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ON CONFORMALLY FLAT LORENTZIAN SPACES SATISFYING A CERTAIN CONDITION ON THE CURVATURE TENSOR

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Abstract

In this paper we prove a local classification theorem for the conformally flat lorentzian spaces satisfying the condition $R(X,Y)R = 0$.

Let M be a Riemannian space and R be the curvature tensor of M. Assume that M has a condition

(*) $R(X,Y)R = 0$, for any tangent vectors X and Y,

where R denotes the Riemannian curvature tensor and $R(X,Y)$ operates on the tensor algebra at each point as a derivation. K. Nomizu [3] studied the effect of this condition for hypersurfaces in the Euclidean spaces. P.J. Ryan [5] treated the same condition for hypersurfaces of spaces of non-zero constant curvature. On the other hand, some authors discussed the effect of the condition

(**) $R(X,Y)Q = 0$, for any tangent vectors X, Y

for hypersurfaces of the Euclidean space, where Q denotes the Ricci tensor (see [6], [7], [8]). In [1], the author and T. Ikawa classified conformally flat Lorentzian spaces satisfying the condition (**).

The purpose of this paper is to consider the condition (*) in Lorentzian space and prove

Theorem. *Let M^n be an n-dimensional ($n > 3$) complete conformally flat Lorentzian space satisfying the condition (*). Then M^n is one of the followings:*
(1) A Lorentzian space of constant curvature.
(2) Locally a product space of an m-dimensional Lorentzian (or resp. Riemannian) space of constant curvature K and an $(n-m)$-dimensional Riemannian (or resp. Lorentzian) space of constant curvature $-K$.
(3) Locally a product space of $(n-1)$-dimensional Lorentzian (or resp. Riemannian) space of constant curvature and a 1-dimensional Riemannian (or resp. Lorentzian) space.

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§1. Preliminaries.

Let M^n be an n-dimensional $(n > 3)$ complete Lorentzian space. The Lorentzian metric of M with signature $(-,+,\cdots,+)$ will be denoted by g. The Riemannian curvature tensor of M will be denoted by R. If M is conformally flat, then the curvature tensor R satisfies

(1.1) $R(X,Y) = (1/(n-2))(QX \cdot Y + X \cdot QY) - (TrQ/(n-1)(n-2))X \cdot Y$

for any tangent vectors fields X and Y, where Q denotes a field of symmetric endomorphism which corresponds to the Ricci tensor Ric, that is $\mathrm{Ric}(X,Y) = g(QX,Y)$ and $X \cdot Y$ denotes the endomorphism which maps Z upon $g(Y,Z)X - g(X,Z)Y$.

The condition (*) gives for all vectors X, Y, Z, V, W tangent to M that

(1.2) $R(X,Y)R(Z,V)W - R(Z,V)R(X,Y)W - R(R(X,Y)Z,V)W - R(Z,R(X,Y)V)W = 0.$

Using (1.1) and (1.2), we then obtain the following equation:

(1.3)

$$\begin{aligned}
&[g(V,W)g(QY,QZ) - g(Z,W)g(QY,QV)]X\\
&+[g(Z,W)g(QX,QV) - g(V,W)g(QX,QZ)]Y\\
&+[g(X,V)g(Q^2Y,W) - g(Y,V)g(Q^2X,W) - g(Y,W)g(QV,QX) + g(X,W)g(QV,QY)]Z\\
&+[g(Y,Z)g(Q^2X,W) - g(X,Z)g(Q^2Y,W) + g(Y,W)g(QZ,QX) - g(X,W)g(QZ,QY)]V\\
&+[g(Y,V)g(Z,W) - g(Y,Z)g(V,W)]Q^2X + [g(X,Z)g(V,W) - G(X,V)g(Z,W)]Q^2Y\\
&-(TrQ/(n-1))\{[g(V,W)g(Y,QZ) - g(Z,W)g(Y,QV)]X\\
&+[g(Z,W)g(X,QV) - g(V,W)g(X,QZ)]Y\\
&+[g(X,W)g(QV,Y) - g(Y,W)g(V,QX) + g(X,V)g(QY,W) - g(Y,V)g(QX,W)]Z\\
&+[g(Y,W)g(QX,Z) - g(X,W)g(QZ,Y) + g(Y,Z)g(QX,W) - g(X,Z)g(QY,W)]V\\
&+[g(Y,V)g(Z,W) - g(Y,Z)g(V,W)QX + [g(X,Z)g(V,W) - g(X,V)g(Z,W)]QY\} = 0.
\end{aligned}$$

Since g is the Lorentzian metric and Q is a symmetric endomorphism of the tangent space T_pM, Q has one of the following four forms [2], [4]:

$$Q_p = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} \qquad (1)$$

$$Q_p = \begin{bmatrix} a & b & & & \\ -b & a & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{bmatrix} \qquad (2)$$

$$(b \neq 0)$$

$$Q_p = \begin{bmatrix} a & 0 & & & \\ 1 & a & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{bmatrix} \tag{3}$$

$$Q_p = \begin{bmatrix} a & 0 & 0 & & & \\ 0 & a & 1 & & & \\ -1 & 0 & a & & & \\ & & & a_4 & & \\ & & & & \ddots & \\ & & & & & a_n \end{bmatrix} \tag{4}$$

In cases (1) and (2), Q_p is represented with respect to an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$; i.e., they satisfy $g(e_1, e_1) = -1$, $g(e_i, e_j) = \delta_{ij}$, $g(e_1, e_j) = 0, (2 \leq i, j \leq n)$. In cases (3) and (4), Q_p is represented with respect to a pseudo-orthonormal frame $\{u_1, u_2, \cdots, u_n\}$; i.e., they satisfy $g(u_1, u_1) = g(u_2, u_2) = g(u_1, u_i) = g(u_2, u_i) = 0$, $g(u_1, u_2) = 1, g(u_i, u_j) = \delta_{ij}(3 \leq i, j \leq n)$.

§2. Proof of Theorem.

The proof of theorem will be divided into four parts, according to the four possible forms of Q.

(1) Suppose that Q_p is of the form (1). If $a = a_j$ for any j, $(1 \leq j \leq n)$, then Q reduces to $Q = aI$, where I is the identity transformation. Hence M^n is Einstein, and from (1.1), it follows that M^n is a space of constant curvature. By taking $X = Z = e_1, V = e_2, Y = W = e_j, (3 \leq j \leq n)$ in (1.3), at each point, it follows that

(2.1) $\quad (a_j^2 - a_1^2) - (TrQ/(n-1))(a_j - a_1) = 0.$

By putting $X = Z = e_1, Y = e_2$ and $V = W = e_j, (3 \leq j \leq n)$ in (1.3), we also get

(2.2) $\quad (a_2^2 - a_1^2) - (TrQ/(n-1))(a_2 - a_1) = 0.$

Then, for any $j (2 \leq j \leq n)$ we have

$$(a_j^2 - a_1^2) - (TrQ/(n-1))(a_j - a_1) = 0.$$

Therefore, if a_j and a_1 are distinct eigenvalues of Q, it follows that

(2.3) $\quad a_j = [TrQ/(n-1)] - a_1.$

Now, if $a_1 = a_2 = \cdots = a_{n-m} =: a$ and $a_{n-m+1} = \cdots = a_n =: b$, then (2.1) implies that

(2.4) $\quad a + b - TrQ/(n-1) = 0$ or equivalently
(2.5) $\quad (n-m-1)b = (1-m)a$.

If $a = 0$, (2.5) implies $b = 0$ or $m = n-1$, that is

$$Q_p = \begin{bmatrix} 0 & & & & \\ & b & & & \\ & & b & & \\ & & & \ddots & \\ & & & & b \end{bmatrix} \tag{2.6}$$

Otherwise, (2.5) implies that

$$Q_p = \begin{bmatrix} a & & & & & \\ & \ddots & & & & \\ & & a & & & \\ & & & b & & \\ & & & & \ddots & \\ & & & & & b \end{bmatrix} \tag{2.7}$$

where $m \neq 1$ and $ab < 0$.

Now, first let us consider the case (2.7). If

$W = \{x \in M^n : Q_x \text{ has the form } (2.7)\}$,

then W is an open set by the continuity argument for the characteristic polynomial of Q. We denote a connected component of W by W_0.
On W_0, two distributions T_1 and T_2 are defined by

$$T_1(x) = \{X \in T_xM : QX = a(x)X\}$$
$$T_2(x) = \{X \in T_xM : QX = b(x)X\}$$

The restrictions of the metric on T_xM to $T_1(x)$ and $T_2(x)$ are nondegenerate. Then m is constant, $a(x)$ and $b(x)$ are smooth functions on W_0, therefore $T_1(x)$ and $T_2(x)$ are $(n-m)$ and m-dimensional distributions of T_xM which are involutive and smooth. Thus, by the theorem of Frobenius, there are maximal integral submanifolds M^{n-m} and M^m of $T_1(x)$ and $T_2(x)$ for every point x of M^n.

For $Z, V \in T_1(x)$, from (1.1), since M^n is conformally flat and $X \in T_1(x)$ satisfies $QX = a(x)X$, we have $R(Z,V) = K(Z \cdot V)$, $K = (a-b)/(n-2)$. Similarily, for $T, W \in T_2(x)$ we have $R(Z,W) = -K(T \cdot W)$. By the second Bianchi identity, we can see that K is constant. Therefore, M^n is locally a product space of an m-dimensional Lorentzian (resp. Riemannian) space of constant curvature K and an $(n-m)$ -dimensional Riemannian (resp. Lorentzian) space of constant curvature $-K$.

Next assume that the rank of Q is $n-1$ at some point x. Namely, let Q_x is given as in the case (2.6).

If $W = \{x \in M :$ the rank of Q is $n-1$ at $x\}$, then W is open and non-zero eigenvalue of Q, say λ, is a smooth function on W. Two distributions T_1 and T_0 on W are defined by

$$\begin{aligned} T_1(x) &= \{X \in T_xM : QX = \lambda(x)X\} \\ T_0(x) &= \{X \in T_xM : QX = 0\}. \end{aligned}$$

Then, it follows that they are smooth, T_1 is involutive and geodesic whose tangent belongs to T_0 is infinitely extendible. Moreover, T_1 and T_0 are parallel. The restrictions of the metric on T_xM to $T_1(x)$ and $T_0(x)$ are non-degenerate. Hence T_1 (resp. T_0) has maximal integral submanifolds M^{n-1} (resp. M^1) of M^n. Since M^n is conformally flat and $X \in T_1(x)$ satisfies $QX = \lambda(x)X$, from (2.2), for $a_1 = 0$ and $a_2 = \lambda$, using (1.1), M^{n-1} has constant curvature $K = \lambda/(n-2)$. Therefore, M^n is locally a product space of $(n-1)$ -dimensional Lorentzian (or resp. Riemannian) space M^{n-1} of constant curvature K and a 1-dimensional Riemannian (or resp. Lorentzian) space M^1.

(2) Let us consider that Q_x is of the form (2). Then it follows that

$$Qe_1 = ae_1 - be_2, Qe_2 = be_1 + ae_2 \text{ and } Qe_j = a_je_j \qquad (j = 3, \cdots, n).$$

By taking $X = Z = e_1$, $V = e_2, Y = W = e_j$, $(3 \leq j \leq n)$ in (1.3), at each point, we get $a_j^2 - a^2 + b^2 - (TrQ/(n-1))(a_j - a) = 0$ and $a = (TrQ)/2(n-1)$. From these equations we obtain that

$$(a_j - a)^2 + b^2 = 0.$$

This contradicts the assumption that $b \neq 0$. Thus, this case can not occur.

(3) Suppose that Q_x is of the form (3). Then it follows that

$$Qu_1 = au_1 + u_2, Qu_2 = au_2, Qu_j = a_ju_j (j = 3, \cdots, n).$$

By taking $X = Z = u_1, V = u_2$ and $Y = W = u_j$ in (1.3), we have that

(2.8) $\qquad 2a = (TrQ)/(n-1).$

Next, again putting $X = Z = u_2, V = u_1$ and $Y = W = u_j$ in (1.3), we also get $(TrQ)/(n-1) = a + a_j$. By virtue of (2.8), it follows that $a_j = a$ for all j. Then, since $TrQ = na$ from the third form of Q_x, using (2.8), we write $2a = (na)/(n-1)$ or $a((n-2)/(n-1)) = 0$. So, for $n = dimM > 3$, this implies that $a = 0$. Therefore, it follows that $Q_x = 0$ and that $R(X, Y) = 0$ for any tangent vectors X and Y by virtue of (1.1).

(4) Finally, we suppose that Q_x is of the form (4). Then we write that

$Qu_1 = au_1 - u_3, Qu_2 = au_2, Qu_3 = u_2 + au_3$ and $Qu_j = a_ju_j, (j = 4, \cdots, n)$. Putting $X = Z = u_j, Y = W = u_1, V = u_2$ in (1.3) we have $2a = (TrQ)/(n-1)$ and taking $X = Z = u_2, V = u_1, Y = W = u_j$ we obtain $a_j + a = (TrQ)/(n-1)$ for all

$j, (4 \leq j \leq n)$. Hence we easily see that $a = 0$ by virtue of $\dim M > 3$. Therefore it follows that $Qu_1 = -u_3, Qu_3 = u_2$ and $Qu_j = 0$ for any j other than 1 and 2. Thus we may conclude that $R(X, Y) = 0$ for any tangent vectors X and Y by virtue of (1.1).

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EĞRİLİK TENSÖRÜ BELİRLİ BİR ŞARTI SAĞLAYAN KONFORMAL FLAT LORENTZ UZAYLARI HAKKINDA

Özet

Bu çalışmada $R(X, Y)R = 0$ şartını sağlayan konformal flat lorentz uzayları için lokal bir sınıflandırma teoremi ispatlanmıştır.

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