

1-1-1996

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### Recommended Citation

KAYA, A. (1996) "RINGS IN WHICH STRONGLY PRIME RIGHT IDEALS ARE FINITELY GENERATED," *Turkish Journal of Mathematics*: Vol. 20: No. 2, Article 8. Available at: <https://journals.tubitak.gov.tr/math/vol20/iss2/8>

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## RINGS IN WHICH STRONGLY PRIME RIGHT IDEALS ARE FINITELY GENERATED

*A. Kaya*

### Abstract

Some properties of rings in which every strongly prime right ideal is finitely generated will be investigated, and sufficient conditions for such rings to be right noetherian will be given.

Let  $R$  be an  $s$ -unital ring (i.e.,  $a \in aR \cap Ra$  for each  $a \in R$ ) and  $I (\neq R)$  a right ideal of  $R$ . If, for each right ideals  $A$  and  $B$  of  $R$ ,  $AB \subseteq I$  implies that either  $A \subseteq I$  or  $B \subseteq I$ , then  $I$  is called a prime right ideal (or equivalently, if  $aRb \not\subseteq I$  whenever  $a$  and  $b$  do not belong to  $I$ ).  $I$  is strongly prime if, for each  $a$  and  $b$  in  $R$ ,  $aIb \subseteq I$  and  $ab \in I$  imply that either  $a \in I$  or  $b \in I$ . It is known that a strongly prime right ideal is prime ([1]).

In [4], Michler has proved that a ring with unity is right noetherian if and only if every prime right ideal of  $R$  is finitely generated. In this paper we shall consider  $s$ -unital rings in which every strongly prime right ideal is finitely generated, and give sufficient conditions for such rings to be right noetherian. In doing so, we shall greatly use Michler's techniques.

In what follows,  $p(R)$  (resp.,  $sp(R)$ ) will denote the set of all prime (resp., strongly prime) right ideals of an  $s$ -unital ring  $R$ , and  $C(S)$  the complement of a nonempty subset  $S$  of  $R$ . Before starting with our main results, let's make some observations on the elements of  $sp(R)$ . As is known,  $I \in p(R)$  if and only if  $C(I)$  is an  $m$ -system. Now we are going to introduce a  $k$ -system which will help us describe the elements of  $sp(R)$ . Let  $S$  be a nonempty subset of a ring  $R$ . We say that  $S$  is a  $k$ -system if  $a, b \in S$  implies that either  $ab \in S$  or there exists  $s \notin S$  such that  $asb \in S$ . Evidently, if  $I$  is an ideal of a ring, then  $C(I)$  is a multiplicative system if and only if it is a  $k$ -system.

**Example** Let

$$R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}, I = \begin{pmatrix} Z_2 & Z_2 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix},$$

where  $Z_2$  is the ring of integers modulo 2. Here  $C(I)$  is a  $k$ -system but  $C(J)$  is not.

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AMS subject classification (1991 revision): 16 D 15, 16 P 40, Secondary: 16 P 70.  
This work has been supported by TÜBİTAK, Ankara (TBAG-C2)

An example of an  $m$ -system which is not a  $k$ -system will be provided by the following proposition, whose proof follows directly from the definition.

**Proposition 1.** *Let  $R$  be a ring and  $I(\neq R)$  a right ideal of  $R$ . Then*

$$I \in sp(R) \Leftrightarrow C(I) \text{ is a } k\text{-system.}$$

We recall that the tertiary radical,  $terM$ , of a right module  $M$  over a ring  $R$  is the ideal of  $R$  consisting of all  $r \in R$  such that  $Nr = 0$  for some essential submodule  $N$  of  $M$ .

We need the following proposition, whose proof readily follows from that of [3, Lemma 2.10].

**Proposition 2.** *Let  $R$  be a ring and  $P$  a prime ideal of  $R$ . Then*

$$P = ter(R/P).$$

When an ideal (right ideal) is finitely generated as a right  $R$ -module, we say that the ideal (right ideal) is finitely generated.

We now come to our main results, which are inspired by [4].

**Theorem 1.** *Let  $R$  be an  $s$ -unital ring. If every strongly prime right ideal and every ideal of  $R$  is finitely generated, then  $R$  is right noetherian.*

**Proof.** Suppose, to the contrary, that  $R$  is not right noetherian. Then the set  $\Gamma$  of right ideals of  $R$  which are not finitely generated is not empty, and so it has a maximal element  $I$ , by Zorn's Lemma. Being an element in  $\Gamma$ ,  $I$  is not in  $sp(R)$ . Hence there exist elements  $a \notin I$  and  $b \notin I$  such that  $aIb \subseteq I$  and  $ab \in I$ . By the maximality of  $I$ ,  $I + aR$  is finitely generated, so we can choose a system of generators

$$u_1 + ar_1, \dots, u_n + ar_n$$

for  $I + aR$ , where  $u_i \in I, r_i \in R$  for  $i = 1, \dots, n$ . It is easy to show that

$$I + aR = (u_1, \dots, u_n, a)_r,$$

where the right ideal on the right is the right ideal generated by the elements  $u_1, \dots, u_n, a$ .

We consider two cases here.

**Case 1:** If  $I$  is a prime right ideal, then from  $aIb \subseteq I$  we get

$$aIbR \subseteq I \Rightarrow aI \subseteq I \text{ or } bR \subseteq I,$$

but  $R$  is  $s$ -unital and  $b \notin I$ , so we must have

$$aI \subseteq I, \text{ that is, } I \subseteq (I : a),$$

where  $(I : a) = \{r \in R \mid ar \in I\}$ . Since  $ab \in I$  we have  $I \subsetneq (I : a)$ . Thus  $(I : a)$  is finitely generated with generators, say,

$$v_1, \dots, v_m, \text{ so } av_i \in I \text{ for } i = 1, \dots, m.$$

Now it can be readily shown that

$$I = (u_1, \dots, u_n, av_1, \dots, av_m)_r.$$

This contradicts  $I \in \Gamma$ .

**Case 2:** If  $I$  is not a prime right ideal, we can complete the proof by following Michler's one in [4, Lemma 3], for, there is a one-to-one correspondence between the set of all strongly prime right ideals of  $R$  containing the kernel of any ring homomorphism from  $R$  onto a ring  $S$  and the set  $sp(S)$  (see [2, Thm. 1]).

□

**Remark 1.** We note here that though the Lemmas 1 and 2 in [4] were stated for rings with identity, their proofs work for  $s$ -unital rings, as well.

The following lemma will play an essential role in what follows. We recall that an ideal  $A$  of a ring  $R$  is completely semiprime when  $c^2 \in A$  implies that  $c \in A$ .

**Lemma 1.** *Let  $R$  be a ring in which every completely prime ideal is a finitely generated right ideal. If there exists an ideal  $I$  of  $R$  that does not contain a product of finitely many completely prime ideals  $P_i$  of  $R$  with  $P_i \supseteq I$ , then there exists a prime ideal  $P$  of  $R$  that is not completely semiprime.*

**Proof.** Consider the nonempty set  $\Gamma$  of all ideals  $J$  of  $R$  not containing a product of finitely many completely prime ideals  $P_i$  with  $P_i \supseteq J$ . By Zorn's Lemma,  $\Gamma$  has a maximal element  $P$ .

We claim that  $P$  is the desired prime ideal.

Clearly  $P$  is not completely prime. If  $P$  were not prime, there would be ideals  $I_1$  and  $I_2$  of  $R$  such that

$$I_1 I_2 \subseteq P, \quad I_1 \not\subseteq P \text{ and } I_2 \not\subseteq P.$$

Since  $P \subset P + I_i$  for  $i = 1, 2$ , the maximality of  $P$  shows that there are completely prime ideals  $U_1, \dots, U_m$  and  $V_1, \dots, V_n$  of  $R$  such that

$$U_1U_2 \cdots U_m \subseteq P + I_1 \subseteq \bigcap_{i=1}^m U_i$$

$$V_1V_2 \cdots V_n \subseteq P + I_2 \subseteq \bigcap_{j=1}^n V_j.$$

But then

$$U_1U_2 \cdots U_mV_1V_2 \cdots V_n \subseteq (P + I_1)(P + I_2) \subseteq P \subseteq \left(\bigcap_{i=1}^m U_i\right) \cap \left(\bigcap_{j=1}^n V_j\right),$$

which implies that  $P \notin \Gamma$ , a contradiction. Therefore,  $P$  is a prime ideal. Since a prime but not completely prime ideal is not completely semiprime, we have completed the proof of Lemma 1.  $\square$

Next, we are going to give a slight generalisation of commutative rings.

Let  $R$  be a ring. We say that  $R$  satisfies (\*) when an element  $c$  is radical over a right ideal not containing  $c$ , then  $c$  is either right or left quasi-central (i.e., for each  $x \in R$  there exists  $x' \in R$  such that either  $cx = x'c$  or  $xc = cx'[5]$ ).

**Examples**

- 1) All commutative rings satisfy (\*).
- 2) Homomorphic images of rings with (\*) satisfy (\*).

3)  $R = \begin{pmatrix} Z_2 & 0 \\ Z_2 & Z_2 \end{pmatrix}.$

is a noncommutative  $s$ -unital ring (in fact, ring with unity) satisfying (\*)

4)  $R = \begin{pmatrix} Z_2 & 0 \\ Z_2 & 0 \end{pmatrix}$

is a noncommutative right  $s$ -unital (i.e.,  $a \in aR$  for each  $a \in R$ ) ring satisfying (\*).

**Theorem 2.** *Let  $R$  be a ring satisfying (\*). If every completely prime ideal of  $R$  is a finitely generated right ideal of  $R$ , then  $R$  satisfies the ascending chain condition on completely prime ideals of  $R$ .*

Before proceeding to prove Theorem 2, we need the following

**Lemma 2.** *Under the hypotheses of Theorem 2, every ideal  $I$  of  $R$  contains a product of finitely many completely prime ideals  $P_i$  of  $R$  with  $P_i \supseteq I$ .*

**Proof.** Suppose, to the contrary, that Lemma 2 is false, then we can find a prime ideal  $P$  that is not completely semiprime (Lemma 1). Hence there exists  $c \in C(P)$  such that

$c^2 \in P$ . Now, as  $R$  satisfies (\*),  $c$  is either right or left quasi-central. Hence, in the prime ring  $\bar{R} = R/P$ ,  $\bar{c}\bar{R}\bar{c} = \bar{0}$ , which implies that  $c \in P$ , a contradiction!

Now the proof of Theorem 2 follows from Lemma 2 and from a simple modification of the proof of [4, Lemma 5].  $\square$

**Lemma 3.** *Let  $R$  be an  $s$ -unital ring with (\*). Suppose that every strongly prime right ideal of  $R$  is finitely generated and that  $R/P$  is a right noetherian ring for every completely prime ideal  $P$  of  $R$ . Then  $R$  is right noetherian.*

**Proof.** By Theorem 1, it is sufficient to show that every ideal of  $R$  is finitely generated. Let  $I$  be an ideal of  $R$ . Since completely prime ideals are strongly prime right ideals ([1]), it follows from Lemma 2 that  $I$  contains a product of finitely many completely prime ideals  $P_i$  of  $R$  with  $P_i \supseteq I$  for  $i = 1, \dots, n$ . So

$$P_1 P_2 \cdots P_n \subseteq I \subseteq \bigcap_{i=1}^n P_i.$$

Besides,  $P_i$  is a finitely generated right ideal of  $R$  and  $R/P_i$  is right noetherian. Hence  $I$  is finitely generated by [4, Lemma 2]. Consequently,  $R$  is right noetherian.  $\square$

**Lemma 4.** *Let  $R$  be an  $s$ -unital ring satisfying (\*). If every strongly prime right ideal of  $R$  is a finitely generated right ideal of  $R$ , then  $R/P$  is right noetherian for every completely prime ideal of  $R$ .*

**Proof.** Since every completely prime ideal is strongly prime right ideal of  $R$  and every strongly prime right ideal of  $R$  is finitely generated,  $R$  satisfies the ascending chain condition on completely prime ideals by Theorem 2. Suppose now that Lemma 4 is false. Then the set  $\Gamma$  of those completely prime ideals of  $R$  such that  $R/P$  is not right noetherian is not empty, so it has a maximal element  $M$  by Zorn's lemma. Since every homomorphic image of an  $s$ -unital ring with (\*) is again an  $s$ -unital ring satisfying (\*) and since every strongly prime right ideal of  $R/M$  is of the form  $A/M$  for some strongly prime right ideal  $A$  of  $R$ , we may assume that  $M = 0$ . This implies that  $R$  is not right noetherian, and hence  $R$  contains an ideal which is not finitely generated right ideal of  $R$  (Theorem 1). Now we can find finitely many completely prime ideals  $P_1, P_2, \dots, P_n$  containing  $I$  such that  $P_1 P_2 \cdots P_n \subseteq I$ . Also,  $M = 0 \neq P_i$  for all  $i$  and the maximality of  $M$  shows that  $R/P_i$  is a right noetherian ring. Thus, by [4, Lemma 2],  $I$  is finitely generated right ideal of  $R$ . This contradiction shows that  $R/P$  is right noetherian for every completely prime ideal of  $R$ .  $\square$

A ring  $R$  is said to be almost commutative ( $AC$ -ring) if for any prime right ideal  $P(\neq R)$  of  $R$  and for any  $a \notin P$  there exists an element  $a'$  such that  $aa'$  is central and  $aa' \notin P$  (cf. [6]).

Noting here that every strongly prime right ideal in an  $s$ -unital ring is a prime right ideal, we readily have the following.

**Proposition 3.** *Let  $R$  be an  $s$ -unital  $AC$ -ring and  $I(\neq R)$  a right ideal of  $R$ . Then*

$$I \in p(R) \Leftrightarrow I \in sp(R).$$

We can now prove our main theorem which will conclude the paper.

**Theorem 3.** *Let  $R$  be an  $s$ -unital ring in which every strongly prime right ideal is finitely generated. Then  $R$  is right noetherian if one of the following holds:*

- a)  $R$  is an  $AC$ -ring,
- b)  $R$  satisfies (\*),
- c)  $ter(R/P) = 0$  for every prime ideal  $P$  of  $R$  (i.e.,  $R$  has no nonzero prime ideal),
- d) every ideal of  $R$  is finitely generated.

**Proof.** According to what we have proved so far, it is sufficient to prove part c) only, for, Theorem 6 in [4] works for  $s$ -unital rings as well. By Proposition 2,  $P = ter(R/P) = 0$  for every prime ideal  $P$  of  $R$ . Hence  $R$  has no nonzero prime ideal (and hence no nonzero completely prime ideal). Now, if  $R$  is not right noetherian, then by part d) (which is Theorem 1) there exists a nonzero ideal  $I$  which is not a finitely generated right ideal of  $R$ . The ideal  $I$  does not contain a product of finitely many completely prime ideals  $P_i$  of  $R$  having  $P_i \supseteq J$  is nonempty. By the proof of Lemma 1, there exists a prime ideal  $P$  of  $R$  which is a maximal element in  $\Gamma$ . Since  $R$  has no nonzero prime ideal,  $P$  should be zero. Hence  $P \subset I$ , which contradicts the maximality of  $P$ . This contradiction completes the proof of Theorem 3. □

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## KUVVETLİ ASAL SAĞ İDEALLERİ SONLU ÜRETİLMİŞ HALKALAR

### Özet

Bu çalışmada her kuvvetli asal sağ ideali sonlu üretilmiş olan halkaların bazı özellikleri araştırılmış ve böyle halkaların sağ Noether halkası olması için yeterli koşullar verilmiştir.

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Received 31.10.1994