

1-1-1996

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### Recommended Citation

RUAN, Yongbin and TIAN, Gang (1996) "Higher Genus Symplectic Invariants and Sigma Model Coupled With Gravity," *Turkish Journal of Mathematics*: Vol. 20: No. 1, Article 6. Available at: <https://journals.tubitak.gov.tr/math/vol20/iss1/6>

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# Higher Genus Symplectic Invariants and Sigma Model Coupled With Gravity

*Yongbin Ruan and Gang Tian*

## 1. Introduction

In this short article, we give an outline of main results. The full version will appear elsewhere. This paper is a continuation of our previous paper [RT]. In [RT], among other things, we build up a mathematical foundation of quantum cohomology ring on semi-positive symplectic manifolds. We also defined higher genus symplectic invariants without gravity (topological sigma model) in terms of inhomogeneous holomorphic maps from a fixed Riemann surface, and proved the composition law they satisfy. Topological gravity, proposed by Witten, concerns the intersection theory of the moduli space of marked Riemann surfaces. Based on the physical intuition, Witten suggested a relation between those intersection numbers and the KdV hierarchy. This relation was clarified by Kontsevich (cf. [Ko]). However, both mathematical and physical phenomenon will become much more interesting if the topological sigma model is coupled with the topological gravity. In fact, in [W2] Witten proposed an approach to the topological sigma model coupled with gravity, and made a very important conjecture on the basic feature of this new model. The purpose of this paper is to establish a mathematical foundation for the theory of topological sigma model coupled with topological gravity over any semi-positive symplectic manifolds. This new theory also provides many more new geometric examples of the topological field theory coupled with gravity. For each semi-positive symplectic manifold  $V$ , we can associate a topological sigma model with gravity, or simply a topological field theory coupled with gravity.

## 2. Gromov-Witten invariants

This theory begins with the GW-invariants

$$\Psi_{(A,g,k)}^V : H_*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q}) \times H_*(V, \mathbf{Z})^k \mapsto \mathbf{Q},$$

for any  $A \in H_2(V, \mathbf{Z})$  and  $2g + k \geq 3$ . Here  $\overline{\mathcal{M}}_{g,k}$  is the Deligne-Mumford compactification of the moduli space of genus  $g$  Riemann surfaces with  $k$  marked points. The GW-invariants are multilinear and supersymmetric on  $H_*(V, \mathbf{Z})^k$ .

At first, we will rigorously define the GW-invariant  $\Psi^V$  on semi-positive symplectic manifolds. From the analytic point of view, it is the most convenient to use the inhomogeneous holomorphic maps from Riemann surfaces in  $\overline{\mathcal{M}}_{g,k}$ , though other equivalent formulations may be possible, such as using stable maps and establishing a more sophisticated intersection theory.

First of all, let's introduce the inhomogeneous Cauchy-Riemann equation, which plays a central role in [RT]. A minor difference, compared to that of [RT], is that we will vary the complex structures on the Riemann surfaces. We would like to define the inhomogeneous term varying continuously as we vary the complex structures of the Riemann surfaces. This can be done as follows: Let  $(V, \omega)$  be a symplectic manifold and  $J$  be a tamed almost complex structure. Let  $\mathcal{M}_{g,k}$  be the moduli space of genus  $g$  Riemann surfaces with  $k$ -marked points and  $\overline{\mathcal{M}}_{g,k}$  be the Deligne-Mumford compactification. Suppose that

$$\overline{\mathcal{U}}_{g,k} \rightarrow \overline{\mathcal{M}}_{g,k}$$

is the universal curve. We use  $\mathcal{U}_{g,k}$  to denote the preimage of  $\mathcal{M}_{g,k}$ .  $\overline{\mathcal{U}}_{g,k}$  is a projective algebraic variety. Therefore, we have an embedding  $\phi : \overline{\mathcal{U}}_{g,k} \rightarrow \mathbf{P}^N$ . There are two relative tangent bundles over  $\mathbf{P}^N \times V$  with respect to  $\pi_i$  ( $i = 1, 2$ ), where  $\pi_i$  is the projection from  $\mathbf{P}^N \times V$  to its  $i$ -th factor. A section  $\nu$  of  $Hom(\pi_1^*T\mathbf{P}^N, \pi_2^*TV)$  is said to be anti- $J$ -linear if for any tangent vector  $v$  in  $T\mathbf{P}^N$ ,

$$(2.1) \quad \nu(j_{\mathbf{P}^N}(v)) = -J(\nu(v))$$

where  $j_{\mathbf{P}^N}$  is the complex structure on  $\mathbf{P}^N$ . Usually, we call such a  $\nu$  an inhomogeneous term.

**Definition 2.1.** *Let  $\nu$  be an inhomogeneous term. A  $(J, \nu)$ -perturbed holomorphic map, or simply a  $(J, \nu)$ -map, is a smooth map  $f : \Sigma \rightarrow V$  satisfying the inhomogeneous Cauchy-Riemann equation*

$$(2.2) \quad (\bar{\partial}_J f)(x) = \nu(\phi(x), f(x)),$$

where  $\bar{\partial}_J$  denotes the differential operator  $d + J \cdot d \cdot j_\Sigma$ .

We denote by  $\mathcal{M}_A(g, k, J, \nu)$  the moduli space of  $(J, \nu)$ -perturbed holomorphic maps from  $(\Sigma, x_1, \dots, x_k)$  into  $V$ , such that  $f_*[\Sigma] = A$  and  $(\Sigma, \{x_i\}) \in \mathcal{M}_{g,k}$  has the trivial automorphism group. There are some important topological properties as follows.

Let  $\pi : \mathcal{U}_A(g, k, J, \nu) \rightarrow \mathcal{M}_A(g, k, J, \nu)$  be the universal family of curves, i.e.,  $\pi^{-1}(f, \Sigma, \{x_i\}) = \Sigma$ . We can define the evaluation map

$$e_A(g, k) : \mathcal{U}_A(g, k, J, \nu) \rightarrow V$$

by

$$(2.3) \quad e_A(g, k)(f, \Sigma, \{x_i\}, y) = f(y).$$

Each marked point  $x_i$  defines a section

$$\sigma_i : \mathcal{M}_A(g, k, J, \nu) \rightarrow \mathcal{U}_A(g, k, J, \nu)$$

by

$$(2.4) \quad \sigma_i((f, \Sigma, \{x_i\})) = x_i.$$

The composition

$$e_i = e_A(g, k) \circ \sigma_i : \mathcal{M}_A(g, k, J, \nu) \rightarrow V.$$

Let

$$\Xi_{g,k}^A = \prod_{i=1}^k e_A(g, k) \circ \sigma_i : \mathcal{M}_A(g, k, I, J, \nu) \rightarrow V^k.$$

Evidently, we have a map  $\Upsilon_A : \mathcal{M}_A(g, k, J, \nu) \rightarrow \mathcal{M}_{g,k}$  by assigning each  $(J, \nu)$ -map to its domain. Together, we get a smooth map

$$\Upsilon_A \times \Xi_{g,k}^A : \mathcal{M}_A(g, k, J, \nu) \rightarrow \mathcal{M}_{g,k} \times V^k.$$

In general,  $\mathcal{M}_A(g, k, J, \nu)$  is not compact. However, there is a natural compactification  $\overline{\mathcal{M}}_A(g, k, J, \nu)$ , which we call GU-compactification.

**Proposition 2.2.** *Suppose that  $(V, \omega)$  is a semi-positive symplectic manifold. Then, there is a Baire set of second category- $\mathcal{H}$  among all the smooth pairs  $(J, \nu)$  such that for any  $(J, \nu) \in \mathcal{H}$*

(1)  $\mathcal{M}_A(g, k, J, \nu)$  is a smooth, oriented manifold of real dimension

$$2c_1(V)(A) + 2(3-n)(g-1) + 2k;$$

(2)  $\Upsilon_A$  and  $\Xi_{g,k}^A$  extends to continuous maps, still denoted by the same symbols, from  $\overline{\mathcal{M}}_A(g, k, J, \nu)$  to  $\overline{\mathcal{M}}_{g,k}$  and  $V^k$ , respectively;

(3) The boundary  $\Upsilon_A \times \Xi_{g,k}^A (\overline{\mathcal{M}}_A(g, k, J, \nu) \setminus \mathcal{M}_A(g, k, J, \nu))$  is of real codimension at least two.

Putting aside technical details for the time being, we can intuitively define the GW-invariants as follows: let  $V$  be any symplectic manifold and  $A \in H_2(V, \mathbf{Z})$ . For any homology classes  $[K] \in H_*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q})$  and  $\alpha_i \in H_*(V, \mathbf{Z})$ , represented by cycles  $K, A_i$ , respectively, we define  $\Psi_{(A,g,k)}^V([K]; \alpha_1, \dots, \alpha_k)$  to be the number of tuples  $(\Sigma; x_1, \dots, x_k; f)$  with appropriate sign, satisfying:  $\Sigma \in K$ ,  $f : \Sigma \mapsto V$  solves a given inhomogeneous Cauchy-Riemann equation, and  $f(x_i) \in A_i$ , whenever

$$\sum \text{cod}(A_i) + \text{cod}(K) = 2c_1(V)(A) + 2(3-n)(g-1) + 2k; \quad (1.5)$$

We simply put  $\Psi_{(A,g,k)}^V([K]; \alpha_1, \dots, \alpha_k)$  to be zero if (1.5) is not satisfied.

This approach towards defining new invariants has been used before in many cases (cf. [R], [R3], [RT], [KM], [Wi]). For complex surfaces, using unperturbed holomorphic maps, the first author already defined the invariant  $\Psi$  in the very important case that  $k = 0$  and  $[K] = \overline{\mathcal{M}}_{g,0}$ . However, in each case, there are specific difficulties to be overcome. Using the techniques we developed in [RT], we will first prove

**Theorem A.** *If  $V$  is a semi-positive symplectic manifold, the GW-invariant  $\Psi_{(A,g,k)}^V$  can be well defined for any  $g, k \geq 0$  with  $2g+k \geq 3$ . Moreover,  $\Psi^V$  depends only the symplectic structure of  $V$  and is independent of any semi-positive symplectic deformation.*

One new consequence of our definition, which was not obvious at all to physicists based on mathematically unjustified path integrals, is the necessity of symplectic structures. The path integral starts from a Lagrangian. The Lagrangian for sigma model or sigma model coupled with gravity is valid for any almost complex manifolds (symplectic or not). Naturally, one would think that its correlation functions will be the invariants of homotopy class of almost complex structures. This is in fact false. Our invariants are symplectic invariants rather than the invariants of almost complex structures. In particular, they can distinguish different symplectic manifolds with the same homotopy class of almost complex structures (see section 5 or [R], [R1]).

### 3. Composition Law

One of the fundamental properties of a topological field theory is the axiom on the decomposition of correlation functions. For a topological field theory coupled with gravity, there is a composition law as well. In our case, the GW-invariants serve as the correlation functions. Therefore, in order to make them more useful, or at least to construct a correct model for the topological field theory, we need to verify that our invariants satisfy the composition law.

The composition law governs how the GW-invariants change during the degeneration of stable curves. Its classical cousin in enumerative algebraic geometry is the degeneration formula, which was only derived in very special cases. It never became a general theory as neat as the composition law describes. One reason might be that the classical counting of holomorphic curves, particularly of higher genus, does not obey the composition laws predicted by physicists, even for the projective plane  $\mathbf{P}^2$ . Namely, the way of counting was not good. In [RT], we found the correct counting in terms of inhomogeneous holomorphic maps and established the composition law at least for the mixed invariants, corresponding to the  $\sigma$ -models without gravity. Based on the same techniques developed in [RT], we are also able to prove the composition law for all GW-invariants.

Assume  $g = g_1 + g_2$  and  $k = k_1 + k_2$  with  $g_i + k_i \geq 2$ . Fix a decomposition  $S = S_1 \cup S_2$  of  $\{1, \dots, k\}$  with  $|S_i| = k_i$ . Then there is a canonical embedding  $\theta_S : \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \mapsto \overline{\mathcal{M}}_{g, k}$ , which assigns to marked curves  $(\Sigma_i; x_1^i, \dots, x_{k_1+1}^i)$  ( $i = 1, 2$ ), their union  $\Sigma_1 \cup \Sigma_2$  with  $x_{k_1+1}^1$  identified to  $x_{k_2+1}^2$  and the remaining points renumbered by  $\{1, \dots, k\}$  according to  $S$ . There is another natural map  $\mu : \overline{\mathcal{M}}_{g-1, k+2} \mapsto \overline{\mathcal{M}}_{g, k}$  by gluing together the last two marked points.

Choose a homogeneous basis  $\{\beta_b\}_{1 \leq b \leq L}$  of  $H_*(V, \mathbf{Z})$  modulo torsion. Let  $(\eta_{ab})$  be its intersection matrix. Note that  $\eta_{ab} = \beta_a \cdot \beta_b = 0$  if the dimensions of  $\beta_a$  and  $\beta_b$  are not complementary to each other. Put  $(\eta^{ab})$  to be the inverse of  $(\eta_{ab})$ . Now we can state the composition law, which consists of two formulas.

**Theorem B.** Let  $[K_i] \in H_*(\overline{\mathcal{M}}_{g_i, k_i+1}, \mathbf{Q})$  ( $i = 1, 2$ ) and  $[K_0] \in H_*(\overline{\mathcal{M}}_{g-1, k+2}, \mathbf{Q})$ . For any  $\alpha_1, \dots, \alpha_k$  in  $H_*(V, \mathbf{Z})$ . Then we have

$$(3.1) \quad \begin{aligned} & \Psi_{(A, g, k)}^V(\theta_{S^*}[K_1 \times K_2]; \{\alpha_i\}) \\ = & \sum_{A=A_1+A_2} \sum_{a, b} \Psi_{(A_1, g_1, k_1+1)}^V([K_1]; \{\alpha_i\}_{i \leq k_1}, \beta_a) \eta^{ab} \Psi_{(A_2, g_2, k_2+1)}^V([K_2]; \beta_b, \{\alpha_j\}_{j > k_1}) \end{aligned}$$

$$(3.2) \quad \Psi_{(A, g, k)}^V(\mu_*[K_0]; \alpha_1, \dots, \alpha_k) = \sum_{a, b} \Psi_{(A, g-1, k+2)}^V([K_0]; \alpha_1, \dots, \alpha_k, \beta_a, \beta_b) \eta^{ab}$$

The composition law is closely associated with the structure of  $\overline{\mathcal{M}}_{g, k}$ . Here we also prove two other properties of  $\Psi$  corresponding to the reduction of marked points  $\pi_k : \overline{\mathcal{M}}_{g, k} \rightarrow \overline{\mathcal{M}}_{g, k-1}$ . These formula will be important later for the generalized string equation and the dilaton equation of the generating function.

There is a natural map  $\pi_k : \overline{\mathcal{M}}_{g, k} \rightarrow \overline{\mathcal{M}}_{g, k-1}$  as follows: For  $(\Sigma, x_1, \dots, x_k) \in \overline{\mathcal{M}}_{g, k}$ , if  $x_k$  is not in any rational component of  $\Sigma$  which contains only three special points, then we define

$$\pi_k(\Sigma, x_1, \dots, x_k) = (\Sigma, x_1, \dots, x_{k-1}).$$

Notes that a distinguished point of  $\Sigma$  is either a singular point or a marked point. If  $x_k$  is in one of the rational components, we contract this component and obtain a stable curve  $(\Sigma', x_1, \dots, x_{k-1})$  in  $\overline{\mathcal{M}}_{g, k-1}$ , and define  $\pi_k(\Sigma, x_1, \dots, x_k) = (\Sigma', x_1, \dots, x_{k-1})$ .

Clearly,  $\pi_k$  is continuous. One should be aware that there are two exceptional cases  $(g, k) = (0, 3), (1, 1)$  where  $\pi_k$  is not well defined. Associated with  $\pi_k$ , we have two *k-reduction formulas* for  $\Psi_{(A, g, k)}^V$ .

**Proposition 3.1.** Suppose that  $(g, k) \neq (0, 3), (1, 1)$ .

(1) For any  $\alpha_1, \dots, \alpha_{k-1}$  in  $H_*(V, \mathbf{Z})$ , we have

$$(3.3) \quad \Psi_{(A, g, k)}^V([K]; \alpha_1, \dots, \alpha_{k-1}, [V]) = \Psi_{(A, g, k-1)}^V([\pi_k(K)]; \alpha_1, \dots, \alpha_{k-1})$$

(2) Let  $\alpha_k$  be in  $H_{2n-2}(V, \mathbf{Z})$ , then

$$(3.4) \quad \Psi_{(A, g, k)}^V([\pi_k^{-1}(K)]; \alpha_1, \dots, \alpha_{k-1}, \alpha_k) = \alpha_k^*(A) \Psi_{(A, g, k-1)}^V([K]; \alpha_1, \dots, \alpha_{k-1})$$

where  $\alpha_k^*$  is the Poincare dual of  $\alpha_k$ .

## 4. Generalized Witten conjecture

In order to formulate the generalized Witten conjecture in terms of our invariants, we need to introduce special cycles in  $\overline{\mathcal{M}}_{g, k}$ . Let  $\pi : \overline{\mathcal{U}}_{g, k} \rightarrow \overline{\mathcal{M}}_{g, k}$  be the universal family of stable curves of genus  $g$  and  $k$  marked points. Each marked point gives rise to a section  $\sigma_i$  ( $1 \leq i \leq k$ ) of this fibration. Following Witten, we let  $\mathcal{L}_i$  be the pull-back of the relative cotangent sheaf of  $\pi : \overline{\mathcal{U}}_{g, k} \rightarrow \overline{\mathcal{M}}_{g, k}$  by  $\sigma_i$ . Then we put  $W_{d_1, \dots, d_k}$  to be the Poincare

dual of the cohomology class  $c_1(\mathcal{L}_1)^{d_1} \cup c_1(\mathcal{L}_2)^{d_2} \cdots \cup c_1(\mathcal{L}_k)^{d_k}$ . We call these  $W_{d_1, \dots, d_k}$  Witten cycles.

For convenience, as Witten did, we use

$$\langle \tau_{d_1, \alpha_1}, \tau_{d_2, \alpha_2}, \dots, \tau_{d_k, \alpha_k} \rangle_{g, k}$$

to denote the GW-invariants  $\Psi_{(A, g, k)}([W_{d_1, \dots, d_k}]; \alpha_1, \dots, \alpha_k)$ . Following Witten, we introduce potential functions

$$F_g = \sum_A \sum_{n_{r, \alpha}} \prod_{r, \alpha} \frac{(t_r^\alpha)^{n_{r, \alpha}}}{n_{r, \alpha}!} \langle \prod_{r, \alpha} \tau_{r, \alpha}^{n_{r, \alpha}} \rangle, \quad g = 0, 1, 2, \dots$$

These functions are only well defined as formal series at this moment. It seems to be a hard problem whether or not this series is convergent in certain region of  $t_r^\alpha$ . We will not address this convergence problem in this paper. We further define

$$F^V = \sum_{g \geq 0} F_g.$$

One of fundamental problems on  $F^V$ , even to physicists, is to find the complete set of equations  $F^V$  satisfies. By imitating the arguments of Witten in [Wi2], we will prove

**Theorem C.**  $F^V$  satisfies the generalized string equation

$$(4.1) \quad \frac{\partial F^V}{\partial t_0^1} = \frac{1}{2} \eta_{ab} t_0^a t_0^b + \sum_{i=0}^{\infty} \sum_a t_{i+1}^a \frac{\partial F^V}{\partial t_i^a}.$$

$F_g$  satisfies the dilation equation

$$(4.2) \quad \frac{\partial F_g}{\partial t_1^1} = (2g - 2 + \sum_{i=1}^{\infty} \sum_a t_i^a \frac{\partial}{\partial t_i^a}) F_g + \frac{\chi(V)}{24} \delta_{g,1},$$

where  $\chi(V)$  is the Euler characteristic of  $V$ .

In general, Witten suggested

$$U = \frac{\partial^2 F^V}{\partial t_{0,1} \partial t_{0,\sigma}}, \quad U' = \frac{\partial^3 F^V}{\partial t_{0,1}^2 \partial t_{0,\sigma}}, \quad \dots, \quad U_\sigma^{(l)} = \frac{\partial^{l+2} F^V}{\partial t_{0,1}^{l+1} \partial t_{0,\sigma}}, \quad \text{for } l \geq 0$$

We will regard  $U^{(l)}$  to be of degree  $l$ . By a differential function of degree  $k$  we mean a function  $G(U, U', U'', \dots)$  of degree  $k$  in that sense. In particular, any function of form  $G(U)$  is of degree zero, and  $(U')^2$  has degree two.

**Generalized Witten Conjecture:** For every  $g \geq 0$ , there are differential functions  $G_{m, \alpha; n, \beta}(U_\alpha, U_\alpha', U_\alpha'', \dots)$  of degree  $2g$  such that

$$\frac{\partial^2 F_g}{\partial \tau_{m, \alpha} \partial \tau_{n, \beta}} = G_{m, \alpha; n, \beta}(U_\alpha, U_\alpha', U_\alpha'', \dots)$$

up to terms of genus  $g$ .

This conjecture was affirmed in case  $V = pt$  by Kontsevich [Ko]. When  $g = 0$ , it is a consequence of the associativity equation proved in [RT]. But the general case is still open. In this paper, we will focus on the lower genus cases. As a corollary of our composition law, we will verify the generalized Witten conjecture for genus 1, 2.

**Theorem D:** *The generalized Witten conjecture holds for genus  $\leq 2$ .*

We call  $\Psi_{(A,g,k)}^V([\overline{\mathcal{M}}_{g,k}]; \dots)$  primitive GW-invariants of genus  $g$ . Those invariants correspond to the enumerative invariants of counting genus  $g$  holomorphic curves passing through generic  $k$  cycles in enumerative algebraic geometry.

**Corollary E.** *For genus  $\leq 2$ , the Witten invariants  $\langle \rangle$  can be reduced to primitive GW-invariants.*

In general, we conjecture that all the Witten invariants can be derived from primitive GW-invariants.

## 5. Stabilizing Conjecture

Our invariant can be also applied to studying the topology of symplectic manifolds. As an example, we will verify the *Stabilizing conjecture* of the first author in the case of simply connected elliptic surfaces. The conjecture claims: *Suppose that  $X, Y$  are simply connected homeomorphic symplectic 4-manifolds. Then  $X, Y$  are diffeomorphic if and only if  $X \times S^2, Y \times S^2$  are deformation equivalent as symplectic manifolds.* He also verified this conjecture for certain complex surfaces homeomorphic to a Del-Pezzo surface (cf. [R1]). By calculating our invariants for the product of simply connected elliptic surfaces with  $S^2$ , we will prove that

**Theorem F: (Theorem 5.1)** *The stabilizing conjecture holds for simply connected elliptic surfaces.*

The proof of the theorem follows from the following calculation of GW-invariants.

**Proposition 5.2.**

$$\Psi_{(mA,1,1)}^{E_{p,q}^n \times \mathbf{P}^1}(\overline{\mathcal{M}}_{1,1}; \alpha) = \begin{cases} 2q(A \cdot \alpha); & m = q(mA = A_p), \\ 2p(A \cdot \alpha); & m = p(mA = A_q), \\ 0; & m \neq p, q \text{ and } m < pq, \end{cases}$$

where  $\alpha$  is a 4-dimensional homology class. In particular,

$$\Psi_{(mA,1,1)}^{E_{p,q}^n \times \mathbf{P}^1}(\overline{\mathcal{M}}_{1,1}; \cdot) \neq 0 \text{ for } m = p, q.$$

$E_{p,q}^n$  are homeomorphic and may have different Chern classes. But the blow-up  $E_{p,q}^n \# \overline{\mathbf{P}^2}$  have the same first Chern class up to a homeomorphism. Hence, by the theorem of Wall,  $(E_{p,q}^n \# \overline{\mathbf{P}^2}) \times S^2$  are diffeomorphic, have the same first Chern class up to a diffeomorphism



and hence their complex structures are homotopic equivalent as the almost complex structures. Our invariants are symplectic deformation invariants and remain unchanged under blowing up. This shows that

**Corollary:** The smooth 6-manifold  $V = (E_{p,q}^n \# \overline{\mathbf{P}^2}) \times S^2$  admits infinitely many deformation classes of symplectic structures which have the same almost complex structure up to homotopy

Some of the results in this paper have been lectured by us in last few years. Also, the main results of this paper were partly announced in the paper [T] of the second author published in the proceeding of the first "Current developments in Mathematics", Boston, May, 1995. All the basic techniques were developed in [RT].

The first author wish to thank S. K. Donaldson who suggested the example of Section 5 to him.

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RUAN & TIAN

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