

1-1-1996

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M. BARAN

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Recommended Citation

BARAN, M. (1996) "T_2-OBJECTS IN CATEGORIES OF FILTER AND LOCAL FILTER CONVERGENCE SPACES," *Turkish Journal of Mathematics*: Vol. 20: No. 2, Article 3. Available at: <https://journals.tubitak.gov.tr/math/vol20/iss2/3>

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T_2 -OBJECTS IN CATEGORIES OF FILTER AND LOCAL FILTER CONVERGENCE SPACES

M. Baran

Abstract

There are various generalizations of the usual T_2 (Hausdorff) axiom of topology to an arbitrary topological category defined in [1]. In this paper, four more new generalizations of Hausdorff spaces are given in the topological categories. Moreover, the relationships among all of these T_2 structures and the ones given in [5] and [7] as well as some invariance properties of them are investigated in the categories of filter and local filter convergence spaces.

Keywords: Topological Categories, T_2 -objects, Filter Convergence Spaces.

Introduction

There are four ways of generalizations of the usual T_2 (Hausdorff) axiom of topology to an arbitrary topological category defined in [1]. The explicit characterizations of each of these T_2 -objects in the categories of Filter and Local Filter convergence Space, FCO and LFCO, are given in [3] and [4].

Nel [7] has showed that a full subcategory, HLFCO of LFCO whose objects determined by the Hausdorff convergence spaces is a cartesian closed initial structured category. We generalize this result to full subcategories, T'_2 LFCO, ΔT_2 LFCO, and ST_2 LFCO of LFCO whose objects are determined by the T'_2 , ΔT_2 , and ST_2 structures, respectively. We have also shown that each of these subcategories is epireflective. Furthermore, we have shown the following:

1. To define four more new T_2 structures in arbitrary topological categories and to characterize each of them in FCO and LFCO.
2. To investigate the relationships among these various T_2 structures in FCO and LFCO.
3. To determine the invariance properties (i.e closed under formation of products, subspaces, and quotient spaces) of each of T_2 structures in FCO and LFCO.

General relationships among these various T_2 for arbitrary topological categories will be established in a subsequent paper.

Let E and B be categories. The functor $U : E \rightarrow B$ is said to be topological if it is concrete (i.e faithful and amnestic (i.e if $U(f) = \text{id}$ and f is an isomorphism, then

$f = \text{id}$)), has small (i.e sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-sink has a final lift [6] p.279.

Recall, [6] p.279, that an object X of E is called indiscrete (resp. discrete) iff every map from an E-object to X (resp. every map from X to an E-object) is an E-morphism.

Let X be a nonempty set and $X^2 = X \times X$ be the cartesian product of X with itself, and $X^2V_\Delta X^2$ be two distinct copies of X^2 identified along the diagonal. A point (x, y) in $X^2V_\Delta X^2$ will be denoted by $(x, y)_1$ $(x, y)_2$ if (x, y) is in the first (resp. the second) component of $X^2V_\Delta X^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$.

1.1 Definitions *The principal axis map $A : X^2V_\Delta X^2 \rightarrow X^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : X^2V_\Delta X^2 \rightarrow X^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$, and the fold map, $\nabla : X^2V_\Delta X^2 \rightarrow X^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$.*

Let $U : E \rightarrow \text{Sets}$, the category of sets, be topological and X an object in E with $UX = B$.

1.2 Definitions 1. X is T_\emptyset iff X does not contain an indiscrete subspace with (at least) two points [9] p.316.

2. X is T'_\emptyset iff the initial lift of the U-source $\{ \text{id} : B^2V_\Delta B^2 \rightarrow U(B^2V_\Delta B^2)' = B^2V_\Delta B^2 \text{ and } \nabla : B^2V_\Delta B^2 \rightarrow UD(B^2) = B^2 \}$ is discrete, where $(B^2V_\Delta B^2)'$ is the final lift of the U-sink $\{ i_1, i_2 : U(X^2) = B^2 \rightarrow B^2V_\Delta B^2 \}$, [1] p.338.

3. X is \bar{T}_\emptyset iff the initial lift of the U-source $\{ A : B^2V_\Delta B^2 \rightarrow U(X^3) = B^3 \text{ and } \nabla : B^2V_\Delta B^2 \rightarrow UD(B^2) = B^2 \}$ is discrete, [1] p.338.

4. X is $\text{pre}T'_2$ iff the initial lift of the U-source $\{ S : B^2V_\Delta B^2 \rightarrow U(X^3) = B^3 \}$ and the final lift of the U-sink $\{ i_1, i_2 : U(X^2) = B^2 \rightarrow B^2V_\Delta B^2 \}$ agree [1] p.338.

5. X is $\text{Pre}\bar{T}_2$ iff the initial lift of the U-sources $\{ S : B^2V_\Delta B^2 \rightarrow U(X^3) = B^3 \}$ and $\{ A : B^2V_\Delta B^2 \rightarrow U(X^3) = B^3 \}$ agree [1] p.338.

6. X is T'_2 iff X is T'_\emptyset and $\text{Pre}T'_2$ [1] p.338.

7. X is \bar{T}_2 iff X is \bar{T}_\emptyset and $\text{Pre}\bar{T}_2$ [1] p.338.

8. X is ΔT_2 iff the diagonal, Δ , is closed in X^2 [1] p.338 or [4].

9. X is ST_2 iff the diagonal, Δ , is strongly closed in X^2 [1] p.338. or [4].

10. X is KT_2 iff X is T'_\emptyset and $\text{Pre}\bar{T}_2$.

11. X is LT_2 iff X is \bar{T}_\emptyset and $\text{Pre}T'_2$.

11. X is MT_2 iff X is T_\emptyset and $\text{Pre}T'_2$.

12. X is NT_2 iff X is T_\emptyset and $\text{Pre}\bar{T}_2$.

1.3 Remark For the category of topological spaces, we have:

1. All of the T_\emptyset 's in 1.2 are equivalent and they reduce to the usual T_\emptyset (see [1] p.338 Theorem 2.2.11(1) and [9] p. 316).

2. All of the T_2 's in 1.2 are equivalent and they reduce to T_2 , the Hausdorff condition. This follows from part (1) and [1] p.338.

Let A be a set and K a function whose value at each x in A is a set $K(x)$ of filters on A "convergent to x " such that:

1. $[x] \in K(x)$ for each $x \in A$ (where $[x] = \{B \subset A : x \in B\}$).
2. $\beta \supset \alpha \in K(x)$ implies $\beta \in K(x)$ for any filter β on A .

Then (A, K) is called a filter convergence space [8], [3] or [7]. A map $f : (A, K) \rightarrow (B, L)$ between filter convergence spaces is called continuous iff $\alpha \in K(x)$ implies $f(\alpha) \in L(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The category of filter convergence spaces and continuous maps is denoted by FCO (see [8] p.354). A filter convergence space (A, K) is said to be a local filter convergence space if $\alpha \cap [x] \in K(x)$ whenever $\alpha \in K(x)$ [7] p.1374. The corresponding full subcategory of FCO is denoted by LFCO. Both FCO and LFCO are topological categories. If α and β are filters on A , then $\alpha \cup \beta = \{U \subset A : V \cap W \subset U \text{ for some } V \in \alpha \text{ and } W \in \beta\}$ is a filter on A [2] p.98

T_2 -OBJECTS IN CATEGORIES OF FILTER AND LOCAL FILTER CONVERGENCE SPACES.

We now, for later use, give the characterizations of each of the separation properties defined in 1.2 for FCO and LFCO.

2.1 Theorem 1. *A local filter convergence (resp. filter convergence space) (A, K) is T_0 iff for any $x, y \in X, [x] \in K(y)$ and $[y] \in K(x)$ (resp. $[x] \cap [y] \in K(x) \cap K(y)$) implies $x = y$ [9] p.318.*

2. *(A, K) in FCO or LFCO is \bar{T}_0 iff for each distinct pair of points x and y in A , $[x] \notin K(y)$ or $[y] \notin K(x)$ [3].*

3. *All objects of FCO and LFCO are T'_0 [3].*

4. *(A, K) in FCO or LFCO is $pre\bar{T}_2$ iff for any pair x and y in A if $K(x) \cap K(y) \neq \{[\phi]\}$, where ϕ is the empty set, then $K(x) = K(y)$ and for any proper filters α and β in $K(x)$ if $\alpha \cup \beta$ is proper, then $\alpha \cap \beta \in K(x)$ [3].*

5. *(A, K) in FCO or LFCO is $preT'_2$ iff (A, K) is discrete i.e for each x in A , $K(x) = \{[x], [\phi]\}$ [3].*

6. *(A, K) in FCO or LFCO is \bar{T}_2 iff for any $x \neq y$ in A , $K(x) \cap K(y) = \{[\phi]\}$ and for any proper filters α and β in $K(x)$ if $\alpha \cup \beta$ is proper, then $\alpha \cap \beta \in K(x)$ [3].*

7. *(A, K) in FCO or LFCO is T'_2 iff (A, K) is discrete [3].*

8. *(A, K) in FCO or LFCO is ΔT_2 iff for each $x \neq y, [x] \notin K(y)$ [4].*

9. *(A, K) in FCO or LFCO is ST_2 iff for any $x \neq y$ in A , $K(x) \cap K(y) = \{[\phi]\}$ [3] or [4].*

10. *Let (A, K) be in FCO or LFCO and F be a nonempty subset of A . F is strongly closed iff for any $a \in A$ if $a \notin F$, then $[a] \notin K(c)$ for all $c \in F$ and if $\alpha \in K(a)$, then $\alpha \cup [F]$ is improper [4].*

11. *(A, K) in FCO or LFCO is T_1 iff for each $x \neq y [x] \notin K(y)$ [3].*

2.2 Corollary *(A, K) in FCO or LFCO is KT_2 iff (A, K) is $pre\bar{T}_2$.*

Proof. It follows from (3) and (4) of 2.1 and definition 1.2. □

2.3 Corollary (A, K) in FCO or LFCO is LT_2 (MT_2) iff (A, K) is discrete.

Proof. Combine (2) and (5) (resp. (1) and (5)) of 2.1 and 1.2. \square

2.4 Corollary (A, K) in FCO or LFCO is NT_2 iff (1) for any pair x and y in A if $K(x) \cap K(y) \neq \{[\phi]\}$, where ϕ is the empty set, then $K(x) = K(y)$ (2) for any proper filters α and β in $K(x)$ if $\alpha \cup \beta$ is proper, then, $\alpha \cap \beta \in K(x)$, (3) for any distinct pair of points x and y in A , $[x] \cap [y] \notin K(x)$ or $[x] \cap [y] \notin K(y)$.

Proof. Combine parts (1) and (4) of 2.1 and definition 1.2. \square

2.5 Remark (1) For the category of FCO, we have \bar{T}_\emptyset implies T_\emptyset , T_\emptyset implies T'_\emptyset , $T'_\emptyset = LT_2 = MT_2$ implies \bar{T}_2 , T_2 implies ST_2 , and NT_2 implies KT_2 but the converse of each implication is not true.

(2) For the category of LFCO, we have $\bar{T}_\emptyset = T_\emptyset$ implies T'_\emptyset , $T'_\emptyset = LT_2 = MT_2$ implies $\bar{T}_2 = NT_2$, NT_2 implies ST_2 , ST_2 implies ΔT_2 , and NT_2 implies KT_2 but the converse of each implication is not true.

(3) In LFCO, we have some relationships among our notions of T_2 's and the ones in [5] and [7]. Theirs T_2 is equivalent to our ST_2 but each of our $T'_2 = LT_2 = MT_2$ and $\bar{T}_2 = NT_2$ implies theirs T_2 , and theirs T_2 implies each of our ΔT_2 's and KT_2 's.

(4) If (A, K) is an indiscrete i.e $\forall x \in A, K(x) = F(A)$, (A has at least two elements), then (A, K) is clearly KT_2 but not NT_2 and ΔT_2 .

(5) In FCO and LFCO, by 2.1, $\Delta T_2 = T_1$ implies each of $\bar{T}_\emptyset, T'_\emptyset$ and T_\emptyset .

Let (A, L) and (X, K) be objects of FCO or LFCO.

2.6 Theorem Let $F : (A, L) \rightarrow (X, K)$ be a monomorphisms.

(1) If (X, K) is ST_2 , then (A, L) is ST_2 .

(2) If f is also initial and (X, K) is KT_2 or \bar{T}_2 , then (A, L) is KT_2 or \bar{T}_2 , respectively.

Proof. (1) follows easily from Theorem 2.1 and f is being mono

(2) Suppose for any a and b in A , $L(a) \cap L(b) \neq \{[\phi]\}$. We show that $L(a) = L(b)$. Let $\alpha \in L(a)$. Then $f(\alpha) \in K(f(a))$. $L(a) \cap L(b) \neq \{[\phi]\}$ implies there exists a proper filter β in $L(a) \cap L(b)$. It follows that $f(\beta)$ is proper and in $K(f(a)) \cap K(f(b))$. Since X is KT_2 , by 2.2 $K(f(a)) = K(f(b))$ and consequently $f(\alpha) \in K(f(b))$. Hence, $\alpha \in L(b)$ since f is initial (see [3] or [7]). Similarly, if $\alpha \in L(b)$, then $\alpha \in L(a)$ and so $L(a) = L(b)$. Suppose for $a \in A$ and α, β proper filters in $L(a)$ with $\alpha \cup \beta$ is proper. Note that $f(\alpha)$ and $f(\beta)$ are in $K(f(a))$ and $f(\alpha) \cup f(\beta) = f(\alpha \cup \beta)$ (since f is mono) is proper. X is KT_2 implies $f(\alpha \cap \beta) \in K(f(a))$ and consequently $\alpha \cap \beta \in L(a)$ since f is initial. By 2.2, (A, L) is KT_2 . The proof for \bar{T}_2 follows. \square

2.7 Theorem *Let $f : (A, L) \rightarrow (X, K)$ be the initial lift (see [3] or [7] p.1374) and (X, K) be ΔT_2 , $T'_2 = LT_2 = MT_2$, or NT_2 , then (A, L) is ΔT_2 , $T'_2 = LT_2 = MT_2$ or NT_2 , respectively, iff f is mono.*

Proof. If f is mono, then the result follows easily. Suppose (A, L) is ΔT_2 but f is not mono, i.e, $\exists x \neq y$ in A such that $f(x) = f(y)$. Note that $f([x]) = [f(x)] = [f(y)] \in K(f(y))$ implies $[x] \in L(y)$ which is a contradiction since (A, L) is ΔT_2 . If (A, L) is $T'_2 = LT_2 = MT_2$, then clearly f is mono. Suppose (A, L) is NT_2 but f is not mono. Then $\exists x \neq y$ such that $f(x) = f(y)$. Note that $f([x] \cap [y]) = [f(x)] \cap [f(y)] = [f(x)] \in K(f(x)) \cap K(f(y))$. Since f is initial, $[x] \cap [y] \in L(x) \cap L(y)$, a contradiction. By 2.4 and 2.6 , we get the result. \square

2.8 Corollary *Suppose (X, K) is one of the our "T₂" spaces. Then every subspace of (X, K) (i.e $f : (A, L) \rightarrow (X, K)$ is mono and initial) is "T₂" space.*

Proof. It follows from 2.6 and 2.7. \square

2.9 Theorem (1) *The cartesian product $(X = \{\prod X_i : i \in I, K)$ is $\bar{T}_2, ST_2, \Delta T_2, KT_2$ or NT_2 iff each (X_i, K_i) is $\bar{T}_2, ST_2, \Delta T_2, KT_2$ or NT_2 , respectively.*

(2) *The finite cartesian product $(X = \{\prod X_i : i \in I = \text{finite } \}, K)$ is $T'_2 = LT_2 = MT_2$ iff each (X_i, K_i) is $T'_2 = LT_2 = MT_2, i = 1, 2, \dots, n$.*

Proof. (1) Suppose (X, K) is "Hausdorff". Then it is easy to see that each (X_i, K_i) is isomorphic to some slice in (X, K) and consequently by 2.8, (X_i, K_i) is "Hausdorff" for all i.

Conversely, suppose each (X_i, K_i) is ΔT_2 but (X, K) is not ΔT_2 i.e by 2.1(8) $\exists x \neq y$ in X such that $[x] \in K(y)$. It follows that $\exists k \in I$ such that $x_k \neq y_k$ in X_k and $\pi_k([x]) = [x_k] \in K_k(\pi_k(y) = y_k)$, a contradiction. Hence (X, K) must be ΔT_2 .

Suppose each (X_i, K_i) is ST_2 . For any $x \neq y$ in X , let $\alpha \in K(x) \cap K(y)$. Note that $\pi_i \alpha \in K_i(\pi_i x = x_i) \cap K_i(\pi_i y = y_i)$. Since $x \neq y, \exists k \in I$ such that $x_k \neq y_k$ in X_k . In particular, by 2.1(9), $\pi_k \alpha = [\phi]$ which implies $\alpha = [\phi]$. Hence by 2.1(9), (X, K) is ST_2 .

Suppose each (X_i, K_i) is \bar{T}_2 . By 2.1 (6) and above result, we need to only show that for any x in X and any proper filters α and β in $K(x)$ if $\alpha \cup \beta$ is proper, then $\alpha \cap \beta \in K(x)$. Note that $\pi_i \alpha$ and $\pi_i \beta$ are in $K_i(x_i = \pi_i x)$ and each $\pi_i \alpha \cup \pi_i \beta$ is proper (since $\pi_i \alpha \cup \pi_i \beta \subset \pi_i(\alpha \cup \beta)$) for all i. Since each (X_i, K_i) is \bar{T}_2 , $\pi_i(\alpha \cap \beta) = \pi_i \alpha \cap \pi_i \beta \in K_i(\pi_i x = x_i)$ for all i and consequently $\alpha \cap \beta \in K(x)$.

Suppose each (X_i, K_i) is KT_2 and for any $x \neq y$ in $X, K(x) \cap K(y) \neq \{[\phi]\}$. Then there exists a proper filter $\beta \in K(x) \cap K(y)$ and consequently $\pi_i \beta \in K_i(x_i) \cap K_i(y_i)$. Since each (X_i, K_i) is KT_2 , $K_i(x_i) = K_i(y_i)$ for all i. Let α be in $K(x)$. Then $\pi_i \alpha \in K_i(x_i) = K_i(y_i)$ for all i and consequently $\alpha \in K(y)$. By interchanging the

role of x and y , we get $K(y) \subset K(x)$ if $K(x) \cap K(y) \neq \{[\phi]\}$. Hence $K(x) = K(y)$ and by 2.2 and above result $\alpha \cap \beta \in K(x)$, (X, K) is KT_2 .

Suppose each (X_i, K_i) in FCO is NT_2 . By 2.4 and above result, we need to only show that for any x and y in X , if $[x] \cap [y] \in K(x) \cap K(y)$, then $x = y$. Note that $\pi_i([x] \cap [y]) = [x_i] \cap [y_i] \in K_i(x_i) \cap K_i(y_i)$ for all i and consequently $x_i = y_i$ for all i i.e $x = y$.

(2) Suppose each (X_i, K_i) is $T'_2 = LT_2$ for $i = 1, 2, \dots, n = I$. Let $x \in X$ and $\alpha \in K(x)$. By definition of the product structure, we get $\pi_i \alpha \in K_i(\pi_i x = x_i)$ for all $i \in I$ and, by 2.1(7) and 2.3, $\pi_i \alpha = [\phi]$ or $[x_i]$. If $\pi_i \alpha = [\phi]$, then $\alpha = [\phi]$. If $\pi_i \alpha = [x_i]$, then let $\beta = \bigcup_{i=1}^n \pi_i^{-1}[x_i] = \bigcup_{i=1}^n \pi_i^{-1}[\pi_i \alpha]$. We now show that $\beta = [x]$. We need to show that

$\{x\} \in \beta$, then Note that $\{x\} = \bigcap_{i=1}^n \pi_i^{-1}\{x_i\} = \{x_1\}x\{x_2\}x \cdots \{x_n\} \in \beta$. If $\cup \in \beta$, then

there exists $\cup_i \in \alpha, i \in I$ such that $\cup \supset \bigcap_{i=1}^n \pi_i^{-1}(\cup_i) \supset \cup_1 \cap \cup_2 \cap \cdots \cap \cup_n$. Since α is a filter, $\cup_1 \cap \cup_2 \cap \cdots \cap \cup_n \in \alpha$ and consequently $\cup \in \alpha$. Hence, $\beta \subset \alpha$. Thus, $\alpha = [x]$ and by 2.1, (X, K) is $T'_2 = LT_2 = MT_2$. If (X, K) is T'_2 , then it is easy to see that (X_i, K_i) is T'_2 for $i \in I$.

We now generalize Nell's, [7], result for some of our " T_2 " structures. Nel has showed that a full subcategory, HLFCO (objects of HLFCO are Hausdorff filter convergence spaces) of LFCO is a cartesian closed initially structured category. Let T'_2LFCO, ST_2LFCO , and ΔT_2LFCO be subcategories of LFCO, where objects of these subcategories are determined by the corresponding " T_2 " local filter convergence spaces. \square

2.10 Lemma *The full subcategories T'_2LFCO, ST_2LFCO , and ΔT_2LFCO of LFCO are cartesian closed initially structured categories.*

Proof. It follows easily from 2.1 and the results 1.13, 1.14, and 2.5 of [7]. \square

2.11 Remark Each of the full subcategories T_2LFCO (T_2FCO) of $LFCO$ (FCO), where T_2 is $\bar{T}_2, ST_2, \Delta T_2, KT_2$, or NT_2 , is epireflective (i.e each subcategories are closed under formation of subspaces and products, and isomorphism -closed) by 2.8 and 2.9.

Let (X, K) be in FCO or LFCO, and F be a nonempty subset of X . Let $q : (X, K) \rightarrow (X/F, L)$ be the quotient map that identifying F to a point, $*$.

Theorem *If (X, K) is $T'_2 = LT_2 = MT_2$, then $(X/F, L)$ is T'_2 .*

Proof. Suppose (X, K) is T'_2 and for any $a \in X/F, \alpha \in L(a)$ Since the map q is quotient, there exists $\beta \in K(x)$ such that $\alpha \supset q\beta$ and $qx = a$. But (X, K) is T'_2 implies

$\beta = [x]$ or $[\phi]$, and consequently $\alpha = [a]$ or $[\phi]$. Hence by 2.1 and 2.3, $(X/F, L)$ is T'_2 . \square

2.13 Lemma *Let α and β be proper filters on X . Then $q\alpha \cup q\beta$ is proper iff either $\alpha \cup \beta$ is proper or $\alpha \cup [F]$ and $\beta \cup [F]$ are proper.*

Proof. Suppose $q\alpha \cup q\beta$ is proper. We show that if $\alpha \cup [F]$ is improper, then $\alpha \cup \beta$ is proper. Suppose it is not proper i.e there exists U in α and V in β such that $U \cap V = \phi$. Since $\alpha \cup [F]$ is improper, $\exists W \in \alpha$ such that $W \cap F = \phi$. Note that $U \cap W \in \alpha$ and $U \cap W = q(U \cap W)$ in $q\alpha$. It follows easily that $\phi = q(U \cap W) \cap q(V) \in q\alpha \cup q\beta$, a contradiction. Similarly, if $\beta \cup [F]$ is improper, then $\alpha \cup \beta$ is proper.

Conversely, suppose $\alpha \cup \beta$ is proper, but $q\alpha \cup q\beta$ is improper. Then $\exists U \in q\alpha$ and $\exists V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. It follows that $\phi = U_1 \cap V_1 \in \alpha \cup \beta$, a contradiction. Suppose $\alpha \cup [F]$ and $\beta \cup [F]$ are proper. Suppose also that $q\alpha \cup q\beta = [\phi]$. Then $\exists U \in q\alpha$ and $\exists V \in q\beta$ such that $\phi = U \cap V \supset q(U_1) \cap q(V_1)$ for some $U_1 \in \alpha$ and $V_1 \in \beta$. But this is a contradiction since $U_1 \cap F \neq \phi \neq V_1 \cap F$, $* \in q(U_1 \cap F)$ and $* \in q(V_1 \cap F)$. \square

2.14 Theorem *If (X, K) is ST_2 and F is strongly closed, then $(X/F, L)$ is ST_2 .*

Proof. Let a and b be any disinct pair of points in X/F and α be in $L(a) \cap L(b)$. If α is improper, then we are done. Suppose α is proper. q is the quotient map implies $\exists \beta \in K(x)$ and $\exists \delta \in K(y)$ such that $\alpha \supset q\beta, \alpha \supset q\delta$, and $qx = a, qy = b$. Note that $q\beta \cup q\delta$ is proper and by 2.13, either $\beta \cup \delta$ is proper or $\beta \cup [F]$ and $\delta \cup [F]$ are proper. The first case can not occur since $x \neq y$ and (X, K) is ST_2 . Since $a \neq b$, we may assume $x \in F$. We have $\beta \in K(x)$ and since F is strongly closed, by 2.1 (10), $\beta \cup [F]$ is improper. This shows that the second case also can not hold. Therefore, α must be improper and, by 2.1(9), we have the result. \square

2.15 Theorem *Let (X, K) be \bar{T}_2 . If F is strongly closed and for each $a \in F, K(a) = \{[a], [\phi]\}$, then $(X/F, L)$ is \bar{T}_2 .*

Proof. Suppose (X, K) is \bar{T}_2 . By 2.1(6) and 2.14, we only need to show that for any $a \in X/F$ and α and β in $L(a)$, if $\alpha \cup \beta$ is proper, then $(\alpha \cap \beta) \in L(a)$. Suppose $a \neq *$ and $\alpha, \beta \in L(a)$ implies $\exists \alpha_1, \beta_1 \in K(x)$ such that $\alpha \supset q\alpha_1, \beta \supset q\beta_1$, and $x = qx = a$. Since $\alpha \cup \beta$ is proper, $q\alpha_1 \cup q\beta_1$ is proper and by 2.13, we have either $\alpha_1 \cup \beta_1$ is proper or $\alpha_1 \cup [F]$ and $\beta_1 \cup [F]$ are proper. If the first case holds, then clearly $\alpha \cap \beta \in L(a)$ since (X, K) is \bar{T}_2 . The second case can not hold since F is strongly closed ($x \in F$ and $\alpha_1 \cup [F]$ is improper) by 2.1 (10).

Suppose $a = *$ and $\alpha_1 \in K(d)$ and $\beta_1 \in K(c)$ for some $c, d \in F$. It follows from the assumption that $\alpha_1 = [d]$ or $[\phi]$ and $\beta_1 = [c]$ or $[\phi]$. If $\alpha_1 = [d]$ and $\beta_1 = [c]$, then

$\alpha = [*] = \beta$ and consequently $\alpha \cap \beta \in L(a)$. This shows that $(X/F, L)$ is \bar{T}_2 . \square

2.16 Theorem *If (X, K) is ΔT_2 and F is strongly closed, then $(X/F, L)$ is ΔT_2 .*

Proof. By 2.1, we need to show that for any distinct pair of points a and b in X/F , $[a] \notin L(b)$. Suppose $[a] \in L(b)$ for some $a \neq b$ and $a \neq *$. Since q is quotient, $\exists \alpha \in K(x)$ such that $[a] \supset q\alpha$ and $qx = b$. By Lemma 3.16 of [2], we get $[a] \supset \alpha$ and consequently $[a] \in K(x)$, a contradiction (X is ΔT_2). Suppose $a = *$. Then $[*] \supset q\alpha$ for some α in $K(x)$ and $x = qx = b$. By Lemma 3.16 of [2], $\alpha \cup [F]$ is proper, a contradiction, since $x \notin F$ and $\alpha \cup [F]$ is improper (F is strongly closed by 2.1 (10)). \square

2.17 Theorem *Let (X, K) be T_1 , F be strongly closed, and for each a in F , $K(a) = \{[a], [\phi]\}$. If (X, K) is KT_2 or NT_2 , then $(X/F, L)$ is KT_2 or NT_2 , respectively.*

Proof. By 2.16 and assumption, $(X/F, L)$ is $T_1 = \Delta T_2$. By 2.5, $(X/F, L)$ and (X, K) are \bar{T}_0 and T'_0 . It follows from 2.15 and assumption that $(X/F, L)$ is \bar{T}_2 and consequently, by 1.2 and 2.5, $(X/F, L)$ is KT_2 and NT_2 . This completes the proof. \square

2.18 Remark (1) Let (X, K) be in FCO or LFCO. For $\phi \neq F \subset X$, let R be the equivalence relation $(Fx F) \cup \{(x, x) : x \in X\}$. Note that the quotient space $(X/R, L)$ is the space (X, K) with F is identified to a point $*$, and is written $(X/F, L)$ which we had before. Let I be a finite set and for $i \in I, F_i$ be nonempty subsets of X such that $\forall i \neq j, F_i \cap F_j = \phi$. Let R be the equivalence relation $(F_1 x F_1) \cup \dots \cup (F_n x F_n) \cup \{(x, x) : x \in X\}$. Note that the quotient map $q : (X, K) \rightarrow (X/R, L)$ is the composition of the quotient maps $q_1 : (X, K) \rightarrow (X/F_1 = Y_1, L_1), q_2 : (Y_1, L_1) \rightarrow (Y_1/F_2 = Y_2, L_2) \dots$, and $q_n : (Y_{n-1}, L_{n-1}) \rightarrow (Y_{n-1}/F_n, L_n) = (X/R, L)$. Then

- (i) By induction, Theorem 2.12 holds for this quotient space $(X/R, L)$.
- (ii) Theorems 2.14 and 2.16 hold for this quotient space $(X/R, L)$ provided that each of F_i is strongly closed for $i = 1, 2, \dots, n$.
- (iii) Theorem 2.15 holds for this quotient space $(X/R, L)$ if for a in $F_i, K(a) = \{[a], [\phi]\}$ and all of F_i are strongly closed, $i = 1, 2, \dots, n$.

(2) In TOP, the category of topological spaces, in general, if X is T_2 and F is closed, then X/F is not necessarily T_2 . However, in FCO or LFCO, by 2.12, the result is true provided that F is closed.

(3) Let $X = \{x, y, z\}$ be three-point set, and define two structures K and L on X as follow: $L = F(X)$, the set of all filters on X i.e L is indiscrete; and $K(x) =$

$\{[x], [y], [z], [x] \cap [y], [x] \cap [z], [\phi]\}K(y) = \{[y], [\phi]\}$, and $K(z) = \{[z], [\phi]\}$. Define $f : (X, K) \rightarrow (X, L)$ by $f(x) = x, f(y) = z$, and $f(z) = y$. Clearly f is mono and continuous, and (X, L) is KT_2 but (X, K) is not KT_2 (let $\alpha = [x] \cap [y]$ and $\beta = [x] \cap [z]$. Note that $\alpha \cup \beta = [x]$ is proper but $\alpha \cap \beta$ is not in $K(x)$). Hence by 1.14 of [7], a subcategory, KT_2 LFCO of LFCO is not quotient-reflective, and by part (1) it is not coreflective. Hence, by 1.13 of [7], it is not an initially structured category. Therefore, the result of Nel [7] can not be extended to KT_2 LFCO.

Let X be an object of E , a topological category over sets, F a nonempty subset of $U(X)$, and $q : X \rightarrow X/F$ be the final lift that identifying F to a point, $*$. Recall, [1] or [4], that F is closed iff $*$ is closed in X/F (see [1] or [4]). F is strongly closed iff X/F is T_1 at $*$. Let (X, K) be in FCO or LFCO. Note that a point p in X is (strongly) closed iff, by [4], $\forall x \neq p, [x] \notin K(p)$ or (and) $[p] \notin K(x)$.

2.19 Corollary (1) (X, K) is ΔT_2 iff points are strongly closed.

(2) If (X, K) is ΔT_2 , then points are closed.

(3) Let F be a nonempty subset of X . If (X, K) is $T'_2 = LT_2 = MT_2$, then F is always (strongly) closed. subset of X .

(4) (X, K) is ΔT_2 and NT_2 iff it is \bar{T}_2 .

Proof. (1) (X, K) is ΔT_2 iff, by 2.1, $\forall x \neq y$ in $X, [x] \notin K(y)$ iff, by [4], points are strongly closed.

(2) It is a special case of (1).

(3) By 2.12, $(X/F, L)$ is T'_2 . By 2.5, $(X/F, L)$ is T_1 and \bar{T}_0 and consequently by [4], F is (strongly) closed subset of X .

(4) It is clear. □

2.20 Remark (1) Let $X = \{x, y\}$ be two-point set, and $K(x) = \{[x], [y], [x] \cap [y], [\phi]\}, K(y) = \{[y], [\phi]\}$. Clearly, the points x and y are closed but they are not strongly closed. Hence by 2.19 (1), (X, K) is not ΔT_2 and consequently, the converse of 2.19 (2) is not true. Furthermore, in FCO or LFCO, T_1 is equivalent to ΔT_2 , (see 2.5) and in TOP, the category of topological spaces, the result: T_1 is equivalent to points are closed, is not true, in general.

(2) Let $X = \{x, y\}$ be two-point set, and $K(x) = \{[x], [y], [\phi], [x] \cap [y]\} = K(y)$. (X, K) is clearly KT_2 but the points x and y are not (strongly) closed. However, in TOP, every point is closed in a Hausdorff space.

(3) Recall, in [1], that there are four various generalizations of the usual T_3 (regular) topological spaces to an arbitrary topological category over sets. In FCO or LFCO, if (X, K) is T'_3 or ST'_3 , then it follows easily from Definition 2.2.15 of [1] p.340 by letting F to be a point that (X, K) is T'_2 . By 2.12, $(X/F, L)$ is T'_2 for any nonempty subset F of X . By 2.5, $(X/F, L)$ is $LT_2, \Delta T_2, ST_2, NT_2$, and KT_2 .

BARAN

Acknowledgement

I would like to thank the referee for his useful suggestions.

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SÜZGEÇ VE YEREL SÜZGEÇ YAKINSAK UZAYLAR KATEGORİLERİNDE T_2 -OBJELERİ

Özet

[1] de Topolojide bilinen T_2 (Hausdorff) aksiyomu herhangi bir topolojik kategoriye değişik yollarla genelleştirilmesi tanımlandı. Bu çalışmada, Hausdorff uzayların dört yeni genelleştirilmesi verildi. Ayrıca. bütün bu T_2 yapıları ve [5] ile [7] de verilenler arasındaki bağıntılar ile bunların bazı invaryant özellikleril süzgeç ve yerel süzgeç yakınsak uzaylar kategorilerinde incelendi.

M. BARAN
Erciyes University,
Department of Mathematics,
38039, Kayseri-TURKEY

Received 25.4.1994