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(σ, τ) - LIE IDEALS IN PRIME RINGS WITH DERIVATION

M. Soytürk

Abstract

Let R be a prime ring, $\text{char } R \neq 2, 3$, $\sigma, \tau : R \rightarrow R$ two automorphisms, U a nonzero (σ, τ) - Lie ideal of R and $d : R \rightarrow R$ a derivation such that $\sigma d = d\sigma, \tau d = d\tau$. In this paper we have proved the following results. (1) If $d(U) \subset Z$ then $U \subset Z$ (2) If $d(U) \subset U$ and $d^2(U) \subset Z$ then $U \subset Z$.

Introduction

Let R be a ring and U an additive subgroup of R , and $\sigma, \tau : R \rightarrow R$ two mappings. We set $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$. The definition of (σ, τ) - Lie ideal was given in [4] as follows: (i) U is called a (σ, τ) -right Lie ideal of R if $[U, R]_{\sigma, \tau} \subset U$. (ii) U is called (σ, τ) - left Lie ideal of R if $[R, U]_{\sigma, \tau} \subset U$. (iii) U is called a (σ, τ) -Lie ideal of R if U is both a (σ, τ) -right Lie ideal and (σ, τ) -left Lie ideal of R . Every Lie ideal of R is a $(1, 1)$ -right(left)Lie ideal of R , where $1 : R \rightarrow R$ is the identity map. Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d, \in I \right\}, U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in I \right\} \subset R, \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and $\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then σ and τ are automorphisms of R , and U is a (σ, τ) -left Lie ideal of R but not Lie ideal of R .

Let d be a nonzero derivation of R . The following results have been proved for Lie ideals in [1] and [3]. (i) If $d(U) \subset Z$ then $U \subset Z$. (ii) If $d^2(U) \subset Z$ then $U \subset Z$. In this paper we generalized the above results in prime rings with (σ, τ) -Lie ideals.

Throughout R will represent a prime ring σ and τ automorphisms of R , d a nonzero derivation of R such that $d\sigma = \sigma d, d\tau = \tau d$ and Z the center of R . Further, we shall often use the relations:

$$\begin{aligned} [xy, z]_{\sigma, \tau} &= x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y \quad \text{and} \\ [x, yz]_{\sigma, \tau} &= \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z). \end{aligned}$$

Results

Lemma *Let R be a prime ring, U a nonzero (σ, τ) -left Lie ideal. If $[R, U]_{\sigma, \tau} \subset Z$ then $U \subset Z$.*

Proof. For all $x \in R, u \in U, [x, u]_{\sigma, \tau} \in [R, U]_{\sigma, \tau}$. We replace x by $x\sigma(u)$, then we get $[x, u]_{\sigma, \tau}\sigma(u) \in [R, U]_{\sigma, \tau}$ and so $[x, u]_{\sigma, \tau}\sigma(u) \in Z$ for all $x \in R, u \in U$. Thus, since $[x, u]_{\sigma, \tau} \in Z$ we get $[x, u]_{\sigma, \tau} = 0$ for all $x \in R$, or $\sigma(u) \in Z$. If $\sigma(u) \in Z$, then $u \in Z$. If $[x, u]_{\sigma, \tau} = 0$ for all $x \in R$, then replacing x by $xy, y \in R$ we get $0 = R[R, \sigma(u)]$ and so by the primeness of R we obtain $u \in Z$. Thus $U \subset Z$ is obtained. \square

Lemma *Let R be a prime ring with $\text{char } R \neq 2, U$ a nonzero (σ, τ) -Lie ideal of R , d a nonzero derivation of R . If $d(U) \subset Z$ then $U \subset Z$.*

Proof. Since $d(U) \subset Z$, for all $x \in R, u \in U$ we have,

$$Z \ni d([x, u]_{\sigma, \tau}) = [d(x), u]_{\sigma, \tau} + [x, d(u)]_{\sigma, \tau} \tag{1}$$

In (1), if we replace x by $xd(v), v \in U$ and using $d(U) \subset Z$ and (1) we get $[x, u]_{\sigma, \tau}d^2(v) \in Z$ for all $x \in R, u, v \in U$. Thus, since $d^2(v) \in Z$, for all $x \in R, u, v \in U$, we get $[x, u]_{\sigma, \tau} \in Z$ or $d^2(v) = 0$. That is $d^2(U) = 0$ or $[R, U]_{\sigma, \tau} \subset Z$. If $d^2(U) = 0$ then by [2, Theorem 2] we get $U \subset Z$. If $[R, U]_{\sigma, \tau} \subset Z$, then by lemma 1 we have $U \subset Z$. \square

Lemma *Let R be a prime ring with $\text{char } R \neq 2, 3$ U a nonzero (σ, τ) - Lie ideal of R , d a nonzero derivation of R such that $d(U) \subset U, d^2(U) \subset Z$. If $d^3(U) = 0$ then $U \subset Z$.*

Proof. Assume that $U \not\subset Z$. Since $[x, u]_{\sigma, \tau} \in U$, replacing x by $\tau(u)x$ we have $\tau(u)[x, u]_{\sigma, \tau} \in U$. For all $x \in R$ and $u \in U$, $0 = d^3(\tau(u)[x, u]_{\sigma, \tau}) = 3(\tau(d^2(u))d([x, u]_{\sigma, \tau}) + \tau(d(u))d^2([x, u]_{\sigma, \tau}))$ and so,

$$\tau(d^2(u))d([x, u]_{\sigma, \tau}) + \tau(d(u))d^2([x, u]_{\sigma, \tau}) = 0 \tag{2}$$

In (2), we replace u by $d(u)$, then for all $x \in R, u \in U$ we get $0 = \tau(d^2(u))d^2([x, d(u)]_{\sigma, \tau})$. Thus, since $d^2(u) \in Z$ we have that

$$d^2(u) = 0 \text{ or } d^2([x, d(u)]_{\sigma, \tau}) = 0 \text{ for all } x \in R. \tag{3}$$

If $d^2([x, d(u)]_{\sigma, \tau}) = 0$ for all $x \in R$; replacing x by $x\sigma(d(u))$ in the last equation and using $\text{char } R \neq 2$ we obtain

$$d([x, d(u)]_{\sigma, \tau}) = 0, \text{ for all } x \in R. \text{ or } d^2(u) = 0.$$

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If $d([x, d(u)]_{\sigma, \tau}) = 0$ for all $x \in R$; then from the relation $0 = d([x\sigma(d(u)), d(u)]_{\sigma, \tau})$ we have, $[x, d(u)]_{\sigma, \tau} = 0$ for all $x \in R$ or $d^2(u) = 0$. Let $[x, d(u)]_{\sigma, \tau} = 0$, for all $x \in R$. If we replace x by $xy, y \in R$ then we see that $d(u) \in Z$. Thus in the second case (3), we have $d(u) \in Z$. Consequently by (3) we obtain that $d^2(u) = 0$ or $d(u) \in Z$. Let $K = \{u \in U | d^2(u) = 0\}$ and $L = \{u \in U | d(u) \in Z\}$. K and L are additive subgroups of U and $U = K \cup L$. Since $d \neq 0$ and $U \not\subseteq Z$, by [2, Theorem 2], $U \neq K$. Thus, by Brauer Trick we have $U = L$. But, if $U = L$ then $d(U) \subset Z$ and so $U \subset Z$ by Lemma 2. This is a contradiction. Therefore we have $U \subset Z$. \square

Theorem *Let R be a prime ring with $\text{char } R \neq 2, 3, U$ a nonzero (σ, τ) -Lie ideal of R , d a nonzero derivation of R such that $d(U) \subset U$ and $d^2(U) \subset Z$ then $U \subset Z$.*

Proof.

Let $d(U) \subset U$ and $d^2(U) \subset Z$. Then, for all $x \in R, u \in U$

$$Z \ni d^2([x, u]_{\sigma, \tau}) = [d^2(x), u]_{\sigma, \tau} + 2[d(x), d(u)]_{\sigma, \tau} + [x, d^2(u)]_{\sigma, \tau} \quad (4)$$

In (4), replacing x by $xd^2(v), v \in U$ and using $d^2(U) \subset Z$ we get

$$2d^3(v)d([x, u]_{\sigma, \tau}) + [x, u]_{\sigma, \tau}d^4(v) \in Z, \text{ for all } x \in R, u, v \in U \quad (5)$$

In the last equation, if we take $xd^2(w), w \in U$ instead of x and use that $\text{char } R \neq 2$, then we have $d^3(v)d^3(w)[x, u]_{\sigma, \tau} \in Z$, for all $x \in R, u, v, w \in U$. Since $d^2(U) \subset Z$, thus we have $d^3(U) = 0$ or $[R, U]_{\sigma, \tau} \subset Z$. If $d^3(U) = 0$, then $U \subset Z$ by Lemma 3. If $[R, U]_{\sigma, \tau} \subset Z$, then $U \subset Z$, by Lemma 1. Consequently we obtain that $U \subset Z$. \square

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TÜREVLİ ASAL HALKALARDA (σ, τ) - LE İDEALLER

Özet

Bu makalede aşağıdaki sonuçlar ispatlanmıştır. R , $\text{char } R \neq 2, 3$ olan bir asal halkla, U, R nin sıfırdan farklı bir ideali, σ ve τ R nin iki otomorfizmi ve $0 \neq d : R \rightarrow R, d\sigma = \sigma d, \tau d = d\tau$ olacak şekilde R nin bir türevi olsun 1) Z, R nin merkezi olmak üzere $d(U) \subset Z$ ise $U \subset Z$. 2) $d(U) \subset U$ ve $d^2(U) \subset Z$ ise $U \subset Z$ dir.

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