi_2^{\pm}(x_i))$, where

$$\varphi_k^{\pm}(x_i) = C_1N^{-1}\{s_k(x_i) + \frac{1}{\sqrt{\varepsilon}}(x_i - (1 - \tau))\} \pm (w_k(x_i) - W_k(x_i)),\ x_i \in [1 - \tau, 1] \cap \bar{\Omega}^N,\ k = 1, 2.$$

From (4.1), it is easy to verify that $\varphi_k^{\pm}(x_{\frac{3N}{8}}) \geq 0,\ k = 1, 2$ and $\varphi_2^{\pm}(x_{\frac{N}{2}}) \geq 0$, for a suitable choice of $C_1 > 0$.

$$P_1^N\bar{\varphi}^{\pm}(x_i) = C_1N^{-1}\{\frac{3}{8}\} \pm (P_1^N - P_1)\bar{w}(x_i) \geq 0,$$

$$P_2^N\bar{\varphi}^{\pm}(x_i) = C_1N^{-1}\{b(x_i)[s_2(x_i) + \frac{1}{\sqrt{\varepsilon}}(x_i - (1 - \tau))] + c(x_i)[s_1(x_i) + \frac{1}{\sqrt{\varepsilon}}(x_i - (1 - \tau))]\} \pm (P_2^N - P_2)\bar{w}(x_i)$$

$$\geq C_1N^{-1}\{\beta(\frac{1}{8} + \frac{1 - \tau}{2}) + 2\gamma + \frac{\beta_0}{\sqrt{\varepsilon}} + \frac{\tau\gamma}{\sqrt{\varepsilon}}\} \mp CN^{-1}\varepsilon^{-\frac{1}{2}} \geq 0.$$

Similarly, one can prove that $P_1^N\bar{\varphi}^{\pm}(x_i) \geq 0$ and $P_2^N\bar{\varphi}^{\pm}(x_i) \geq 0$, when $x_i \in [1, 1 + \tau]$ and $x_i \in [2 - \tau, 2]$. Then by the Lemma 3.1, we have $\varphi_1^{\pm}(x_i) \geq 0$ and $\varphi_2^{\pm}(x_i) \geq 0$, $x_i \in \bar{\Omega}^N$. Therefore,

$$|\ w_k(x_i) - W_k(x_i)\ | \leq CN^{-1}\ln^2 N,\ i \in I_N,\ k = 1, 2.$$

Hence, the proof □

Theorem 4.3 *Let $\bar{U}(x_i)$ be the numerical solution of (2.1) defined by (3.1)-(3.2). Then*

$$|\ u_k(x_i) - U_k(x_i)\ | \leq CN^{-1}\ln^2 N,\ i \in I_N,\ k = 1, 2.$$

Proof Using the above Theorem 4.1 and Theorem 4.2, one can prove the desired result. □

5. Nonlinear problem

Consider the nonlinear BVP

$$-\varepsilon u'''(x) = F(x, u(x), u'(x), \tilde{u}'(x)),\ x \in \Omega, \tag{5.1}$$

$$u(x) = \varphi(x),\ u'(x) = \varphi'(x),\ x \in [-1, 0],\ u'(2) = \ell, \tag{5.2}$$

where $\tilde{u}'(x) = u'(x - 1)$, with

$$F_{u'}(x, u, u', \tilde{u}') \leq -\beta \leq 0, \; F_u(x, u, u', \tilde{u}') \geq -\gamma \geq 0, \; F_{\tilde{u}'}(x, u, u', \tilde{u}') \geq -\eta \geq 0.$$

Assume that the reduced problem

$$F(x, u_0(x), u_0'(x), \tilde{u}_0'(x)) = 0,$$
$$u_0(x) = \varphi(x), \; x \in [-1, 0]$$

has a solution. Using the Newton's linearisation technique defined in [5], the sequence of iterates $\{\bar{u}^{[k+1]}(x)\}$ is obtained. Let $\bar{u}^{[k+1]}(x) = (u_1^{[k+1]}, u_2^{[k+1]})$ be the solution of the linear problem for each fixed nonnegative integer k:

$$P_1^{[k]}\bar{u}^{[k+1]} = u_1'^{[k+1]}(x) - u_2^{[k+1]}(x) = 0, \; x \in (0, 2], \tag{5.3}$$

$$P_2^{[k]}\bar{u}^{[k+1]} = -\varepsilon u_2''^{[k+1]}(x) + b^k(x)u_2^{[k+1]}(x) + c^k(x)u_1^{[k+1]}(x) + d^k(x)\tilde{u}_2^{[k+1]}(x) = G^k(x), \; x \in \Omega, \tag{5.4}$$

where

$$b^k(x) = -F_{u_2}(x, u_1^{[k]}, u_2^{[k]}, \tilde{u}_2^{[k]}), \; c^k(x) = -F_{u_1}(x, u_1^{[k]}, u_2^{[k]}, \tilde{u}_2^{[k]}), \; d^k(x) = -F_{\tilde{u}_2}(x, u_1^{[k]}, u_2^{[k]}, \tilde{u}_2^{[k]}),$$
$$G^k(x) = F(x, u_1^{[k]}, u_2^{[k]}, \tilde{u}_2^{[k]}) + b^k(x)u_2^{[k]} + c^k(x)u_1^{[k]} + d^k(x)\tilde{u}_2^{[k]}.$$

For simplicity, the following are denoted $F(x, u_1(x), u_2(x), \tilde{u}_2(x))$, $F(x, u_1^{[k]}(x), u_2^{[k]}(x), \tilde{u}_2^{[k]}(x))$, $F_{u_1}(x, u_1^{[k]}(x), u_2^{[k]}(x), \tilde{u}_2^{[k]}(x))$, $F_{u_2}(x, u_1^{[k]}(x), u_2^{[k]}(x), \tilde{u}_2^{[k]}(x))$, and $F_{\tilde{u}_2}^k(x, u_1^{[k]}(x), u_2^{[k]}(x), \tilde{u}_2^{[k]}(x))$ respectively by F, F^k, $F_{u_1}^k$, $F_{u_2}^k$ and $F_{\tilde{u}_2}^k$. To prove the convergence of the successive iteration, the following theorem is established.

Theorem 5.1 *Suppose* $|F_{u_1u_1}|$, $|F_{u_2u_2}|$, $|F_{u_1u_2}|$, $|F_{u_2\tilde{u}_2}|$, $|F_{\tilde{u}_2u_1}|$ *and* $|F_{\tilde{u}_2\tilde{u}_2}|$ *are bounded above by* $M < 1$. *Let* $\{\bar{u}^{[k]}\}_0^\infty$ *be the Newton sequence defined by (5.3)-(5.4). Then, for all* $x \in \bar{\Omega}$, *we have*

$$\| \bar{u}^{[k+1]} - \bar{u} \| \leq M \| \bar{u}^{[k]} - \bar{u} \|^2$$

Proof It is proved that

$$P_1^k(\bar{u}^{[k+1]} - \bar{u}) = 0,$$

$$\begin{aligned}
P_2^k(\bar{u}^{[k+1]} - \bar{u}) &= F^k - u_1^{[k]}F_{u_1}^k - u_2^{[k]}F_{u_2}^k - \tilde{u}_2^{[k]}F_{\tilde{u}_2}^k - (F - u_1F_{u_1}^k - u_2F_{u_2}^k - \tilde{u}_2F_{\tilde{u}_2}^k)\\
&= F^k - F + (u_1 - u_1^{[k]})F_{u_1}^k + (u_2 - u_2^{[k]})F_{u_2}^k + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k\\
&= F^k - \Big\{(F^k + (u_1 - u_1^{[k]})F_{u_1}^k + (u_2 - u_2^{[k]})F_{u_2}^k + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k) + \frac{1}{2}\Big[((u_1 - u_1^{[k]})^2F_{u_1u_1}(\bar{\theta})\\
&+ (u_2 - u_2^{[k]})^2F_{u_2u_2}(\bar{\theta}) + (\tilde{u}_2 - \tilde{u}_2^{[k]})^2F_{\tilde{u}_2\tilde{u}_2}(\bar{\theta})) + 2(u_1 - u_1^{[k]})(u_2 - u_2^{[k]})F_{u_1u_2}(\bar{\theta})\\
&+ 2(u_2 - u_2^{[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]})F_{u_2\tilde{u}_2}(\bar{\theta}) + 2(\tilde{u}_2 - \tilde{u}_2^{[k]})(u_1 - u_1^{[k]})F_{\tilde{u}_2u_1}(\bar{\theta})\Big]\Big\} + (u_1 - u_1^{[k]})F_{u_1}^k\\
&+ (u_2 - u_2^{[k]})F_{u_2}^k + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k,
\end{aligned}$$

where $\bar{\theta} = (x, \theta, \theta', \tilde{\theta}')$ is such that $(x, u_1, u_2, \tilde{u}_2) > \bar{\theta} > (x, u_1^{[k]}, u_2^{[k]}, \tilde{u}_2^{[k]})$.

$$\begin{aligned}P_2^k(\bar{u}^{[k+1]} - \bar{u}) = -\frac{1}{2}\Big\{&(u_1 - u_1^{[k]})^2 F_{u_1u_1}(\bar{\theta}) + (u_2 - u_2^{[k]})^2 F_{u_2u_2}(\bar{\theta}) + (\tilde{u}_2 - \tilde{u}_2^{[k]})^2 F_{\tilde{u}_2\tilde{u}_2}(\bar{\theta})\\ &+ 2(u_1 - u_1^{[k]})(u_2 - u_2^{[k]}) F_{u_1u_2}(\bar{\theta}) + 2(u_2 - u_2^{[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]}) F_{u_2\tilde{u}_2}(\bar{\theta})\\ &+ 2(\tilde{u}_2 - \tilde{u}_2^{[k]})(u_1 - u_1^{[k]}) F_{\tilde{u}_2u_1}(\bar{\theta})\Big\}.\end{aligned}$$

Then,

$$\begin{aligned}\mid P_2^k(\bar{u}^{[k+1]} - \bar{u}) \mid \le M\Big\{ &\mid u_1^{[k]} - u_1 \mid^2 + \mid u_2^{[k]} - u_2 \mid^2 + \mid \tilde{u}_2^{[k]} - \tilde{u}_2 \mid^2 + \mid u_1^{[k]} - u_1 \mid\mid u_2^{[k]} - u_2 \mid\\ &+ \mid u_2^{[k]} - u_2 \mid\mid \tilde{u}_2^{[k]} - \tilde{u}_2 \mid + \mid \tilde{u}_2^{[k]} - \tilde{u}_2 \mid\mid u_1^{[k]} - u_1 \mid \Big\}\\ &\le M \parallel \bar{u}^{[k]} - \bar{u} \parallel^2 .\end{aligned}$$

Then by the Corollary 2.2, this completes the proof. □

The numerical method discussed in this article can be applied to the sequence of iterates of the nonlinear problem (5.1)-(5.2).

6. Numerical examples

To illustrate the efficiency of the method discussed in this article, three examples are given in this section. For the purpose of calculating the maximum point-wise error, we use the idea of two mesh principle (when exact solution is not known) and evaluate the convergence experiment rate in our computed solution. For this, we put

$$D_\varepsilon^M = \max_{0 \le i \le M} \mid U_i^M - U_{2i}^{2M} \mid,$$

where U_i^M and U_{2i}^{2M} are the i^{th} components of the numerical solutions on meshes of M and $2M$ points, respectively. We compute the uniform error and rate of convergence as

$$D^M = \max_\varepsilon D_\varepsilon^M \text{ and } \quad p^M = \log_2\Big(\frac{D^M}{D^{2M}}\Big).$$

The numerical results for the values of the perturbation parameter $\varepsilon \in \{2^{-4}, 2^{-5}, \ldots, 2^{-23}\}$ are described in the following examples.

Example 6.1

$$\begin{cases} -\varepsilon u'''(x) + 5u'(x) - \frac{5\sin x}{2} u(x) - x^2 u'(x-1) = 1 + \cos x, \; x \in \Omega,\\ u(x) = 2x + 1, \; x \in [-1, 0], \; u'(2) = 2.\end{cases}$$

Table 1 presents the values of D_k^M and p_k^M, $k = 1, 2$ corresponding to the solution components u_1 and u_2. Figure 1 represents the numerical solution, Figure 2 represents the maximum error plot for solution components u_1 and u_2.

Table 1. Maximum uniform error and rate of convergence of Example 6.1.

N (Number of grid points)							
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
D_1^M	7.2048 e - 2	3.5071 e - 2	1.7303 e - 2	8.5939 e - 3	4.2824 e - 3	2.1374 e - 3	1.0677 e - 3
p_1^M	1.0387	1.0192	1.0097	1.0049	1.0025	1.0013	-
D_2^M	3.9860 e - 2	1.9327 e - 2	9.5198 e - 3	4.7243 e - 3	2.3532 e - 3	1.1743 e - 3	6.5349 e - 4
p_2^M	1.0443	1.0216	1.0108	1.0055	1.0028	8.4551e - 1	-

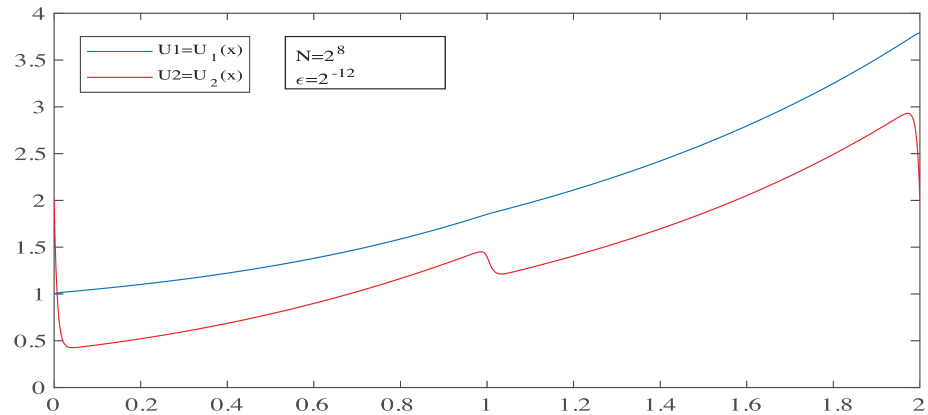

Figure 1. Numerical solution of Example 6.1.

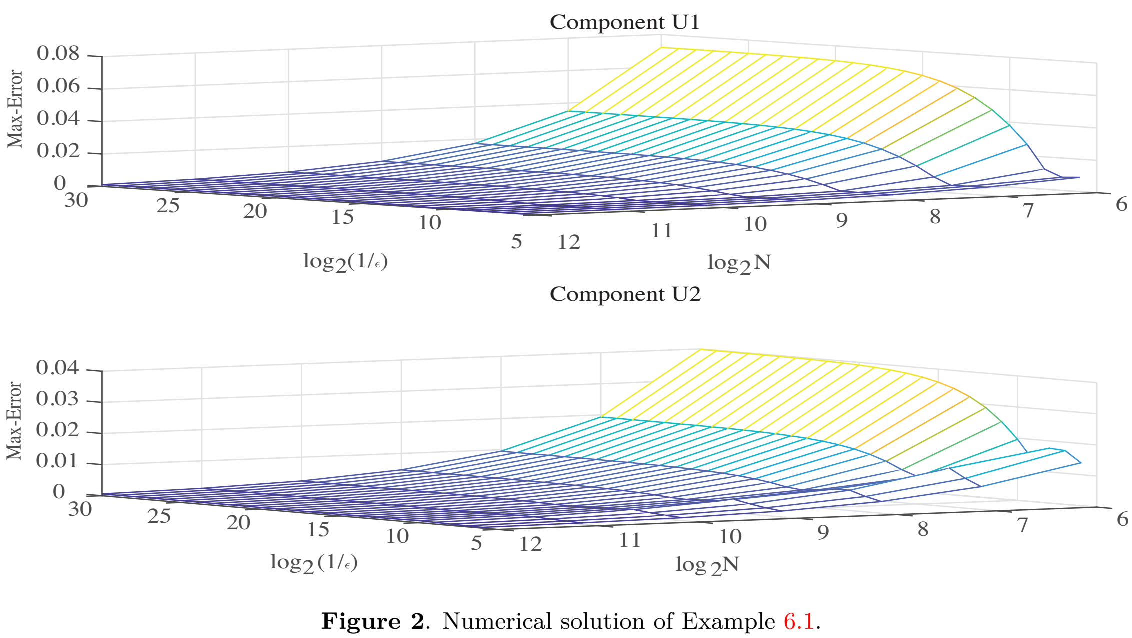

Figure 2. Numerical solution of Example 6.1.

Example 6.2

$$\begin{cases} -\varepsilon u'''(x) + (5+x)u'(x) - \frac{1}{2}u(x) - u'(x-1) = \begin{cases} 5, & x \in \Omega^-, \\ 0, & x \in \Omega^+. \end{cases} \\ u(x) = 1, \ x \in [-1,0], \ u'(2) = 0. \end{cases}$$

Table 2 presents the values of D_k^M *and* p_k^M, $k = 1, 2$ *corresponding to the solution components* u_1 *and* u_2. *Figure 3 represents the numerical solution, and Figure 4 represent the maximum error plot for solution components* u_1 *and* u_2.

Table 2. Maximum uniform error and rate of convergence of Example 6.2.

N (Number of grid points)							
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
D_1^M	1.9058 e - 2	9.3345 e - 3	4.5305 e - 3	2.2282 e - 3	1.1093 e - 3	5.5135 e - 4	2.7484 e - 4
p_1^M	1.0298	1.0429	1.0238	1.0062	1.0087	1.0044	-
D_2^M	1.6476 e - 2	8.3607 e - 3	3.6815 e - 3	2.3756 e - 3	1.2264 e - 3	6.6515 e - 4	3.6912 e - 4
p_2^M	9.7866 e - 1	1.1833	6.3198 e - 1	9.5393 e - 1	8.8263 e - 1	8.4960 e - 1	-

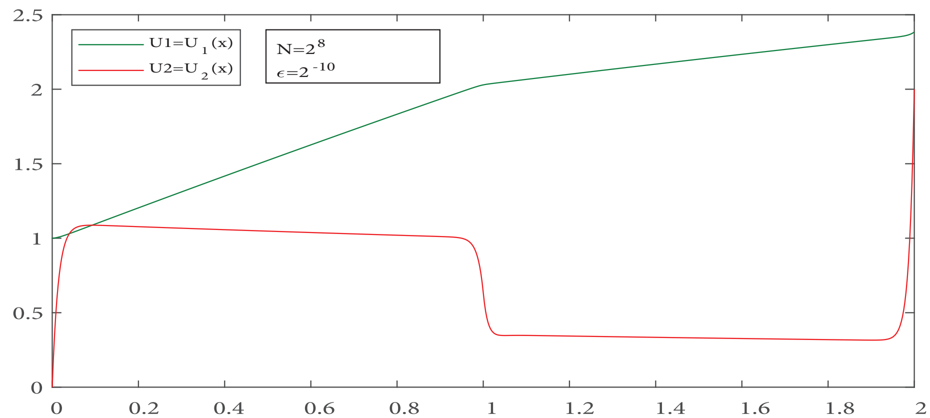

Figure 3. Numerical solution of Example 6.2.

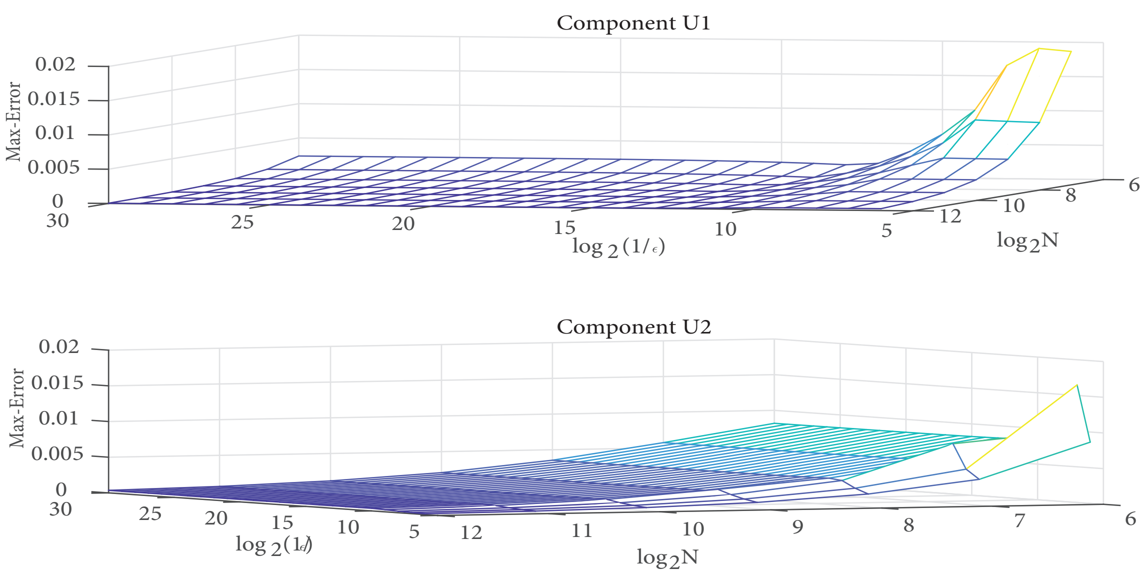

Figure 4. Numerical solution of Example 6.2.

Example 6.3 *Consider the nonlinear BVP*

$$\begin{cases} -\varepsilon u'''(x) + 2u'(x) = [u'(x-1)]^2, \ x \in \Omega, \\ u(x) = 1 + x, \ x \in [-1, 0], \ u'(x) = 1, \ x \in [-1, 0], \ u'(2) = 1. \end{cases}$$

Tables 3 and 4 present the iterative numerical solutions for u_1 and u_2, and Table 5 presents the values of D_k^M and p_k^M, $k = 1, 2$ corresponding to the solution components u_1, u_2. Figures 5 and 6 represent the numerical solution of iteratives $\bar{u}_1^{[k]}$ and $\bar{u}_2^{[k]}$ for fixed $\varepsilon = 2^{-9}$ and $N = 1024$.

Table 3. The iterative values of u_1 of the Example 6.3.

x_i	$u_1^{[0]}$	$u_1^{[1]}$	$u_1^{[2]}$	$u_1^{[3]}$	$u_1^{[4]}$	$u_1^{[5]}$	$u_1^{[6]}$
0.1250	1.0625	1.0792	1.0792	1.0792	1.0792	1.0792	1.0792
0.2500	1.1250	1.1462	1.1462	1.1462	1.1462	1.1462	1.1462
0.3750	1.1875	1.2099	1.2099	1.2099	1.2099	1.2099	1.2099
0.5000	1.2500	1.2726	1.2726	1.2726	1.2726	1.2726	1.2726
0.6250	1.3125	1.3349	1.3348	1.3348	1.3348	1.3348	1.3348
0.7500	1.3750	1.3962	1.3959	1.3959	1.3959	1.3959	1.3959
0.8750	1.4375	1.4542	1.4532	1.4532	1.4532	1.4532	1.4532
1.0000	1.5000	1.4999	1.4960	1.4960	1.4960	1.4960	1.4960
1.1250	1.5156	1.5332	1.5243	1.5243	1.5243	1.5243	1.5243
1.2500	1.5313	1.5562	1.5447	1.5447	1.5447	1.5447	1.5447
1.3750	1.5469	1.5746	1.5621	1.5621	1.5621	1.5621	1.5621
1.5000	1.5625	1.5915	1.5786	1.5787	1.5787	1.5787	1.5787
1.6250	1.5781	1.6089	1.5962	1.5962	1.5962	1.5962	1.5962
1.7500	1.5938	1.6311	1.6188	1.6190	1.6190	1.6190	1.6190
1.8750	1.6094	1.6735	1.6620	1.6625	1.6625	1.6625	1.6625
2.0000	1.6250	1.7985	1.7870	1.7875	1.7875	1.7875	1.7875

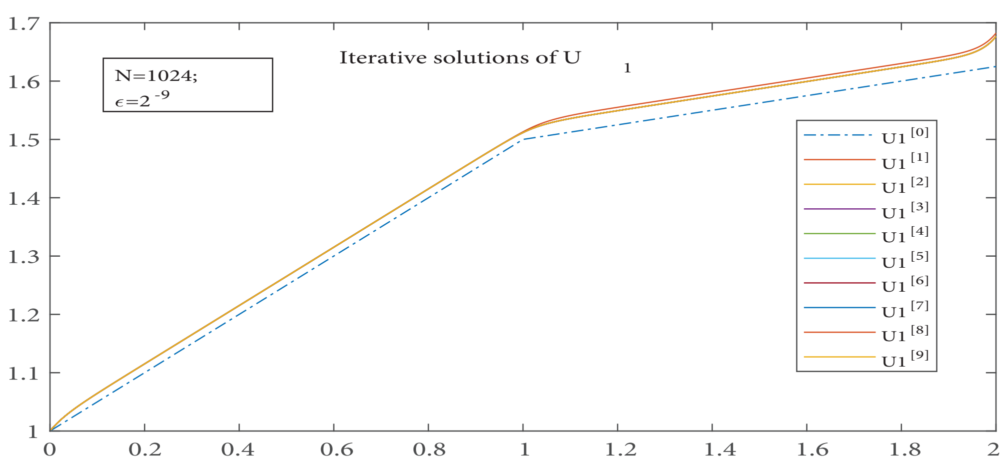

Figure 5. Iterative numerical solutions of u_1 stated in Example 6.3.

Table 4. The iterative values of u_2 of the Example 6.3.

x_i	$u_2^{[0]}$	$u_2^{[1]}$	$u_2^{[2]}$	$u_2^{[3]}$	$u_2^{[4]}$	$u_2^{[5]}$	$u_2^{[6]}$
0.1250	0.5000	0.6340	0.6340	0.6340	0.6340	0.6340	0.6340
0.2500	0.5000	0.5358	0.5358	0.5358	0.5358	0.5358	0.5358
0.3750	0.5000	0.5094	0.5094	0.5094	0.5094	0.5094	0.5094
0.5000	0.5000	0.5019	0.5018	0.5018	0.5018	0.5018	0.5018
0.6250	0.5000	0.4981	0.4977	0.4977	0.4977	0.4977	0.4977
0.7500	0.5000	0.4905	0.4889	0.4889	0.4889	0.4889	0.4889
0.8750	0.5000	0.4640	0.4578	0.4578	0.4578	0.4578	0.4578
1.0000	0.5000	0.3654	0.3423	0.3423	0.3423	0.3423	0.3423
1.1250	0.1250	0.2668	0.2269	0.2269	0.2269	0.2269	0.2269
1.2500	0.1250	0.1839	0.1632	0.1632	0.1632	0.1632	0.1632
1.3750	0.1250	0.1470	0.1388	0.1389	0.1389	0.1389	0.1389
1.5000	0.1250	0.1351	0.1327	0.1328	0.1328	0.1328	0.1328
1.6250	0.1250	0.1397	0.1402	0.1406	0.1406	0.1406	0.1406
1.7500	0.1250	0.1774	0.1809	0.1819	0.1819	0.1819	0.1819
1.8750	0.1250	0.3388	0.3459	0.3479	0.3479	0.3479	0.3479
2.0000	0.1250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5. Maximum uniform error and rate of convergence of Example 6.3.

N (Number of grid points)							
	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
D_1^M	2.6773 e - 2	1.4862 e - 2	7.8663 e - 3	4.0725 e - 3	2.0707 e - 3	1.0454 e - 3	5.2571 e - 4
p_1^M	8.4910 e - 1	9.1789 e - 1	9.4977 e - 1	9.7583 e - 1	9.8601 e - 1	9.9175 e - 1	-
D_2^M	2.1838 e - 2	2.0830 e - 2	1.8547 e - 2	1.1587 e - 2	8.2617 e - 3	4.4837 e - 3	2.5014 e - 3
p_2^M	6.8157 e - 2	1.6746 e - 1	6.7869 e - 1	4.8803 e - 1	8.8174 e - 1	8.4195 e - 1	-

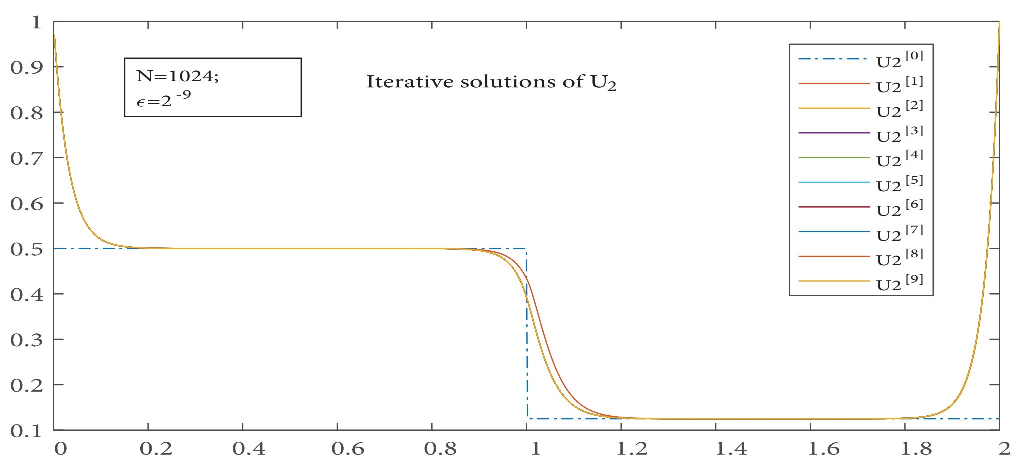

Figure 6. Iterative numerical solutions of u_1 stated in Example 6.3.

7. Conclusion

In this paper, we presented the first order convergent computational method to solve reaction-diffusion type third order SPDDEs. We observed that the second component u_2 of the problem (2.2)-(2.3) exhibit strong interior twin layers at $x = 1$ and strong boundary layers at $x = 0$ and $x = 2$. Therefore, we divided the domain into six subdomains $[0, \tau]$, $[\tau, 1 - \tau]$, $[1 - \tau, 1]$, $[1, 1 + \tau]$, $[1 + \tau, 2 - \tau]$, $[2 - \tau, 2]$. Each subdomain has been discretized by some set of mesh points with equal mesh size, whereas the mesh sizes between subdomains are different. The present method is of almost first order convergence (see Tables 1 and 2). The maximum pointwise error for the problems considered in the Examples 6.1 and 6.2 are given in Tables 1 and 2. From Figures 1 and 3, one can see that the U_2 component exhibits strong interior twin layers at $x = 1$ and strong boundary layers at $x = 0$ and $x = 2$. The maximum pointwise error plots of the Examples 6.1 and 6.2 have been plotted in Figures 2 and 4. Furthermore, the nonlinear problems are linearized by using Newton's method of quasilinearization technique, this technique gives a sequence of successive approximations $u_1^{[m]}$ and $u_2^{[m]}$ with proper choice of initial guess $u_{0,1}^{[0]}$ and $u_{0,2}^{[0]}$. Next, the linearized problems are solved using the fitted numerical methods presented in this article. Tables 3–5 present the 6 iterations of the numerical solutions and maximum pointwise error of Example 6.3, respectively.

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