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Yu. F. KOROBEINIK

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ABSOLUTELY REPRESENTING SYSTEMS AND CONVOLUTION OPERATORS IN THE COMPLEX DOMAIN*

Yu.F. Korobeinik

Abstract

The author suggests a method of characterizing of the kernel of a convolution-type operator. The method is based on the exact description of nontrivial expansions of zero of an absolutely representing system consisting of eigen vectors of some linear operator with continuous spectrum. The method suggested in the paper is applied to the characterization of the kernel of the convolution-type operator defined on some space of entire functions and to the factorization of such an operator.

According to [1] a sequence $\{x_k\}_{k=1}^{\infty}$ of elements of a complete locally convex space (CLCS) H is said to be *an absolutely representing system* (ARS) in H if each element x from H can be represented in the form of a series

$$x = \sum_{k=1}^{\infty} c_k x_k$$

absolutely converging in H.

Absolutely representing systems of exponents and of generalized exponents have proved to be very useful tools in the investigation of convolution and convolution type equations. For example, the construction of a partial solution of a nonhomogeneous convolution equation in the space $H(G)$ of functions analytic in a complex domain G has been made with the help of ARS of exponents by A.F. Leont'ev [2, Ch. V, $\oint 2$] and Yu. F. Korobeǐnik [3, $\oint 8$]. This method has been extended in [4] to other spaces of analytic functions.

A new method of the investigation of a homogeneous convolution equation in the space $H(G)$ has been suggested in $\oint 1$ of the paper [5]. The idea of this method has been developed in [5-8] and has given an opportunity to characterize the kernel of a ρ-convolution operator in the space of function analytic in a ρ-convex domain and to solve a problem of factorization of such operator (see[6-8]).

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We can describe a far more general situation when the method of the papers [5-8] is yet applicable.

Let T be a linear continuous operator from one CLCS H_1 into another CLCS H_2. Let A be some set of indices. Suppose that some totality $E = \{e(t) : t \in A\}$ of nonzero elements has the following properties:

(1) $\forall t \in A \quad e(t) \in H_1 \cap H_2$;

(2) $\forall t \in A \quad Te(t) = a(t)e(t)$ where $a(t)$ is a scalar function defined on A;

(3) span E is dense both in H_1 and in H_2.

Let us suppose that some sequence $E_0\{e(t_k)\}_{k=1}^{\infty}$ of elements from E is an ARS in H_1. Suppose also that one can describe a general form of *an absolutely converging nontrivial expansion of zero* (ACNEZ) for the system E_0 in the space H_2, that is one can characterize the space D of sequences $\{d_k\}_{k=1}^{\infty}$ formed by coefficients of all series

$$\sum_{k=1}^{\infty} d_k e(t_k)$$

converging absolutely to zero in H_2.

Under these assumptions we can characterize the kernel $T^{-1}(0)$ of the operator T. If $x \in H_1$, then $x = \sum_{k=1}^{\infty} c_k e(t_k)$ and the series converges absolutely in H_1. Suppose that $x \in T^{-1}(0)$. Then

$$0 = Tx = \sum_{k=1}^{\infty} c_k Te(t_k) = \sum_{k=1}^{\infty} c_k a(t_k) e(t_k),$$

whence $\{c_k a(t_k)\}_{k=1}^{\infty} \in D$. On the other hand, if for the sequence $\{w_k\}_{k=1}^{\infty}$ the corresponding series $y = \sum_{k=1}^{\infty} w_k e(t_k)$ converges obsolutely in H_1 and if $\{w_k a(t_k)\}_{k=1}^{\infty} \in D$ then

$$Ty = \sum_{k=1}^{\infty} w_k Te(t_k) = 0 \text{ and } y \in T^{-1}(0).$$

So, if Q is the space of all sequences $\{c_k\}_{k=1}^{\infty}$ such that the series

$$\sum_{k=1}^{\infty} c_k e(t_k)$$

converges absolutely in H_1, then

$$T^{-1}(0) = \left\{ x = \sum_{k=1}^{\infty} b_k e(t_k) : \ \{b_k\}_{k=1}^{\infty} \in Q, \ \{b_k a(t_k)\}_{k=1}^{\infty} \in D \right\}.$$

We apply here this method to the convolution operator in the spaces of entire functions the growth of which is determined by their indicators.

Everywhere below $\rho > 1$, $\rho_1 = \frac{\rho}{\rho-1}$, $k = 1, 2$. Denote by T_ρ the set of all bounded 2π-periodic trigonometrically ρ-convex functions (see [9,Ch.I,§16]). Let $[\rho, \infty)$ be the class of all entire functions of finite type with respect to the order ρ. For each function f from $[\rho, \infty)$ we can define its indicator

$$Ind_f(\theta) = \lim_{r\to\infty} \sup (r^{-\rho} \ln |f(r\exp(i\theta))|), \quad \theta \in \mathbb{R}.$$

If $h(\theta)$ is an arbitrary 2π-periodic real-valued bounded function, then $[\rho, h(\theta))$ is the space of all functions from $[\rho, \infty]$ with indicators $\le h(\theta)$ and with the Frechet topology introduced by the countable set of norms

$$|y|_n = \sup_{z\in\mathbb{C}} |y(z)| \exp\left\{-h(arg\ z) - \frac{1}{n}\right\} |z|^\rho, \quad n = 1, 2, \cdots$$

For any $g \in T_\rho$ let us denote by $\Gamma(g)$ the union of all intervals (α, β) of ρ-trigonometricity: $\forall\theta \in (\alpha, \beta)$, $g(\theta) = L\cos\rho\theta + M\sin\rho\theta$. We put $R(g) = \mathbb{R}\backslash\Gamma(g)$. Let B_ρ be the set of all positive functions $g(\theta)$ from T_ρ such that $|z|^\rho g(arg\ z)$ is a convex fucntion of z. For any function g from B_ρ one can define its Young transformation:

$$g^0(\varphi) = \sup\{R(z\exp i\varphi) - |z|^\rho g(arg\ z) : \quad z \in \mathbb{C}\}.$$

Suppose $h_k \in B_\rho$. Let μ be a linear continuous functional on $H(\mathbb{C})$ such that its Fourier-Borel transform $a(\lambda) := \mu(\exp\ \lambda z)$ belongs to $[\rho_1, \infty)$. Let $h(\theta)$ be the indicator of $a(z)$. Suppose that the equality

$$h_1^0(\theta) = h(\theta) + h_2^0(\theta) \tag{1}$$

is valid.

Of concern is the convolution operator

$$(T_a f)(z) = \mu[f(z+w)], \quad z \in \mathbb{C}.$$

According to [10] T_a is continuous from $[\rho, h_1(\theta)]$ into $[\rho, h_2(\theta)]$ and T_a is an epimorphism iff $a(z)$ has a completely regular growth (c.r.g.) on all rays from $R(h_2^0)$. It is worth reminding that the function $y(z)$ from $[\rho_1, \infty)$ has a c.r.g. with respect to the order ρ_1 on the ray arg $z = \theta$ if there exists a finite limit

$$\lim_{\substack{r\to\infty \\ r\notin\Lambda}} (r^{-\rho_1} \ln |y(r\exp i\theta)| = Ind_y(\theta).$$

Here Λ is a subset of $(0, +\infty)$ with zero relative measure:

$$\lim_{r\to\infty} \frac{mes\Lambda \cap [0,r]}{r} = 0.$$

We put in this situation $A = \mathbb{C}; e(z) = \exp zw;\ H_k = [\rho, h_k]$.

Let E_0 be an arbitrary fixed subset of $(0,+\infty)$ of zero relative measure. V.A. Savel'ev [4] has constructed the function $M(\lambda)$ from $[\rho_1,\infty)$ with the following properties:

(1) $M(\lambda)$ has a c.r.g. on each ray arg $z = \theta, \theta \in [0,2\pi]$;

(2) all zeros $\{\lambda_k\}_{k=1}^{\infty}$ of $M(\lambda)$ are simple and

$$\lim_{n\to\infty} [|\lambda_n|^{-\rho_1} \ln |M'(\lambda_n)| - h_1^0(arg\lambda_n)] = 0;$$

(3) $Ind_M(\theta) = h_1^0(\theta)$;

(4) $\forall l \geq 1\ |\lambda_l| \notin E_0$;

(5) $\forall l \geq 1\ \arg \lambda_l \in R(h_1^0)$.

Previously the function M with the properties (1)-(4) has been constructed by A.F. Leont'ev [11].

Since according to our assumptions $|z|^\rho h_1($ arg $z)$ is a convex function, the sequence $E_0 := \{\exp \lambda_k z\}_{k=1}^{\infty}$ due to Lemma from [8, §4] is an ARS in $[\rho, h_1]$. Besides, the Theorem 3 from [8] characterizes an ACNEZ for the system E_0 in the space $[\rho, h_2]$, and we can apply the above described method. Suppose that the symbol $a(z)$ of the operator T_a has a c.r.g. on each ray from $R(h_1^0)$. This assumption is a bit stronger than the assumption of T_a being an epimorphism, since $R(h_1^0) = R(h_2^0) \cup R(h)$.

After repeating almost literally the proof of the Theorems 7,8 from [8] we obtain the following results.

Theorem 1. *Let*

$$k = 1,2;\ h_k \in B_\rho, \rho > 1,\ \rho_1 = \frac{\rho}{\rho - 1}.$$

Let the function $a(z)$ from $[\rho,\infty)$ have a c.r.g. on all rays from $R(h_1^0)$ and let the equality (1) be valid. Then the kernel $T_a^{-1}(0)$ of the convolution operator T_a is isomorphic topologically to the quotient space $[\rho_1, h]/I_1(a)$ where

$$I_1(a) = \{v(z) = a(z)b(z):\ b \in [\rho_1, 0]\}.$$

If, additionally, $\forall\theta\ h(\theta) > 0$, then the kernel of the operator T_a is isomorphic to a nuclear space of power series of finite type and consequently $T_a^{-1}(0)$ has an absolute Schauder basis.

Remark 1. It is rather interesting that the characterization of the kernel T_a^{-1} given in this theorem does not depend on $h_k(\theta)$.

Remark 2. The proof of the last part of the Theorem 1 is also based on the application of interpolational topological invariants of Dragilev's type (see, [12],[13]).

Remark 3. This theorem has been announced before in [14] but its formulation [14] has one essential misprint. Namely the symbol $R(h_1^0)$ in that formulation ([14, p.22]) must denote the set of all ν from $[0, 2\pi]$ such that the function

$$(h_1^0)'(\theta) + \rho^2 \int_\nu^\theta h_1^0(t)dt$$

(not $h_1^0(\theta)$ as in [14]!) is nonconstant in $(\nu - \delta, \nu + \delta)$, $\forall \delta > 0$.

It would be interesting to find out whether the representation of $T_a^{-1}(0)$ from the Theorem 1 is valid under more mild assumptions, for example, when T_a is an epimorphism or when T_a is only continuous from $[\rho, h_1]$ into $[\rho, h_2]$.

The Theorem 1 may be applied to the problem of the factorization of the operator T_a. As above

$$k = 1, 2; \rho > 1, \ \rho_1 = \frac{\rho}{\rho - 1}, \ \ h_k \in B_\rho.$$

Let $a_k(z) \in [\rho_1, \infty)$. We put

$$a(z) = a_1(z)a_2(z); \ g_k(\theta) = Ind_{a_k}(\theta).$$

Let q_k be arbitrary functions from B_ρ. Suppose that the following equalities are valid for all θ and for $k = 1, 2$:

$$\begin{aligned} h_2^0(\theta) + g_1(\theta) + g_2(\theta) &= h_1^0(\theta); \\ h_2^0(\theta) + g_k(\theta) &= q_k^0(\theta). \end{aligned}$$

Then the operator T_a is continuous from $[\rho, h_1]$ into $[\rho, h_2]$ and for all y from $[\rho, h_1]$

$$T_a y = T_{a_1}(T_{a_2} y) = T_{a_2}(T_{a_1} y),$$

where the operator T_{a_k} is continuous both from $[\rho, h_1]$ into $[\rho, q_{3-k}]$ and from $[\rho, q_k]$ into $[\rho, h_2]$. We put

$$\begin{aligned} T_a^{-1}(0) &= \{y \in [\rho, h_1] : \ (T_a y)(z) = 0, \quad \forall z \in \mathbb{C}\}; \\ T_{a_k}^{-1}(0) &= \{y \in [\rho, h_1] : \ (T_{a_k} y)(z) = 0, \quad \forall z \in \mathbb{C}\}. \end{aligned}$$

Let us say that T_a admits a factorization in the space $[\rho, h_1]$ if

$$T_a^{-1}(0) = T_{a_1}^{-1}(0) + T_{a_2}^{-1}(0). \tag{2}$$

Theorem 2. *If the functions $a_k(z)$ have completely regular growth on all rays from $R(h_1^0)$ then the convolution operator T_a admits a factorization in $[\rho, h_1]$ iff*

$$\forall\Phi \in [\rho_1, g_1 + g_2]\exists\Phi_k \in [\rho_1, g_k] : \forall z \in \mathbb{C} \quad \Phi(z) = a_1(z)\Phi_1(z) + a_2(z)\Phi_2(z). \tag{3}$$

The proof of the Theorem 2 is based on the same scheme as the proof of the Theorem 8 from [8].

One can easily derive a criterion of the validity of (3) from general results of the paper [15]. Thus one can obtain the following

Corollary *Under the above mentioned assumptions the convolution operator T_a admits a factorization in $[\rho, h_1]$ iff* $\forall\epsilon > 0\ \exists d > 0\ \forall z \in \mathbb{C}$

$$|a_1(z)| \exp[-g_1(\arg z)|z|^{\rho_1}] + |a_2(z)| \exp[-g_2(\arg z)|z|^{\rho_1}] \geq d \exp(-\epsilon|z|^{\rho_1}).$$

It would be also interesting to prove the Theorem 2 under more mild assumptions.

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KOMPLEKS DÜZLEMDE KONVOLOSYON OPERATÖRLERİ VE MUTLAK TEMSİL SİSTEMLERİ

Özet

Bu çalışmada bazı lineer operatörlerin özdeğerlerinden oluşan mutlak temsil sistemleri kullanılarak konvolosyon operatörlerinin çekirdekleri incelenmiş ve bunların bazı uygulamaları verilmiştir.

Yu.F. KOROBEINIK
Rostov State University,
Zorge St. 5
Rostov-on-Don
344104 RUSSIA

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