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O. MUCUK

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O. Mucuk

Abstract

In [4] a construction on topological groups was given. In this paper we generalize this costruction to more general topological groupoids and have a similar structure on the topological groupodis.

Introduction

Let G be a topological group and W an open neighbourhood of the identity e in G. Let $F(W)$ be the free group on W and N the normal subgroup of $F(W)$ generated by the elements of $F(W)$ in the form $[vu]^{-1}[v][u]$ such that u, v and vu belong to W, where for example $[u]$ is the equivalence class of u in $F(W)$. Note that where vu is the group multiplication in G while $[v][u]$ is the free group multiplication in $F(W)$. Suppose that $\tilde{G}$ is the quotient group of $F(W)$ by N. Thus we have an inclusion map $\tilde{i} : W \to \tilde{G}$ and a projection map $p : \tilde{G} \to G$.

In [4] it was proved that the group $\tilde{G}$ can be given a topology such that right translations are homeomorphisms. In general $\tilde{G}$ is not a topological group. However they prove that the projection map $p : \tilde{G} \to G$ is a covering map.

In this paper we generalize this construction to the topological groupoid case and from a topological groupoid G and an open subset W of G have a groupoid $M(G,W)$ which is not a topological groupoid but each fibre $(MG)_x$ of the initial point map has a topology such that right translations are homeomorphisms. This construction on $M(G,W)$ is called star topological groupoid (Definition 2). For groupoids and topological groupoids we refer to [6].

MONODROMY GROUPOID

A *groupoid* is a small category in which each morphism has a both sided inverse. In a groupoid G we write α and β for the initial and final maps respectively, write G_x for $\alpha^{-1}(x)$ and write $G(x,y)$ for the set of all morphisms from x to y. For a groupoid G let $I(G)$ be the set of all identities. We identify the set of object O_G with $I(G)$ and so sometimes write O_G for $I(G)$.

A *topological groupoid* is a groupoid G in which the set of morphisms G and the set of objects O_G are both topological spaces and all groupoid structure maps are continuous.

Definition 1. *Let G and H be groupoids. A local morphism of groupoids is a map $f : W \to H$ from a subset $W \subseteq O_G$ of G such that for $u \in W$, $\alpha_H(fu) = f(\alpha_G u), \beta_H(fu) = f(\beta_G u)$ and $f(vu) = f(v)f(u)$ whenever $u, v \in W$ and vu is defined and belongs to W. Suppose G and H are both topological groupoids and W is an open neighborhood of O_G. We call a continuous map $f : W \to H$ local morphism of topological groupoids if it is a local morphism on underlying groupoids.*

The following construction is similar to that of [4] stated in the introduction (see also [7] for more details and background of this area).

Let G be a topological groupoid and let W be an open subset of G such that $O_G \subseteq W$. The graph structure on W inherited from the groupoid structure gives a free groupoid $F(W)$. Let N be the normal subgroupoid of $F(W)$ generated by the elements of the form $[vu]^{-1}[v][u]$ for $(v, u) \in W \times W$ such that vu is defined and in W. The quotient groupoid $F(W)/N$ is denoted by $M(G, W)$ and called *monodromy groupoid* of G for W. The elements of $F(W)$ are written as $[u]$ and those of $M(G, W)$ are written as $< u >$ when $u \in W$ (See [3] and [5] for the details of free groupoids and quotient groupoids).

Thus by the construction of the monodromy groupoid the inclusion map $i : W \to G$ determines an injection $\tilde{i} : W \to M(G, W)$ and hence a morphism of groupoids $p : M(G, W) \to G$ such that $p\tilde{i} = i$. Write $\tilde{W}$ for $\tilde{i}(W)$. Further if $f : W \to H$ is a local morphism of topological groupoids then there exists a unique morphism $\phi : M(G, W) \to H$ of groupoids such that $\phi\tilde{i} = i$.

STAR TOPOLOGICAL GROUPOIDS

The following definition was first given in [8] under the name *"un morceau α-structuré de groupoide"* and then restated in [1] under the name "α-topological groupoid".

Definition 2. *A locally star topological groupoid is a pair (G, W) of a groupoid G and a topological space W such that:*

i) $O_G \subseteq W \subseteq G$, that is W contains all identities;

ii) W is the topological sum of the subspaces $W_x = W \cap G_x, x \in O_G$;

iii) if $g \in G(x, y)$, then for the right translation

$$R_g : W_y \to W_y g, h \mapsto hg$$

the sets $R_g^{-1}(W)$ and $W \cap W_y g$ are open in W and the map

$$R_g : R_g^{-1}(W) \to W \cap W_y g, h \mapsto hg,$$

the restriction of the right translation R_g, is a homeomorphism.

A locally star topological groupoid (G, W) is said to be a *star topological groupoid* if G and W have the same underlying set. Thus a star topological groupoid is a groupoid in which each fibre G_x has a topology such that right translations are homemorphisms.

Let G be a star topological groupoid. A subset W of G is called *star – open* if for each $x \in O_G$, the fibre $W_x = W \cap G_x$ is open in G_x.

We now give a theorem which was proved in [7] in a different way using a result given in [1] on the extendibility of locally star topological groupoids to star topological groupoids. The following proof is completely different.

Earlier we have written $\tilde{W}$ for $\tilde{i}(W)$. Now impose on $\tilde{W}$ the topology such that the bijection $\tilde{i} : W \to \tilde{W}$ is a homemorphism. We then give the following main theorem.

Theorem 1. *The groupoid $M(G,W)$ is a star topological groupoid such that $\tilde{W}$ is star open in $M(G,W)$. Further the projection map $p : M(G,W) \to G$ is a covering map on each fibre $M(G,W)_x$.*

Proof. Let $\sigma_g : W_{\beta g} \to \tilde{W}g$ be defined by $u \to < u > g$. Then σ_g is a bijection. For each $g \in M(G,W)_x$, impose on $\tilde{W}g$ the topology induced from that of W by the bijection $\sigma_g : W_{\beta g} \to \tilde{W}g$ and then impose on $M(G,W)_x$ the final topology with respect to the inclusions $i : \tilde{W}g \to M(G,W)_x$, for all $g \in M(G,W)_x$.

We now show that for $h \in M(G,W)(x,y)$, the right translation

$$R_h : M(G,W)_y \to M(G,W)_x, g \mapsto gh$$

is a homeomorphism. It is obvious that R_h is bijective and the right translation $R_h - 1$ is the inverse of R_h. Hence it is enough to prove that R_h is continuous. Since the topology on $M(G,W)_y$ is the final topology with respect to the inclusions $i : \tilde{W}g \to M(G,W)_y$, R_h is continuous if and only if the composition map $R_h i : \tilde{W}g \to M(G,W)_x$ is continuous. But $R_h i$ is continuous, since $R_h i|\tilde{W}g, (\tilde{W}g)h$, the restriction of $R_h i$, is a homeomorphism by the homeomorphisms $\sigma_g : W_{\beta g} \to \tilde{W}g$ and $\sigma_{gh} : W_{\beta g} \to \tilde{W}gh$. So that $M(G,W)$ becomes a star topological groupoid.

Further by the construction of the topology on $M(G,W)_x$, obviously $\tilde{W}_x$ is open in $M(G,W)_x$. Hence $\tilde{W}$ is star open in $M(G,W)$.

Next we prove the following

I) for $g, h \in M(G,W)_x$, the set $\tilde{U}_{g,h} = \tilde{W}g \cap \tilde{W}h$ is open in both $\tilde{W}g$ and $\tilde{W}h$;

II) the topologies on $\tilde{U}_{g,h}$ induced by $\tilde{W}g$ and $\tilde{W}h$ as subspaces coincide.

In order to prove these we need the following lemma which is given in group case in [4]. □

Lemma 1. *Let $g \in M(G,W)_x$ and U_g be the set of elements $u \in W$ such that $< u > \in \tilde{W}g$, that is, $U_g = (\tilde{i})^{-1}(\tilde{W}_g)$. Then U_g is open in W_x and the map $\psi_g : U_g \to W$,*

$u \mapsto u'$ *with* $<u>=<u'>g$, *is continuous.*

Proof. Let $u_0 \in U_g$ and let $u_0' \in W$ such that $\psi_g(u_0) = u_0'$. So that $<u_0>=<u_0'>g$. We choose a subset V of W as $(Wu_0^{-1}) \cap (Wu_0'^{-1}) \cap W$. Then V is open in G_{u_0} using the right translations $R_{u_0^{-1}}$ and $R_{u'^{-1}_0}$. Hence $1_{\beta u_0} \in V$, and for $v \in V$ we have $(u_0, v) \in D$ and $(u_0', v) \in D$. Where D is the set $\gamma^{-1}(W) \cap (W_\alpha X_\beta W)$ and

$$W_\alpha X_\beta W = \{(v,u) \in W \times W : \alpha v = \beta u\}.$$

For $u \in Vu_0$, we have $u = vu_0$ with a $v \in V$ so that

$$<u>=<vu_0>=<v><u_0>=<v><u_0'>g=<vu_0'>g.$$

Hence $u \in U_g$ so that $u_0 \in Vu_0 \subset U_g$ (We notice that $U_g \subseteq W_x$). But Vu_0 is open in G_x by the right translation

$$R_{u_0} : G_{u_0} \to G_x.$$

Hence Vu_0 is open in W_x, so U_g is open in W_x. Further $<u>=<vu_0'>g$ and $<u>=<u'>g$ so that

$$<u'>=<vu_0'>, u' = vu_0' = uu_0^{-1}u_0'.$$

Hence ψ_g is continuous at $u \in Vu_0$. But since Vu_0 is open in U_g, so ψ_g is continuous on all U_g which completes the proof of Lemma 1. □

Proof. of (I) and (II)

$$\begin{aligned}\sigma_g^{-1}(\tilde{U}_{g,h}) &= \{u \in W :<u>g \in \tilde{U}_{g,h}\} \\ &= \{u \in W :<u>g \in \tilde{W}g \cap \tilde{W}h\} \\ &= \{u \in W :<u>g \in \tilde{W}h\} \\ &= \{u \in W :<u>\in \tilde{W}hg^{-1}\} \\ &= U_{hg^{-1}}\end{aligned}$$

and similarly we can see that $\sigma_h^{-1}(\tilde{U}_{gh}) = U_{gh^{-1}}$. By Lemma 1 $U_{hg^{-1}}$ (resp. $U_{gh^{-1}}$) is open in $W_{\beta g}$ (resp. $W_{\beta h}$). So $\tilde{U}_{g,h}$ is open in both $\tilde{W}g$ and $\tilde{W}h$ which proves (I).

For (II) let $\tilde{U}$ be a subset of $\tilde{U}_{g,h}$ and let $w \in \tilde{U}$. Then there are $u, u' \in W$ such that $w =<u>g$ and $w =<u'>h$. So

$$<u>g=<u'>h, <u>=<u'>hg^{-1} \text{ or } <u'>=<u>gh^{-1}.$$

So that

$$\begin{aligned}\tilde{U} &= \{< u > g : \text{ for some } u \in U_{hg^{-1}}\} \\ &= \{< u' > h : \text{ for some } u' \in U_{gh^{-1}}\}\end{aligned}$$

and

$$\begin{aligned}\sigma_g^{-1}(\tilde{U}) &= \{u \in W :< u > g \in \tilde{U}\} \\ &= \{u \in W :< u > g \in \tilde{U} \text{ and } < u > g =< u' > h \text{ for a } u' \in W\} \\ &= \{u \in W :< u > g \in \tilde{U} \text{ and } < u >=< u' > hg^{-1} \text{ for a } u' \in W\} \\ &= \{u \in U_{hg^{-1}} :< u > g \in \tilde{U}\};\end{aligned}$$

and similarly

$$\sigma_h^{-1}(\tilde{U}) = \{u \in U_{gh^{-1}} :< u > h \in \tilde{U}\}$$

Hence,

$$\psi_{hg^{-1}}^{-1}(\sigma_h^{-1}(\tilde{U})) = \sigma_g^{-1}(\tilde{U}), \text{ and } \psi_{gh^{-1}}^{-1}(\sigma_g^{-1}(\tilde{U})) = \sigma_h^{-1}(\tilde{U}).$$

So if $\tilde{U}$ is open in $\tilde{W}g$, then $\sigma_g^{-1}(\tilde{U})$ open in $W_{\beta g}$. Since $\psi_{gh^{-1}} : U_{gh^{-1}} \to W_{\beta g}$ is continuous, $\psi_{gh^{-1}}^{-1}(\sigma_g^{-1}(\tilde{U})) = \sigma_h^{-1}(\tilde{U})$ is open in $U_{gh^{-1}}$. But by Lemma 1 $U_{gh^{-1}}$ is open in $W_{\beta h}$. Hence $\tilde{U}$ is open in $\tilde{W}h$. Conversely (with the same idea) we can show that if $\tilde{U}$ is open in $\tilde{W}h$ then it is open in $\tilde{W}g$. This shows that the topologies on $\tilde{U}_{g,h} = \tilde{W}g \cap \tilde{W}h$ induced by $\tilde{W}g$ and $\tilde{W}h$ coincide, which is required in (II).

So by the Proposition 8, page 33 in [2] the star $M(G,W)_x$ together with the topology imposed on it as above satisfies the following conditions.
A) for each $g \in M(G,W)_x$, $\tilde{W}g$ is open in $M(G,W)_x$;
B) for $g \in M(G,W)_x$, the topologies on $\tilde{W}g$ induced by the bijection $\sigma_g : W_{\beta g} \to \tilde{W}g$ and by $M(G,W)_x$ as a subspace coincide.

Finally we prove that $p_x : M(G,W)_x \to G_x$, the restriction of the projection map $p : M(G,W) \to G$, is a covering map. Let $g \in G(x,y)$. Since W_y is open in G_y then by the right translation $R_g : G_y \to G_x$, $(W_y)g$ is a open neighbourhood of g in G_x, and

$$\begin{aligned}p^{-1}(W_y g) &= p^{-1}(W_y)p^{-1}(g), \\ &= U\{(\tilde{W}_y)h : h \in p^{-1}(g)\}.\end{aligned}$$

We notice that since O_p is identity, $\beta h = y$. But for each $h \in p^{-1}(g)$, the restriction $p : (\tilde{W}_y)h \to (W)_y)g$ is a homeomorphism by the homeomorphisms $R_g : W_y \to (W_y)g$ and $R_h\tilde{i} : W_y \to (\tilde{W}_y)h$. Hence $p_x : M(G,W)_x \to G_x$ is a covering morphism. □

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YILDIZ TOPOLOJİK GROUPOİDLER

Özet

Referanslardan [4]'de topolojik gruplar hakkında bir kavram geliştirilmiştir. Bu makalede bu kavramı topolojik gruplardan daha geniş olan topolojik groupoidlere genelleştiriyor ve topolojik grupoidlerde bir kavram elde ediyoruz.

O. MUCUK,
Erciyes University,
Faculty of Science,
Department of Mathematics,
38039, Kayseri-TURKEY

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