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RELATIONS BETWEEN CLASSES OF DRAGILEV SPACES (f_1)_{\sigma} AND (f_2)_{\sigma} FOR ARBITRARY FUNCTION f^{-1}_{1}\circ f_2}

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RELATIONS BETWEEN CLASSES OF DRAGILEV SPACES $(f_1)_\sigma$ AND $(f_2)_\sigma$ FOR ARBITRARY FUNCTION $f_1^{-1} \circ f_2$

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Abstract

Necessary and sufficient condition under wich Dragilev classes $(f_1)_\sigma$ and $(f_2)_\sigma$ coincide have been found for an arbitrary function $f_1^{-1} \circ f_2$.

Introduction

A matrix (a_{pn}) of non-negative scalars satisfying:
1. $\forall n, p \quad a_{pn} \leq a_{p+1,n}$; 2.$\forall n \sup a_{pn} > 0$.
is called a Kothe matrix and the sequence space

$$\ell_1(a_{pn}) = \left\{ (\xi_n) : \| \xi_n \|_p = \sum_n |\xi_n| \cdot a_{pn} < \infty \forall p \right\}$$

topologized by seminorms $(\| \cdot \|_p)$ is called a Kothe space.

For each fixed function $f(u), u \in \mathbf{R}$, which is odd, increasing and logarithmically convex for $u \geq 0$, and for $\lambda \in \{-1, 0, 1, \infty\}$ we define the class $(f)_\lambda$ of Kothe spaces $L_f(a_n, \lambda) = \ell_1(a_{pn})$, where $a_{pn} = \exp f(\lambda_p, a_n)$, $(p, n = 1, 2, ...)$, $a_n \uparrow \infty, \lambda_p \uparrow \lambda$ (introduced by Dragilev [1] and called Dragilev spaces or generalized power series spaces). In [1] M. Dragilev proved that if logarithmically convex functions f_1, f_2, φ are connected by relation $f_1(u) = f_2(\varphi(u))$ then the classes $(f_1)_{6_1}$ and $(f_2)_{6_2}$ coincide or do not intersect. In [2] $\Big([3]\Big)$ M. Kocatepe obtained the necessary and sufficient condition when the classes $(f_1)_\infty$ and $(f_2)_1$ $\Big($correspondingly, $(f_1)_0$ and $(f_2)_{-1}\Big)$ intersect, but $f_1^{-1} \circ f_2$ is not logarithmically convex.

Function $g(u)$ is called rapidly increasing, if for $\forall \alpha > 1 \lim\limits_{u \to \infty} g(\alpha u)/g(u) = \infty$:

We shall consider rapidly increasing functions f, f_1, f_2 only but function $f_1^{-1} \circ f_2$ is arbitrary.

Following V. Zahariuta [4] we shall say that for locally convex spaces X and Y, $(X,Y) \in R$ if all continuons linear maps from X into Y are also compact.

Results.

Lemma 1. *Suppose $\ell_1(a_{pn})$ is a Schwarts Kothe space,*

$$a_{pn} \leq 1 \textit{ and}$$

$$\forall p \exists q : \log(a_{qn}) / \log(a_{pn}) \to 0 \textit{ for } n \to \infty \tag{1}$$

$$\textit{If } \exists q_0 \forall p \liminf_n \frac{f^{-1}(\log a_{pn})}{f^{-1}(\log a_{q_0 n})} \geq 1 \tag{2}$$

then the relation $(L_f(b_n, -1), \ell_1(a_{pn})) \in R$ takes place for each $(b_n) \uparrow \infty$.

Proof. We shall use methods [4], [5]. Let $b_{pn} = \exp f\left((-1-\frac{1}{p}) \cdot b_n\right)$ and $T : \ell_1(b_{pn}) \to \ell_1(a_{pn})$ be linear and continuous, then T is represented by a matrix (t_{ij}) and continuity of T gives a strictly increasing function $q = q(p)$ of $\mathbb{N} \to \mathbb{N}$ and positive constant $C(p)$ such that:

$$\sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{qn}} \leq C(p) < \infty. \tag{3}$$

We shall now use the fact that $\ell_1(a_{pn})$ is a Montel space, and for the conclusion of the lemma it is enough to show:

$$\exists q_0 \forall p \exists M(p) : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \leq M(p) < \infty. \tag{4}$$

First pick p_1 using (2) so that:

$$\forall p \liminf_n \frac{f^{-1}(\log a_{pn})}{f^{-1}(\log a_{p_1 n})} \geq 1 \tag{5}$$

For the given p_1 we pick q_1 so that (3) takes place. Let $q_0 > q_1$, we can pick $C > 0$ such that:

$$\left(1 + \frac{1}{q_0}\right) \cdot (1 + C) < \left(1 + \frac{1}{q_1}\right). \tag{6}$$

Let p be fixed and set

$$N_n = N_n(C,p) = \left\{ i : -\left(1 + \frac{1}{q_0}\right)(1 + C) \cdot b_n \leq f^{-1}(\log a_{pi}) \right\} \tag{7}$$

Now consider

$$\sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} = \sum_1 + \sum_2 ,$$

where $\sum_1$ is for $i \notin N_n$ and $\sum_2$ is for $i \in N_n$.

First we estimate $\sum_1$. Using (1) pick p_2 such that

$$\log(a_{p_2 n})/\log(a_{pn}) \to 0 \tag{8}$$

For the given p_2, pick q_2 for which the relation (3) takes place.
Consider

$$\sum_i = \sum_{i \notin N_n} |t_{in}| \frac{a_{pi}}{a_{p_2 i}} \frac{a_{p_2 i}}{b_{q_2 n}} \frac{b_{q_2 n}}{b_{q_0 n}} \leq C(p_2) \sup_{i \notin N_n} \exp 6^{(1)}_{in}(p),$$

where

$$\begin{aligned} 6^{(1)}_{in}(p) &= \log a_{pi} - \log a_{p_2 i} + f\left[\left(-1 - \frac{1}{q_2}\right) \cdot b_n\right] - \\ &- f\left[\left(-1 - \frac{1}{q_0}\right) \cdot b_n\right] \leq \log a_{pi} - \log a_{p_2 i} + \\ &+ f\left[\left(1 + \frac{1}{q_0}\right) \cdot b_n\right] \leq \log a_{pi} - \log a_{p_2 i} - \\ &- f\left[\left(\frac{1}{C+1}\right) \cdot f^{-1}(\log a_{pi})\right]. \end{aligned}$$

Write $\phi = f^{-1} \circ \log$, we have now

$$6^{(1)}_{in} \leq \log a_{pi} \left[1 - \frac{\log a_{p_2 i}}{\log a_{pi}} - \frac{f\left(\frac{1}{1+C}\phi(a_{pi})\right)}{f\left(\phi(a_{pi})\right)}\right]$$

$f(u)$ is rapidly increasing, that is $6^{(1)}_{in} \leq 0$ for $i \geq i_0, i \notin N_n; 6^{(1)}_{in}(p)$ is bounded, and thus $\sup_n \sum_1 \leq M_1(p) < \infty$.

Now consider $\sum_2$:

$$\sum_2 = \sum_{i \in N_n} |t_{in}| \frac{a_{pi}}{a_{p_1 i}} \frac{a_{p_1 i}}{b_{q_1 n}} \frac{b_{q_1 n}}{b_{q_0 n}} \leq C(p_1) \sup_{i \in N_n} \exp 6^{(2)}_{in},$$

where

$$6^{(2)}_{in} = \log a_{pi} - \log a_{p_1 i} + f\left[\left(-1 - \frac{1}{q_1}\right) \cdot b_n\right] - f\left[\left(-1 - \frac{1}{q_0}\right) \cdot b_n\right].$$

If $p < p_1$, then $6^{(2)}_{in} \leq 0$. Otherwise, we can find $\epsilon > 0$ from (6) such that

$$\left(1 + \frac{1}{q_0}\right)] \cdot \frac{1+C}{1-\epsilon} < 1 + \frac{1}{q_1}.$$

For $i \geq i_0$ from (5) we have

$$f^{-1}(\log a_{pi}) \leq (1-\epsilon) \cdot f^{-1}(\log a_{p_1 i}).$$

In other words

$$\begin{aligned} 6^{(2)}_{in} &\leq -\log a_{p_1 i} - f\left[\left(1 + \frac{1}{q_1}\right) \cdot b_n\right] + f\left[\left(1 + \frac{1}{q_0}\right) \cdot b_n\right] \leq \\ &\leq -f\left[\left(\frac{1}{1-\epsilon}\right) \cdot f^{-1}(loga_{pi})\right] + f\left[\left(1 + \frac{1}{q_0}\right) \cdot b_n\right] - \\ &- f\left[\left(1 + \frac{1}{q_1}\right) \cdot b_n\right] \leq f\left[\left(1 + \frac{1}{q_0}\right) \frac{1+C}{1-\epsilon} \cdot b_n\right] + \\ &+ f\left[\left(1 + \frac{1}{q_0}\right) \cdot b_n\right] - f\left[\left(1 + \frac{1}{q_1}\right) \cdot b_n\right]. \end{aligned}$$

$f(u)$ is rapidly increasing, also $6^{(2)}_{in}$ is bounded, and thus $\sup_n \sum_2 \leq M_2(p) < \infty$. □

Lemma 2. *Let $\ell_1(b_{pn})$ be a Schwarts K"othe space, $b_n \equiv 1 (n = 1, 2, ...)$, $f(u)$ be rapidly increasing function and*

$$\forall p \exists q : \log(b_{pn}) / \log(b_{qn}) \to 0 \; for \; n \to \infty \tag{9}$$

$$If \; \exists_{q_0} \forall_p \limsup_n \frac{f^{-1}(\log b_{pn})}{f^{-1}(\log b_{q_0 n})} \leq 1 \tag{10}$$

then the relation $(\ell, (b_{pn}), L_f(a_n, 1)) \in R$ takes place for each $(a_n), a_n \uparrow \infty$.

Proof. We pick out arbitrary sequence $(a_n), a_n \uparrow \infty$ and let

$$a_{pn} = \exp f((1 - \frac{1}{p})a_n) \qquad (n, p = 1, 2, ...).$$

Let further $T : \ell_1(b_{pn}) \to \ell_1(a_{pn})$ be a linear continuous operator. Then

$$\forall_p \exists_q : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{q_n}} \le C(p) < \infty. \tag{11}$$

For Montel space $\ell_1(a_{pn})$ we need to prove that

$$\exists q_0 \forall_p \exists_{M(p)} : \sup_n \sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \le M(p) < \infty. \tag{12}$$

For fixed index p_1 (for example, $p_1 = 1$) we can find q_1 such that condition (3) takes place. From (1), (2) and using the Schwarts property we can pick up $q_0 > q_1$ such that when $n \to \infty$ we have

$$\log(b_{q_1,n}) / \log(b_{q_0,n}) \to 0, \tag{13}$$

$$f^{-1}(\log b_{qn}) / f^{-1}(\log b_{q_0,n}) \to 1 \tag{14}$$

for every $q > q_0$, and $b_{q_0,n} \to \infty$

Further $\forall p \exists C \quad (0 < C < 1)$ such that

$$(1 - \frac{1}{p})/(1 - C) < 1 - \frac{1}{p+1} \tag{15}$$

Let

$$N_h = N_h(C,p) = \{i : (1 - \frac{1}{p})a; < (1 - Cf^{-1}(\log b_{q_0 n})\} \tag{16}$$

Let further

$$\sum_i |t_{in}| \frac{a_{pi}}{b_{q_0 n}} = \sum_1 + \sum_2$$

where

$$\sum_1 = \sum_{i \notin N_h} |t_{in}| \frac{a_{pi}}{b_{q_0 n}} \text{ and } \sum_2 = \sum_{i \in N_u} |t_{in}| \frac{a_{pi}}{b_{q_0 n}}$$

We begining from the sum $\sum_1$.

For every p exists P_2 such that $p_2 > p+1$. Using (3) for p_2 we pick q_2. Further we have

$$\sum_1 = \sum_{i \notin N_n} |t_{in}| \frac{a_{pi}}{a_{p_2 i}} \cdot \frac{a_{p_2 i}}{b_{q_2 n}} \cdot \frac{b_{q_2 n}}{b_{q_0 n}} \le C(p_2) \sup_{i \notin N_n} \exp 6_{in}^{(1)}(p),$$

where

$$
\begin{aligned}
6_{in}^{(1)}(p) &= f((1-\frac{1}{p})a_i) - f((1-\frac{1}{p_2}a_i + \log b_{q_2n} - \log b_{q_0n} \leq \\
&\leq f((1-\frac{1}{p})a_i - f((1-\frac{1}{p_2})a_i) + \log b_{q_2n}
\end{aligned}
$$

Using (6), (7) we can find ϵ such that

$$
f^{-1}(\log b_{q_2n}) \leq (1+\epsilon)f^{-1}(\log b_{q_0n} \tag{17}
$$

for sufficiently large n and

$$
\frac{(1-\frac{1}{p})\cdot(1+\epsilon)}{1-c} < 1 - \frac{1}{p+1} < 1 - \frac{1}{p_2} \tag{18}
$$

The inequality (9) we rewrite in more convenient form

$$
\log b_{q_2n} \leq f((1+\epsilon)f^{-1}(\log b_{q_0n})) \tag{19}
$$

Using (11) and the inequalities (8), (10) we have

$$
\begin{aligned}
6_{in}^{(1)} &\leq f((1-\frac{1}{p})a_i) - f((1-\frac{1}{p_2}a_i) + \\
&+ f((1+\epsilon)f^{-1}(\log b_{q_0n})) \leq \\
&\leq f((1-\frac{1}{p})a_i) - f((1-\frac{1}{p_2})a_i) + f((1-\frac{1}{p+1})a_i) \leq \\
&\leq 0
\end{aligned}
$$

for sufficiently large n, becausse f is a rapidly increasing function.

Now we consider $\sum_2$.

$$
\begin{aligned}
\sum_2 &= \sum_{i\in N_u} |t_i n| \frac{a_{pi}}{a_{p_1 i}} \cdot \frac{a_{p_1 i}}{b_{q_1 n}} \cdot \frac{b_{q_1 n}}{b_{q_0 n}} \leq \\
&\leq C(P_1) \sup_{i\in N_n} \exp 6_{in}^{(2)},
\end{aligned}
$$

where

$$
\begin{aligned}
6_{in}^{(2)} &= f((1-\frac{1}{p})a_i) - f((1-\frac{1}{p_1}a_i) + \log b_{q_1n} - \log b_{q_0n} \leq \\
&\leq f((1-\frac{1}{p})a_i) + \log b_{q_1n} - \log b_{q_0n}
\end{aligned}
$$

Using inequality (18) we have

$$6_{in}^{(2)} \leq f((1-C)f^{-1}(\log b_{q_0 n})) + \log b_{q_1 n} - \log b_{q_0 n}$$

Introducing new function $\phi(u) = f^{-1}(\log u)$, we obtain

$$\begin{aligned} 6_{in}^{(2)} &= f((1-C)\phi(b_{q_0 n})) + f(\phi(b_{q_1 n})) - f(\phi(b_{q_0 n})) = \\ &= f(\phi(b_{q_0 n})) \cdot \left(\frac{f((1-C)\phi(b_{q_0 n}))}{f(\phi(b_{q_0 n}))} + \frac{\log b_{q_1 n}))}{\log b_{q_0 n}))} - 1 \right) \end{aligned}$$

Using the relation (5) we obtain that $6_{in}^{(2)} \leq 0$ for sufficiently large n, because f is a rapidly increasing function. □

Theorem 1. *The classes $(f_1)_{-1}, (f_2)_{-1}$ coincide iff $\forall \alpha > 0$ the two inequalities take place:*

$$\liminf_{u} \frac{f_2^{-1}\left(f_1((1+\alpha)\cdot u)\right)}{f_2^{-1}(f_1(u))} > 1 \tag{20}$$

and

$$\liminf_{u} \frac{f_1^{-1}\left(f_2((1+\alpha)\cdot u)\right)}{f_1^{-1}(f_2(u))} > 1 \tag{21}$$

Proof. Sufficiency. For arbitrary sequence $(a_n) \uparrow \infty$ we consider sequence $b_n = f_2^{-1}(f_1(a_n))$. It is obvious that $(b_n) \uparrow \infty$. From (9) we have

$$\forall p \exists q, q > 1 : \frac{f_2^{-1}\left(f_1((1+\frac{1}{p})\cdot u)\right)}{f_2^{-1}(f_1(u))} > 1 + \frac{1}{q}$$

for sufficiently large u. That is

$$f_2^{-1}\left(f_1((1+\frac{1}{p})\cdot u)\right) \geq \left(1+\frac{1}{q}\right) f_2^{-1}(f_1(u)),$$

$$f_1((1+\frac{1}{p}).u) \geq f_2\left((1+\frac{1}{q})f_2^{-1}(f_1(u))\right),$$

$$f_1((-1-\frac{1}{p}).a_n) \leq f_2\left((-1-\frac{1}{q})\cdot b_n\right).$$

Analogusly, from (10) we have

$$\forall p \exists q, q > 1 : f_1^{-1}\left(f_2((1+\frac{1}{p})\cdot u)\right) \le \left(1+\frac{1}{q}\right) f_1^{-1}(f_2(u))$$

or $f_2((-1-\frac{1}{p}).b_n) \le f_1\left((-1-\frac{1}{q})\cdot a_n\right)$. for sufficiently large n.
Necessity.

Assume, for example, that condition (20) does not take place, that is $\exists \alpha > 0$ and $(a_n) \uparrow \infty$ such that

$$\lim_{n\to\infty} \frac{f_2^{-1}\left(f_1((1+\alpha)a_n)\right)}{f_2^{-1}(f_1(a_n))} = 1.$$

Let $\ell_1(a_p n)$ be a Kothe space with $a_{pn} = \exp f_1\left[(-1-\frac{1}{p})a_n\right]$.
The function $f_1(u)$ is rapidly increasing, therefore the condition (1) takes place. Let $q_0 = 1/\alpha$, then

$$\liminf_n \frac{f_2^{-1}(\log a_{pn})}{f_2^{-1}(\log a_{q_0 n})} = \liminf_n \frac{f_2^{-1}\left(f_1((-1-\frac{1}{p})a_n)\right)}{f_2^{-1}\left(f_1((-1-\frac{1}{q_0})a_n)\right)} \ge \liminf_n \frac{f_2^{-1}\left(f_1(a_n)\right)}{f_2^{-1}\left(f_1((1+\frac{1}{q_0})a_n)\right)} = 1.$$

It means that the condition (2) of lemma 1 takes place and relation $(L_{f_2}(b_n,-1), L_{f_1}(a_n,-1)) \in R$ for every sequence (b_n). The last means that $L_{f_1}(a_n,-1)$ is not isomorphic to some $L_{f_2}(b_n,-1)$ or classes $(f_1)_{-1}$ and $(f_2)_{-1}$ is not coicide. □

Theorem 2. *The classes $(f_1)_{-1}$ and $(f_2)_{-1}$ do not intersect if one of the two conditions takes place:*

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_2^{-1}\left(f_1((1+\frac{1}{q})\cdot a_{n(j)})\right)}{f_2^{-1}(f_1(a_{n(j)}))} = 1 \tag{22}$$

or

$$\forall (a_n) \exists (n(j)) \exists q > 1 : \lim_j \frac{f_1^{-1}\left(f_1((1+\frac{1}{q})\cdot a_{n(j)})\right)}{f_1^{-1}(f_2(a_{n(j)}))} = 1.$$

Proof. Assume that there are two isomorphic spaces $L_{f_1}(a_n,-1)$ and $L_{f_2}(b_n,-1)$ but the condition (11) holds.

Let $\exists q \exists (n(j))$:

$$\liminf_j \frac{f_2^{-1}\Big(f_1((-1-\frac{1}{p})\cdot a_{n(j)})\Big)}{f_2^{-1}\Big(f_1((-1-\frac{1}{q})\cdot a_{n(j)})\Big)} \geq$$

$$\geq \liminf_j \frac{f_2^{-1}(f_1(a_{n(j)})}{f_2^{-1}\Big(f_1((1+\frac{1}{q})\cdot a_{n(j)})\Big)} = 1.$$

Using lemma 1 for a $a_{pj} = \exp f_1((-1-\frac{1}{p})a_{n(j)}$ we have $(L_{f_2}(b_n,-1), L_{f_1}(a_{n(j)},-1)) \in R$. Now it is easy to show that $L_{f_1}(a_n,-1)$ is not isomorphic to $L_{f_2}(b_n,-1)$ and we get a contradiction.
The next theorems are proved analogously. □

Theorem 3. *The classes $(f_1)_1, (f_2)_1$ coincide iff $\forall\alpha > 0$ the two inequalities take place:*

$$\liminf_u \frac{f_2^{-1}\Big(f_1((1+\alpha)\cdot u)\Big)}{f_2^{-1}(f_1(u))} > 1$$

and

$$\liminf_u \frac{f_1^{-1}\Big(f_2((1+\alpha)\cdot u)\Big)}{f_1^{-1}(f_2(u))} > 1$$

Theorem 4. *The classes $(f_1)_1$ and $(f_2)_1$ do not intersect if one of the two conditions takes place:*

$$\forall(a_n)\exists(n(j))\exists q > 1 : \lim_j \frac{f_2^{-1}\Big(f_1((1-\frac{1}{q})\cdot a_{n(j)})\Big)}{f_2^{-1}(f_1(a_{n(j)}))} = 1$$

or

$$\forall(a_n)\exists(n(j))\exists q > 1 : \lim_j \frac{f_1^{-1}\Big(f_1((1-\frac{1}{q})\cdot a_{n(j)})\Big)}{f_1^{-1}(f_2(a_{n(j)}))} = 1$$

The same theorems for classes $6 = 0, \infty$ were obtained in [6].

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DRAGİLEV UZAYLARI ARASINDAKİ BAZI BAĞLANTILAR

Özet

Bu çalışmada keyfi f_1 ve f_2 için $(f_1)_\sigma$ ve $(f_2)_\sigma$ Dragilev uzaylarının çalışmaları için yeter ve gerek koşullar verilmektedir.

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