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A. O. ASAR

S. ATLIHAN

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A.O. Asar & S. Atlıhan

Abstract

In this work it is shown that the direct product of two Meldrum groups does not satisfy the normalizer condition. It is not known yet if the same result holds true for any two HM-groups.

Introduction

Let G be the direct product of two groups H and K each satisfying the normalizer condition. In Lemma 4 of [4] of a necessary and sufficient condition is given for G to satisfy the normalizer condition. In particular if H and K are isomorphic, then this the case if and only if H and K are hypercentral. However it is not known yet wether the direct product of two HM groups (for a prime p) satisfy the normalizer condition or not. In this work this question is answered negatively for Meldrum groups which are HM-groups (see [5]).

Theorem. *Let $H = G \times \bar{G}$, where G and $\bar{G}$ are Meldrum groups for a prime p. Then H does not satisfy the normalizer condition.*

By definition a group G satisfies the **normalizer condition** if for all $H < G, H < N_G(H)$. It is well-known that the normalizer condition is subgroup inherited.

2. Some Properties of HM*-Groups

A locally nilpotent p-group T is called an HM^*-group if T' is nilpotent and T/T' is isomorphic to a finite direct product of $C_{p}\infty$ groups (see [2]). If $T/T' \cong C_{p}\infty$ and T satisfies the subnormality condition but is not nilpotent, then T is called a **group of Heineken-Mohammed type (HM-group)**. (It can be shown easily that if T is an HM-group, then T' is not properly supplemented in T and every proper subgroup of T is nilpotent as well as subnormal.

The proof of the following lemma is essentially contained on p. 120 of [2].

Lemma *2.1. Let X be an HM^*-group for a prime p. Then $X' = [X, X']$. Furthermore X contains no proper normal subgroup N such that X/N has finite exponent.*

Proof. Let $\bar{X} = X/[X, X']$. Then $\bar{X}' \leq Z(\bar{X})$ and so $\bar{X}$ is nilpotent. Hence $\bar{X}$ is abelian by Theorem 9.23 of [6] which implies that $X' = [X, X']$.

Next let N be a normal subgroup of X such that X/N has finite exponent. Then it is easy to see that $X = NX'$ since X/NX' is both radicable abelian and has finite exponent. So now by the first part of the proof

$$X' = [NX', X'] = ([N, X']X'') \cap X' \\ \leq N$$

since X' is nilpotent. Hence $X' \leq N$ and so it follows that $X = N$. □

Lemma 2.2. *Let H be a locally nilpotent p-group such that $H = ST$, $[S, T] = 1$ and $S'T'$ is abelian of finite exponent p^m for some $m \geq 1$, where S and T are HM^*-subgroups with $S/S' \cong T/T' \cong C_{p}\infty$. Let S/S' and T/T' have generating sets $\{s_iS'\}$ and $\{t_iT'\}$ such that.*

$s_1 \notin S', s_1^p \in S', s_{i+1}^p \in$ for all $i \geq 1$ and the t_i obey the same rules in T.
Then

$$R =< (s_i t_i)^{p^m} : i \geq 1 >$$

is an HM^*-subgroup of H.

Proof. For each $i \geq 1$ there exist $a_i \in S'$ and $b_i \in T'$ such that

$$s_{i+1}^p = s_i a_i \text{ and } t_{i+1}^p = t_i b_i$$

Put $v_i = s_i t_i, c_i = a_i b_i$ and define

$$V =< v_i : i \geq 1 > .$$

Then $v_{i+1}^p = v_i c_i$ for all $i \geq 1$. Let $\bar{V} = V/V'$. Then since
$[\bar{v}_{i+1}, \bar{v}_i] = 1$ and $\bar{c}_i^{p^m} = 1$, we get

$$\bar{v}_{i+1}^{p^{m+1}} = \bar{v}_i^{p^m}, i \geq 1$$

So if we put $\bar{r}_i = \bar{v}_i^{p^m}$ for all $i \geq 1$, then

$$\bar{r}_{i+1}^p = \bar{r}_i, i \geq 1$$

Define $\bar{R} = R/V' =< \bar{r}_i : i \geq 1 >$. Then $\bar{R} \cong C_{p}\infty$. But also $V/V \cap H' \cong VH'/H' \cong C_p\infty$ by definition of V. Hence it follows that $V = R(V \cap H')$. Next let $\bar{V} = V/R'$. Then $\bar{V} = \bar{R}(\overline{V \cap H'})$ Clearly $\bar{V}' \leq Z(\bar{V})$ since it is contained in the intersection of the abelian subgroups $\bar{R}$ and $\overline{V \cap H}'$ Therefore $\bar{V}$ is nilpotent which implies that $\bar{R}^\circ \leq Z(\bar{V})$ by Lemma 2.2 of [1]. But also $\bar{R} = \bar{R}^\circ \bar{V}'$ by Lemma 3.1 of [2] hence it follows that $\bar{R} \leq Z(\bar{V})$ and so $\bar{V}$ is abelian. Consequently $V' = R'$ and so $R/R' \cong C_p\infty$. Therefore it follows that R is an HM^*-group and generated by $\{r_i : i \geq 1\}$. □

Lemma 2.3. *Let H, R, S, T be defined as in Lemma 2.2 If R is normal in H, then* $R' = S'T'$.

Proof. Since R is normal in H, R' is also normal in H. Let $\bar{H} = H/R'$. Then $\bar{R} \cong C_p\infty$ and so $\bar{T}/C_{\bar{T}}(\bar{R})$ is finite by Theorem 3.29(2) of [6] which implies that $\bar{T} = C_{\bar{T}}(\bar{R})$ by Lemma 2.1. Thus $\bar{T}$ centralizes

$$\bar{t}_i^{p^m} = \bar{r}_i(\bar{s}_i^{p^m})^{-1}$$

for all $i \geq 1$. Hence $< t_i^{p^m} : i \geq 1 > \leq Z(\bar{T})$ which implies that $\bar{T}/Z(\bar{T})$ has finite exponent $\leq p^m$ and so $\bar{T} = Z(\bar{T})$ by Lemma 2.1 Similarly $\bar{S} = Z(\bar{S})$. Therefore $\bar{H} = \bar{S}\bar{T}$ is abelian and so $S'T' = R'$ which was to be shown. □

3. Proof of the Theorem

First we briefly summarize the construction of a Meldrum group G given in [5]. We use exactly the same notations used by Meldrum. For each $k \geq 3$ an integer x_k is chosen such that

$$p^{k-1} + p < x_k \leq p^k - 1$$

and an elementary abelian p-group A_k with basis elements $a(i,k)$ for $1 \leq i \leq x_k$ is defined such that A_k has an automorphism b_k of order p^k with the property that

$$\begin{aligned} [b_k, a(i,k)] &= a(i-1,k) \quad && \text{for } i > 1 \\ &= 1 && \text{for } i = 1 \end{aligned} \tag{1}$$

Let $B_k =< b_k >$. In the group algebra $Z_p[B_k]$ the basis elements corresponding to $1, \cdots, b_k^{p^k-1}$ are denoted by $1, \cdots, \beta_k^{p^k-1}$ to avoid confusion. Now for all integers $s \geq 1$,

$$a(i,k)(1-\beta_k)^s = a(i-s,k) \tag{2}$$

where $a(i,k)=1$ if $i\leq 0$.

Next we define the semidirect product $C_k = A_k B_k$. Let

$$D_k =< a(x_k-1,k), b_k > \text{ and } E_k = D_k \cap A_k.$$

Then

$$E_k =< a(i,k) : 1\leq i\leq x_k-1 > .$$

Now C_k is embedded in C_{k+1} by θ_k as follows.

$$\begin{aligned}\theta_k(b_k) &= b_{k+1}^{p} a(x_{k+1}, k+1)(1-\beta_{k+1})^{p-1} \\ &= b_{k+1}^{p} a(x_{k+1}-p+1, k+1)\end{aligned} \quad (3)$$

$$\theta(a(i,k)) = a(pi-p+r_{k+1}, k+1)$$

where $1\leq r_{k+1}\leq p$. Indentifying the elements with their images in all the groups C_k gives $C_k\leq C_{k+1}$ for all $k\geq 3$. Thus the groups

$$H=\cup_{k=3}^{\infty} C_k' \qquad G=\cup_{k=3}^{\infty} D_k$$

are well-defined. In fact it is easy to see that

$$H' = G' = \cup_{k=3}^{\infty} E_k'.$$

Furthermore since G is a group of HM-type, G' cannot be properly supplemented in G which yields immediately that

$$G =< b_k : k\geq 3 > .$$

Also it can be shown that the sequence (x_k) can be defined such that

$$x_{k+1} = px_k - p + r_{k+1} \quad (4)$$

for all $k\geq 3$. In the rest of this paper it is assumed that (x_k) is defined by the above equality.

Lemma 3.1. *The following hold in G for all $k\geq 3$.*

(i) $b_k^{p^s} = b_{k+1}^{p^{s+1}} a(x_{k+1}-p^{s+1}+1, k+1)$
for all $s\geq 0$.

(ii) $b_k = b_{k+s}^{p^s}\prod_{j=0}^{s-1} a(x_{k+s-j}-p^{s-j}+1, k+s-j)$

for all $s \geq 1$.
(iii) The image of $a\ (x_{k+s-j} - t, k+s-j)$ *in* D_{k+s} *is*

$$a(x_{k+s} - p^j t, k+s)$$

Proof. We prove (i) by induction on s. Of course each element is identified by its image in the various groups D_k. Thus by (3)

$$b_k = b_{k+1}^p a(x_{k+1} - p + 1, k+1) \tag{5}$$

so (i) holds for $s = 0$. Assume that it holds for $s = t \geq 0$. Thus

$$b_k^{p^t} = b_{k+1}^{p^{t+1}} a(x_{k+1} - p^{t+1} + 1, k+1)$$

Hence

$$\begin{aligned} b_k^{p^{t+1}} &= \left[b_{k+1}^{p^{t+1}} a(x_{k+1} - p^{t+1} + 1, k+1) \right]^p \\ &= b_{k+1}^{p^{t+2}} \left[a(x_{k+1} - p^{t+1} + 1, k+1) \right]^{(1 - b_{k+1}^{p^{t+1}})^{p-1}} \\ &= b_{k+1}^{p^{t+2}} a(x_{k+1} - p^{t+2} + 1, k+1) \end{aligned}$$

which completes the induction and verifies (i).

Next we show (ii). For $s = 1$ this follows from (5). Assume that it holds for $s = t \geq 1$. Then

$$b_k = b_{k+1}^{p^t} a(x_{k+1} - p^t + 1, k+t) \cdots a(x_{k+1} - p + 1, k+1)$$

Also by (i)

$$b_{k+t}^{p^t} = b_{k+t+1}^{p^{t+1}} a(x_{k+t+1} - p^{t+1} + 1, k+t+1)$$

So substituting this above completes the induction and the proof of (ii).

(iii) is an easy consequence of the second equality in (3) and equality (4). □

Lemma 3.2. *For all integers* i, n, s *with* $n \geq 3$, $s \geq 1$ *the following holds*

$$a(i,n)^{b_n^s} = \prod_{j=0}^{s} a(i-j, n)^{(-1)^j \binom{s}{j}}.$$

Proof. For $s = 1$, it follows from (1) that

$$a(i,n)^{b_n} = a(i,n)a^{-1}(i-1,n)$$

So we assume that the assertion is true for $s = t \geq 1$ and show it for $s = t+1$. Thus

$$a(i,n)^{b_n^t} = \prod_{j=0}^{t} a(i-j,n)^{(-1)^j \binom{t}{j}}$$

Hence

$$\begin{aligned}
a(i,n)^{b_n^{t+1}} &= \prod_{j=0}^{t} \left[a(i-j,n)^{(-1)^j \binom{t}{j}} \right]^{b_n} \\
&= \prod_{j=0}^{t} \left[a(i-j,n)^{(-1)^j \binom{t}{j}} a^{-1}(i-j-1,n)^{(-1)^j \binom{t}{j}} \right] \\
&= a(i,n) \left(\prod_{j=0}^{t} \left[a^{-1}(i-j,n)^{(-1)^{j-1} \binom{t}{j-1}} a(i-j,n)^{(-1)^j \binom{t}{j}} \right] \right) \\
&\quad a(i-t-1,n)^{(-1)^{t+1}} \\
&= a(i,n) \prod_{j=0}^{t} \left[a(i-j,n)^{(-1)^j \binom{t+1}{j}} a(i-t-1,n)^{(-1)^{t+1}} \right] \\
&= \prod_{j=0}^{t+1} a(i-j,n)^{(-1)^j \binom{t+1}{j}}
\end{aligned}$$

which was to be shown. □

Proof of the theorem. Let G and $\bar{G}$ be two Meldrum groups with defining sequences $(x_k), (r_k)$ and $(\bar{x}_k), (\bar{r}_k)$ respectively. Let $H = G \times \bar{G}$. By the remark in the introduction we may suppose that G and $\bar{G}$ are not isomorphic. Assume that H satisfies the normalizer condition. Since G and $\bar{G}$ are not isomorphic, the sequences (x_k) and $(\bar{x}_k)$ are distinct (see pages 442-443 of [5]). Thus there exists a $k_0 \geq 3$ such that $x_{k_0} \neq \bar{x}_{k_0}$. Without loss of generality we may suppose that $x_{k_0} > \bar{x}_{k_0}$.

First we show that $x_k > \bar{x}_k$ for all $k \geq k_0$. For $k = k_0$ this is obvious. Now assume that $x_k > \bar{x}_k$ for some $k \geq k_0$. Then $x_k \geq \bar{x}_k + 1$. Using this is in (4) we get

$$\begin{aligned} x_{k+1} &\geq p(\bar{x}_k + 1) - p + r_{k+1} \\ &= p\bar{x}_k + p - p + r_{k+1} - \bar{r}_{k+1} + \bar{r}_{k+1} \\ &= \bar{x}_{k+1} + p - \bar{r}_{k+1} + r_{k+1} \\ &\geq \bar{x}_{k+1} + r_{k+1} \\ &> \bar{x}_{k+1} \end{aligned}$$

since $1 \leq r_k$ and $\bar{r}_k \leq p$ for all $k \geq 3$.

Now let $R =< (b_k\bar{b}_k)^p : k \geq k_0 >$. Then R is an HM^*-subgroup of H by Lemma 2.2 In fact R is an HM-group by Lemma 3.6 of [2] since G and $\bar{G}$ are HM-groups and H satisfies the normalizer condition. Moreover $N_H(R) = H$ by Lemma 3.2 of [2] and so R is normal in H. In particular $R' = G' \times \bar{G}'$ by Lemma 2.3.

Let $d_k = (b_k\bar{b}_k)^p$ for all $k \geq k_0$. Then by Lemma 3.1(i) we have

$$d_k = d_{k+1}^p a(x_{k+1} - p^2 + 1, k+1)\bar{a}(\bar{x}_{k+1} - p^2 + 1, k+1) \tag{6}$$

Hence

$$\begin{aligned} [d_{k+1}, d_k] &= \left[a(x_{k+1} - p^2 + 1, k+1)\bar{a}(\bar{x}_{k+1} - p^2 + 1, k+1)\right]^{1-d_{k+1}} \\ &= a(x_{k+1} - p^2 - p + 1, k+1)\bar{a}(\bar{x}_{k+1} - p^2p + 1, k+1) \end{aligned} \tag{7}$$

Now let

$$X = \langle [d_{k+1}, d_k]^r : r \in R, k \geq k_0 \rangle$$

We claim that $X = R'$. Put $\bar{R} = R/X$. (Note that "-" is also used to denote the second Meldrum group but it causes no problem here). Since $\bar{d}_k$ and $\bar{d}_{k+1}$ commute we get from (6) that.

$$\bar{d}_k^p = \bar{d}_{k+1}^{p^2}$$

for all $k \geq k_0$ which implies that

$$Y = \left\langle \bar{d}_k^p : k \geq k_0 \right\rangle \cong C_{p^\infty}$$

This implies that $\bar{R} = \overline{YR'}$, and hence $\bar{R} = \bar{Y}$ since $\bar{R}$ is an HM-group. In particular it follows from this that $\bar{R}' = 1$ and so $R' = X$ as claimed. Now let $u \geq k_0$. Then

$$\bar{a}(\bar{x}_u - 1, u) \in \bar{G}' \leq G'\bar{G} = R'$$

Therefore there exist positive integers $n \geq e \geq k_0$ such that

$$\bar{a}(\bar{x}_u - 1, u) = [d_{k_0+1}, d_{k_0}]^{v_{k_0}} \cdots [d_{e+1}, d_e]^{v_e} \tag{8}$$

where $v_{k_0}, \cdots, v_e \in Z_p[\langle d_n \rangle R']$

Let $k_0 \leq i \leq e$. Then by (7) and Lemma 3.1 (iii) we get

$$[d_{i+1}, d_i] = a(x_n - p^{u_i+2} - p^{u_i+1} - p^{u_i}, n)\bar{a}(\bar{x}_n - p^{u_i+2} - p^{u_i+1} - p^{u_i}, n) \tag{9}$$

where $u_i = n - (i+1)$. Now it is easy to see from (9) and Lemma 3.2 that the element

$$[d_{i+1}, d_i]^{v_i}$$

is a product of powers of certain elements from the set

$$\{a(x_n - j, n)\bar{a}(\bar{x}_n - j, n) : j \geq 1\}$$

Since this is true for any $k_0 \leq i \leq e$ we can find positive integers $1 \leq m \leq x_n$. $1 \leq t_1 < \cdots < t_m$ and $z_1, \cdots, z_m$ such that (8) can be written as

$$\begin{aligned}\bar{a}(\bar{x}_u - 1, u) \;=\; & (a(x_n - t_1, n)\bar{a}(\bar{x}_n - t_1, n))^{z_1} \cdots \\ & (a(x_n - t_m, n)\bar{a}(\bar{x}_n - t_m, n))z_m\end{aligned}$$

which is equivalent to

$$\begin{aligned}&1 = a(x_n - t_1, n)^{z_1} \cdots a(x_n - t_m, n)^{z_m} \\ &\bar{a}(\bar{x}_u - 1, u) = \bar{a}(\bar{x}_n - t_1, n)z_1 \cdots \bar{a}(\bar{x}_n - t_m, n)^{z_m}\end{aligned}$$

However it follows from the definition of A_n that the elements

$$a(x_n - 1, n), \cdots, a(1, n)$$

are linearly independent over Z_p. Therefore for ech $1 \leq i \leq m$ the following holds in the first equality of (10). Either

$$a(x_n - t_i, n) = 1$$

or $p | z_i$. But in the first case also

$$\bar{a}(\bar{x}_n - t_i, n) = 1$$

since $\bar{x}_k < x_k$ for all $k \geq k_0$. Thus in any case

$$\bar{a}(\bar{x}_n - t_i, n)^{z_i} = 1$$

for all $1 \leq i \leq m$. This a contradiction since $\bar{a}(\bar{x}_u - 1, u) \neq 1$ This completes the proof of the theorem.

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MELDRUM GRUPLARININ BİR ÖZELLİĞİ

Özet

Bu çalışmada iki Meldrum grubunun direkt çarpımının normalleyen şartını sağlamadığı gösterilmiştir. Bu sonucun herhangi iki HM-grubu için doğru olup olmadığı henüz bilinmemektedir.

A.O. ASAR & S. ATLIHAN
Gazi Üniversitesi,
Gazi Eğitim Fakültesi,
Teknikokullar, Ankara-TURKEY

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