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## WEAK $\star$ - INVARIANTLY COMPLEMENTED SUBSPACES OF $L^\infty(1/\omega)$ AND IDEALS OF $L^1(\omega)$ WITH A BOUNDED APPROXIMATE IDENTITY

*Z. Argün & C. Tonyali*

### Abstract

For a locally compact group  $G$  let  $L^1(\omega)$  be the weighted group algebra and let  $X$  be a weak  $\star$ -closed translation invariant subspace of  $L^\infty(1/\omega)$ . In this paper for a certain class of functions we show that the following conditions are equivalent: (i)  $X$  is topological invariantly complemented in  $L^\infty(1/\omega)$ ; (ii)  $X$  is invariantly complemented in  $L^\infty(1/\omega)$ ; (iii) The left ideal  $X_\perp$  has a bounded right approximate identity.

### Introduction

Let  $G$  be a locally compact group with fixed left Haar measure  $dx$ . By a weighted function on  $G$  we mean a positive and locally bounded measurable function  $\omega$  on  $G$  such that  $\omega(st) \leq \omega(s)\omega(t)$  and  $\inf \omega(s) = \gamma > 0 (s, t \in G)$ . We may assume that  $\omega$  is upper semi-continuous on  $G$  (see, for example, ([7], page 83)). If we set

$$L^1(\omega) = \left\{ f : \|f\|_\omega^1 = \int_G |f(t)|\omega(t) dt < \infty \right\}$$

then,  $L^1(\omega)$  is a Banach Space: as usual, we equate functions equal  $dx$  almost everywhere. under convolution product defined by the equation

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy \quad (f, g \in L^1(\omega))$$

$L^1(\omega)$  becomes a Banach algebra. Since  $\omega$  is locally bounded,  $L^1(\omega)$  contains the space of functions of compact supports. We also indicate that it has a bounded approximate identity (in the sequel we abbreviated it as a.i.) (see, for example ([1], Lemma 1.4.1)).

Recall that the isometric involution “ $\star$ ” on  $L^1(G)$  defined by  $f^\star(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ , does not in general, apply to  $L^1(\omega)$ . In fact,  $f^\star$  need not even be in  $L^1(\omega)$ . This is shown by examples in ([7], chapter 1, § 6.1). In this study we therefore assume the

weighted function  $\omega$  satisfies that  $\text{ess sup } \{\omega(x^{-1})/\omega(x) : x \in G\} = \delta < \infty$ . This situation does not imply that  $\omega$  is bounded. For example, the weighted function  $\omega$  defined by,  $\omega(x) = 1 + \frac{1}{2} (\frac{x}{2} + |x|)$  is not bounded on  $(\mathbb{R}, +)$ , but  $\text{sup } \omega(-x)/\omega(x) = 3$  and  $1 \leq \omega(x)$  for all  $x \in \mathbb{R}$ . Note also that the involution “\*” is continuous with  $\| f^* \| \leq \delta \| f \| \leq \delta^2 \| f^* \|$ . In this case the involution “\*” becomes a topological algebraic anti-isomorphism of  $L^1(\omega)$ .

Using the notation of [5] and [6], for  $f \in L^1(\omega)$  and  $x \in G$   ${}_x f(y) = f(xy), f_x(y) = f(yx)(y \in G)$ . Then  ${}_x f, f_x$  belong to  $L^1(\omega)$  and

$$\| {}_x f \|_{\omega}^1 \leq \omega(x^{-1}) \| f \|_{\omega}^1, \| f_x \| \leq \Delta(x^{-1})\omega(x^{-1}) \| f_x \|_{\omega}^1 .$$

The dual space of  $L^1(\omega)$  is  $L^\infty(\omega)$ , Banach space of complex valued measurable functions  $\varphi$  on G for which  $\varphi/\omega$  is essentially bounded; that is  $\varphi/\omega \in L^\infty(G)$ . The norm in  $L^\infty(1/\omega)$  is;

$$\| \varphi \|_{\omega}^\infty = \text{ess sup } \{ |\varphi(t)| / \omega(t) : t \in G \} \quad (\varphi \in L^\infty(1/\omega))$$

and the duality is implemented by

$$\langle f, \varphi \rangle = \int_G f(x) \overline{\varphi(x)} dx \quad (f \in L^1(\omega), \varphi \in L^\infty(1/\omega)) .$$

Furthermore for each  $\varphi \in L^\infty(1/\omega)$  and  $x \in G$   ${}_x \varphi(y) = \varphi(xy), \varphi_x(y) = \varphi(yx), \tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$   $y \in G$ , and  ${}_x \varphi, \tilde{\varphi}$  belong to  $L^\infty(1/\omega)$  with  $\| {}_x \varphi \|_{\omega}^\infty \leq \omega(x) \| \varphi \|_{\omega}^\infty, \| \tilde{\varphi} \|_{\omega}^\infty \leq \delta \| \varphi \|_{\omega}^\infty \leq \delta \| \varphi \|_{\omega}^\infty$ . Moreover, for  $f, g \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$  we have

$$\langle g, f^* * \varphi \rangle = \langle f * g, \varphi \rangle$$

Where  $f * \varphi(x) = \int_G f(y) \varphi(y^{-1}x) dy$  and  $f^* * \varphi(x) = \int_G \overline{f(y)} \varphi(yx) dy$ . Let RUC  $(1/\omega)$  be the closed subspace of  $L^\infty(1/\omega)$  consisting of continuous functions  $\varphi \in L^\infty(1/\omega)$  for which the mapping

$$G \longrightarrow L^\infty(x/\omega), \quad x \longrightarrow {}_x \varphi \quad (x \in G)$$

is continuous.

H. Rosenthal classified in [8] the closed ideals in  $L^1(G)$  for which  $I^\perp$  is complemented in  $L^\infty(G)$  in the case of an abelian group G. By Theorem 4 in [6] it turns out that the closed ideals  $X_\perp$  of  $L^1(G)$  has a bounded a.i. when X is invariantly complemented in  $L^\infty(G)$ , where

$$X_\perp = \{ f \in L^1(G) : \langle f, \varphi \rangle = 0, \varphi \in X \}$$

Recently Bekka in [2] has proved that a weak \*-closed left translation invariant subspace X of  $L^\infty(G)$  is invariantly complemented if and only if  $X_\perp$  has a bounded right a.i. when G is locally compact group.

In this work, our main purpose is to generalize Bekka's results in [2] for a certain class of weighted functions.

**Definition 1.** A weak  $*$ -closed left translation invariant subspace  $X$  of  $L^\infty(1/\omega)$  is said to be invariantly complemented in  $L^\infty(1/\omega)$  if  $X$  has a closed left translation invariant complemented in  $L^\infty(1/\omega)$  or equivalently, if  $X$  is the range of a bounded projection on  $L^\infty(1/\omega)$  commuting with left translation. Indeed, if  $X$  has a closed left translation invariant complement in  $L^\infty(1/\omega)$  then there exists a closed subspace  $Y$  of  $L^\infty(1/\omega)$  such that if  $\varphi \in Y$ , then  ${}_x\varphi \in Y$  ( $x \in G$ ) and  $L^\infty(1/\omega) \cong X \oplus Y$ . Define a mapping  $P$  on  $L^\infty(1/\omega)$  such that  $P(\varphi_1 + \varphi_2) = \varphi_1$  where  $\varphi = \varphi_1 + \varphi_2 \in L^\infty(1/\omega)$ . It's routine to show that  $P$  satisfies the conditions of a bounded projection commuting with left translations.

On the other hand if  $P$  is a bounded projection commuting with left translations on  $L^\infty(1/\omega)$  with its range  $X$ , then the kernel of  $P$  is a left translation invariant subspace and complement of  $X$  in  $L^\infty(1/\omega)$ .

**Definition 2.** We say that  $X$  is topologically invariantly complemented in  $L^\infty(1/\omega)$  if  $X$  is the range of a bounded projection  $P$  on  $L^\infty(1/\omega)$  such that  $P(f * \varphi) = f * P(\varphi)$  for all  $f \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$  where  $X$  is a weak  $*$ -closed left translation invariant subspace of  $L^\infty(1/\omega)$ .

A close examination of the proof of Theorem 4.1 in [9] shows that a bounded projection on  $L^\infty(1/\omega)$  commuting with left translations need not commute with left convolution by functions from  $L^1(\omega)$ . The notion of (topologically) invariant complemented subspace of RUC  $(1/\omega)$  is defined in a similar way.

Now, we need the following lemmas.

**Lemma 1.** If  $\varphi \in RUC(1/\omega)$ , the  ${}_x\varphi \in RUC(1/\omega)$  for all  $x$  in  $G$ .

**Proof.** Let  $\varphi \in RUC(1/\omega)$  and  $x \in G$ . Then for given any  $\varepsilon > 0$  there exists a relatively compact neighborhood  $U$  of the unit  $e_G$  of  $G$  such that;

$$\|{}_s\varphi - {}_t\varphi\|_\omega^\infty < \varepsilon$$

when  $st^{-1} \in U$ . let  $W$  be an open subset of  $U$  which contains  $e_G$ . If  $uv^{-1} \in x^{-1}Wx$ , then we have  $(xu)(xv)^{-1} \in W \subseteq U$  and so,

$$\|{}_{xu}\varphi - {}_{xv}\varphi\|_\omega^\infty = \|{}_u(x\varphi) - {}_v(x\varphi)\|_\omega^\infty < \varepsilon;$$

that is, we have found an open neighborhood  $x^{-1}Wx$  of  $e_G$  such that;

$$\|{}_u(x\varphi) - {}_v(x\varphi)\|_\omega^\infty < \varepsilon$$

when  $uv^{-1} \in x^{-1}Wx$ . By definition of the right uniform continuity,  ${}_x\varphi \in RUC(1/\omega)$ , as required (see, for example ([7], ch.3, § 1.8) for the definition of the right and left uniform

continuity). □

**Lemma 2.** *If  $g \in L^1(\omega)$ , then for each  $\varepsilon > 0$  there exists a symmetric neighborhood  $V$  of  $e_G$  (then unit of  $G$ ) such that;*

$$\|xg - yg\|_{\omega}^{\infty} < \varepsilon \quad \text{when } xy^{-1} \in V.$$

**Proof.**  $f \in L^1(\omega)$  with compact support  $K$ . By Lemma 20.4 (i) in [5] and ([7], p. 85, § 7.2) for given any  $\varepsilon' > 0$ , there exists a relatively compact neighborhood  $U$  of  $e_G$  such that  $\|uf - f\|_1 < \varepsilon'$  when  $u \in U$ . Let  $V$  be a symmetric neighborhood of  $e_G$  such that  $V \subseteq U$ . Then for all  $v \in V$ ,  $\|vf - f\|_1 < \varepsilon'$ .

On the other hand, for all  $v \in V$  we have;

$$\text{supp}(vf - f) \subseteq V^{-1}K \cup K = VK \cup K \subseteq \bar{U}K \cup K$$

and  $\bar{U}K$  is compact (hence  $\bar{U}K \cup K$  is compact).

Now set  $Q = \sup\{\omega(z) : z \in \bar{U}K \cup K\} < \infty$ . If  $v \in V$ , then we obtain

$$\begin{aligned} \|vf - f\|_{\omega}^1 &= \int_G |(vf - f)(y)| \omega(y) dy \\ &= \int_{\text{supp}(vf - f)} |(vf - f)(y)| \omega(y) dy \\ &\leq \int_{VK \cup K} |(vf - f)(y)| \omega(y) dy \\ &\leq \int_{\bar{U}K \cup K} |(vf - f)(y)| \omega(y) dy \\ &\leq Q \|vf - f\|_1 < Q\varepsilon' = \varepsilon \end{aligned}$$

which completes the proof of this Lemma. □

**Lemma 3.** *If  $f \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$ , then  $f * \varphi \in RUC(1/\omega)$ .*

**Proof.** Let  $f \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$ . Thus

$$\begin{aligned} |f * \varphi(vx) - f * \varphi(x)| &= \left| \int_G f(t)\varphi(t^{-1}vx)dt - \int_G f(t)\varphi(t^{-1}x)dt \right| \\ &= \left| \int_G f(vxz^{-1})\varphi(z)\Delta(z^{-1})dz - \int_G f(xz^{-1})\varphi(z)\Delta(z^{-1})dz \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \int_G (f(vxz^{-1}) - f(xz^{-1})) \overline{\tilde{\varphi}(z^{-1})} \Delta(z^{-1}) dz \right| \\
 &= \left| \int_G ({}_v x f - {}_x f)(z) \overline{\tilde{\varphi}(z)} dz \right| \\
 &\leq \| {}_v x f - {}_x f \|_\omega^1 \| \tilde{\varphi} \|_\omega^\infty \\
 &\leq \delta \| {}_v x f - {}_x f \|_\omega^1 \| \varphi \|_\omega^\infty
 \end{aligned}$$

where for the first term  $z = t^{-1}vx$  and for the second term  $z = t^{-1}x$ .

Now by the Lemma 2, for given any  $\varepsilon > 0$  there exists symmetric neighborhood  $V$  of  $e_G$  such that;  $\| {}_v x f - {}_x f \|_\omega^1 < \varepsilon$  when  $v \in V$ . This inequality and boundedness from below of  $\omega$  imply that  $f * \varphi \in RUC(1/\omega)$  for each  $f \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$ , as required.  $\square$

**Lemma 4.** *let  $G$  be a locally compact group and let  $L^1(\omega)$  be its weighted group algebra. If  $I$  is a closed left ideal in  $L^1(\omega)$  then, its annihilator*

$$I^\perp = \{ \varphi \in L^\infty(1/\omega) : \langle f, \varphi \rangle = 0 \text{ for all } f \in I \}$$

*is a weak \*-closed left translation invariant subspaces of  $L^\infty(1/\omega)$ . Conversely, if  $X$  is a weak \*-closed left translation invariant subspaces of  $L^\infty(1/\omega)$  then, its annihilator  $X_\perp$  is a closed left ideal in  $L^1(\omega)$ .*

**Proof.** Recall that  $\langle g, f * \varphi \rangle = \langle f * g, \varphi \rangle$  for all  $f, g \in L^1(\omega)$  where;  $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$  and  $\Delta$  denotes the modular function of  $G$ . If  $I$  is a left ideal in  $L^1(\omega)$  then obviously,  $I^\perp$  is a weak \*-closed subspace of  $L^\infty(1/\omega)$  and if  $\{u_\alpha\}$  is a bounded a.i. of  $L^1(\omega)$  then, we have

$$\begin{aligned}
 \langle f, {}_x \varphi \rangle &= \lim_\alpha \langle u_\alpha * f, {}_x \varphi \rangle = \lim_\alpha \langle f, u_\alpha^* * {}_x \varphi \rangle \\
 &= \lim_\alpha \langle \Delta(x)f, (u_\alpha^*)_x * \varphi \rangle \\
 &= \lim_\alpha \langle \Delta(x) ((u_\alpha^*)_x)^* * f, \varphi \rangle \\
 &= 0 \text{ (since } f \in I \text{)}
 \end{aligned}$$

for all  $x \in G$  and  $f \in I$  which implies that  $I^\perp$  is left translation invariant.

Conversely, let  $X$  be a weak \*-closed left translation invariant subspace of  $L^\infty(1/\omega)$ . Then for each  $x \in G$  and  $\varphi \in X$ , we have  ${}_x \varphi \in X$ . Let  $g \in X_\perp$  and  $f \in L^1(\omega)$ . Then

$$\langle f * g, \varphi \rangle = \int_G (f * g)(x) \overline{\varphi(x)} dx$$

$$\begin{aligned}
 &= \int_G \left\{ \int_G \Delta(t^{-1}) f(t^{-1}) g(tx) dt \right\} \overline{\varphi(x)} dx \\
 &= \int_G \left\{ \int_G {}_t g(x) \overline{\varphi(x)} dx \right\} \Delta(t^{-1}) f(t^{-1}) dt \\
 &= \int_G \langle {}_t g, \varphi \rangle \Delta(t^{-1}) f(t^{-1}) dt \\
 &= \int_G \langle g, {}_{t^{-1}} \varphi \rangle \Delta(t^{-1}) f(t^{-1}) dt = 0.
 \end{aligned}$$

It follows that  $X_{\perp}$  is a left ideal in  $L^1(\omega)$ . Obviously, is closed. Hence  $X_{\perp}$  is a closed left ideal of  $L^1(\omega)$ , as required.  $\square$

**Theorem 5.** *Let  $G$  be a locally compact group,  $X$  be a weak  $*$ -closed left translation invariant subspace of  $L^{\infty}(1/\omega)$  and let  $X_{\perp}$  be the annihilator of  $X$  in  $L^1(\omega)$ . Then the following statements are equivalent:*

- (i)  $X$  is topologically invariantly complemented in  $L^{\infty}(1/\omega)$ ;
- (ii)  $X$  is invariantly complemented in  $L^{\infty}(1/\omega)$ ;
- (iii)  $X \cap RUC(1/\omega)$  is invariantly complemented in  $RUC(1/\omega)$ ;
- (iv)  $X \cap RUC(1/\omega)$  is topologically invariantly complemented in  $RUC(1/\omega)$ ;
- (v) The closed left ideal  $X_{\perp}$  has a bounded right a.i..

**Proof.** (i)  $\implies$  (ii); Let  $P : L^{\infty}(1/\omega) \rightarrow X$  be a projection with  $p(f * \varphi) = f * P\varphi$  for all  $f \in L^1(\omega), \varphi \in L^{\infty}(1/\omega)$ . Let  $\{e_{\alpha}\}$  be an a.i. for  $L^1(\omega)$ . By the equation  $\langle g, f * \varphi \rangle = \langle f * g, \varphi \rangle$  and lemma 3 we have

$$\begin{aligned}
 \langle f, P({}_x \varphi) \rangle &= \lim_{\alpha} \langle e_{\alpha} * f, P({}_x \varphi) \rangle \\
 &= \lim_{\alpha} \langle f, e_{\alpha}^* * P({}_x \varphi) \rangle \\
 &= \lim_{\alpha} \langle f, \Delta(x) P((e_{\alpha}^*)_x * \varphi) \rangle \\
 &= \lim_{\alpha} \langle f, \Delta(x) (e_{\alpha}^*)_x * P\varphi \rangle \\
 &= \lim_{\alpha} \langle f, e_{\alpha}^* * {}_x (P\varphi) \rangle \\
 &= \langle f, {}_x P\varphi \rangle
 \end{aligned}$$

which implies that  ${}_x(P\varphi) = P({}_x \varphi)$  and so,  $x$  is invariantly complemented in  $L^{\infty}(1/\omega)$ .

To show (ii)  $\implies$  (iii); let  $P : L^\infty(1/\omega) \rightarrow X$  be a bounded projection on  $X$  commuting with left translations. If  $\varphi \in RUC(1/\omega)$  and  $\varepsilon > 0$

$$\begin{aligned} \|_x P\varphi -_y P(\varphi) \|_\omega^\infty &= \| P(x\varphi) - P(y\varphi) \|_\omega^\infty = \| P(x\varphi -_y \varphi) \|_\omega^\infty \\ &\leq \| P \| \|_x \varphi -_y \varphi \|_\omega^\infty \\ &< \| P \| \varepsilon \end{aligned}$$

whenever  $xy^{-1} \in U$  where  $U$  is a neighborhood of  $e_G$  the unit of  $G$ . Hence  $P\varphi \in RUC(1/\omega)$ ; that is,  $P\varphi \in X \cap RUC(1/\omega)$ . On the other hand, if  $\varphi \in X \cap RUC(1/\omega)$  then  $\varphi \in X$  and so,  $P\varphi = \varphi$ . Moreover,  $\varphi$  being an element of  $RUC(1/\omega)$  the restriction  $P|_{RUC(1/\omega)}$  of  $P$  to  $RUC(1/\omega)$  is onto. By the definition of invariantly complemented of  $X \cap RUC(1/\omega)$ , it is an invariantly complemented subspace of  $RUC(1/\omega)$ , as required.

For the proof of implication (iii)  $\implies$  (iv) it is enough to show that the mapping  $P$  which is taken in the implication (ii)  $\implies$  (iii) satisfies the equality  $P(f * \varphi) = f * P\varphi$  for all  $f$  in  $L^1(\omega)$  and  $\varphi$  in  $RUC(1/\omega)$ . To this end, let  $\varphi \in RUC(1/\omega)$  and, set  $\Psi(y) = \int_G f(x)_{x^{-1}} \varphi(y) dx$ .

Then we have  $\Psi \in L^\infty(1/\omega)$  and

$$\begin{aligned} \|_y \Psi -_z \Psi \|_\omega^\infty &= \text{ess sup}_{u \in G} |{}_y \Psi(u) -_z \Psi(u) / \omega(u)| \\ &= \text{ess sup}_{u \in G} |\Psi(yu) - \Psi(zu) / \omega(u)| \\ &= \text{ess sup}_{u \in G} \left| \int_G f(x) \{ \varphi(x^{-1}yu) - \varphi(x^{-1}zu) \} dx \right| / \omega(u) \\ &= \text{ess sup}_{u \in G} \left| \int_G \{ f(yuv^{-1}) - f(zuv^{-1}) \} \varphi(v) \Delta(v^{-1}) dv \right| / \omega(u) \\ &= \text{ess sup}_{u \in G} \left| \int_G (yuf -_z u f)(v^{-1}) \overline{\varphi(v^{-1})} \Delta(v^{-1}) dv \right| / \omega(u) \\ &\leq \text{ess sup}_{u \in G} \int_G |{}_u (y f -_z f)(v)| |\varphi(\tilde{v})| dv / \omega(u) \\ &\leq \text{ess sup} \omega(u^{-1}) / \omega(u) \cdot \|_y f -_z f \|_\omega^1 \cdot \| \tilde{\varphi} \|_\omega^\infty \\ &\leq \delta^2 \|_y f -_z f \|_\omega^1 \| \varphi \|_\omega^\infty \end{aligned}$$

□

By Lemma 2 for give any  $\varepsilon > 0$ , there exists a symmetric neighborhood  $U$  of  $e_G$  such that  $\|_y f -_z f \| < \varepsilon$  ( $yz^{-1} \in U$ ), and so, if  $yz^{-1} \in U$ , then

$$\|_y \Psi -_z \Psi \|_\omega^\infty < \varepsilon \delta^2 \| \varphi \|_\omega^\infty$$



which implies that  $\Psi$  belongs to  $RUC(1/\omega)$ . By the definition of the convolution product  $\int_G f(x)_{x^{-1}}\varphi dx$  represents  $f * g$ . Thus it is not difficult to see that  $P(f * \varphi) = f * P(\varphi)$ .

To prove (iv)  $\implies$  (v), let  $P : RUC(1/\omega) \rightarrow X \cap RUC(1/\omega)$  be a bounded projection onto  $X \cap RUC(1/\omega)$  with  $P(f * \varphi) = f * P\varphi$  for all  $f \in L^1(\omega), \varphi \in RUC(1/\omega)$ . Define  $P' : L^\infty(1/\omega) \rightarrow L^\infty(1/\omega)$  by

$$\langle p, P'\varphi \rangle = \overline{P(f^* * \varphi)(e_G)} \quad (\varphi \in L^1(1/\omega), f \in L^\infty(\omega))$$

To see that the range of  $P'$  is  $X$ , first observe that when  $\varphi \in RUC(1/\omega)$   $\overline{P(f^* * \varphi)(e_G)} = \langle f, P\varphi \rangle$ ; that is,  $P' \big|_{RUC(1/\omega)} = P$ . Let  $\{e_\alpha\}$  be a bounded a.i. in  $L^1(\omega)$ . By Lemma 3, we obtain

$$\begin{aligned} \langle f, P\varphi \rangle &= \lim_\alpha \langle e_\alpha * f, P\varphi \rangle = \lim_\alpha \overline{P(f^* * e_\alpha^* * \varphi)(e_G)} \\ &= \lim_\alpha \overline{P(f^* * P(e_\alpha^* * \varphi))(e_G)} \\ &= \lim_\alpha \langle f, P(e_\alpha^* * \varphi) \rangle = 0 \end{aligned}$$

which implies that  $P'\varphi \in (X_\perp)^\perp = X$ . Thus the range of  $P'$  is a subset of  $X$ .

If  $\varphi \in X, g \in X_\perp$ , then  $\langle g, f^* * \varphi \rangle = \langle f * g, \varphi \rangle = 0$  and so  $f^* * \varphi \in (X_\perp)^\perp = X$ . Hence  $f^* * \varphi \in X \cap RUC(1/\omega)$  and

$$\langle f, P'\varphi \rangle = \overline{P(f^* * \varphi)(e_G)} = \overline{f^* * \varphi(e_G)} = \langle f, \varphi \rangle;$$

that is,  $P'\varphi = \varphi$  for all  $\varphi \in X$  and so, the range of  $P'$  is  $X$ . Obviously,  $P'$  is linear and since

$$\begin{aligned} |\langle f, P'\varphi \rangle| &= |P(f^* * \varphi)(e_G)| \\ &= \omega(e_G) |P(f^* * \varphi)(e_G)| / \omega(e_G) \\ &\leq \omega(e_G) \operatorname{ess\,sup}_{x \in G} |P(f^* * \varphi)(x)| / \omega(x) \\ &= \omega(e_G) \|P(f^* * \varphi)\|_\omega^\infty \\ &\leq \omega(e_G) \|P\| \|f^* * \varphi\|_\omega^\infty \\ &\leq \delta\omega(e_G) \|P\| \|f\|_\omega^1 \|\varphi\|_\omega^\infty \end{aligned}$$

$P'$  is continuous with  $\|P'\| \leq \delta\omega(e_G) \|P\|$ . Consequently  $P'$  is an extension of  $P$  to  $L^\infty(1/\omega)$  as a bounded projection.

Now let  $\{e_\alpha\}$  be an a.i. for  $L^1(\omega)$  bounded by  $M$ . Set  $I = X_\perp$  and  $C = (1 + \delta\omega(e_G) \|P\|)M$ . Let  $E'$  be an element of  $(L^\infty(1/\omega))'$  which is  $\sigma((L^\infty(1/\omega))', L^\infty(1/\omega))$  closure of  $\{e_\alpha\}$  in  $L^1(\omega)$ . Define a linear functional  $E$  on  $L^\infty(1/\omega)$  such that

$$\langle \varphi, E \rangle = \langle \varphi - P'\varphi, E' \rangle$$

for all  $\varphi \in L^\infty(1/\omega)$ . Then  $\|E\| \leq (1 + \|P'\|) \|E'\| \leq (1 + \delta\omega(e_G) \|P\|)M = C$  and  $\langle \varphi, E \rangle = 0$  for all  $\varphi \in I^\perp$ . Thus  $E \in B_c(I'') = \{F \in I'' : \|F\| \leq C\}$  where  $I''$  denotes the continuous bidual of  $I$ . By the Alaoglu Theorem  $B_c(I)$  is weak \*-dense in  $B_c(I'')$ , and so, there exists a net  $\{u_\beta\}$  in  $B_c(I)$  such that  $u_\beta \rightarrow E$  with respect to  $\sigma(I'', I')$  topology. Since  $X$  is complemented in  $L^\infty(1/\omega)$ , it is easy to see that, also  $u_\beta \rightarrow E$  with respect to  $\sigma(L^1(\omega)'', L^\infty(1/\omega))$ . It has a right a. i. bounded by  $C$  if and only if it has a weak right a.i. bounded by  $C$  (see, for example ([3])) and so, we need only to show that  $\{u_\beta\}$  is a weak right a.i. for  $I$ . To this end, let  $f \in I, \varphi \in L^\infty(1/\omega)$ . Then we have

$$\begin{aligned} \lim_{\beta} \langle f * u_\beta - f, \varphi \rangle &= \lim_{\beta} \langle f * u_\beta, \varphi \rangle - \langle f, \varphi \rangle \\ &= \lim_{\beta} \langle u_\beta, f^* * \varphi \rangle - \langle f, \varphi \rangle \\ &= \langle f^* * \varphi - P'(f^* \varphi), E' \rangle - \langle f, \varphi \rangle \\ &= \langle f^* * \varphi, E' \rangle - \langle f, \varphi \rangle - \langle P'(f^* * \varphi), E' \rangle \\ &= \lim_{\alpha} \langle e_\alpha, f^* * \varphi \rangle - \langle f, \varphi \rangle - \lim_{\alpha} \langle e_\alpha, P'(f^* * \varphi) \rangle \\ &= \lim_{\alpha} \langle f * e_\alpha, \varphi \rangle - \overline{\lim_{\alpha} P(e_\alpha^* * f^* * \varphi)(e_G)} - \langle f, \varphi \rangle \\ &= -\overline{\lim_{\alpha} P((f * e_\alpha)^* * \varphi)(e_G)} \\ &= -\langle f, P'\varphi \rangle = 0 \quad \text{since } P'\varphi \in X \end{aligned}$$

which implies that  $\{u_\beta\}$  is a bounded weak right a.i.. This completes the proof of the implication (iv)  $\implies$  (v).

Finally to show (v)  $\implies$  (i), let  $\{u_\alpha\}$  be a bounded right a.i. for  $X_\perp$  and  $E$  be the weak \*-limit point of the net  $\{u_\alpha\}$  in  $L^1(\omega)''$ . Then the mapping  $P : L^\infty(1/\omega) \rightarrow L^\infty(1/\omega)$ , defined by,

$$\langle f, P\varphi \rangle = \langle f, \varphi \rangle - \langle f^* * \varphi, E \rangle$$

For all  $f \in L^1(\omega)$  and  $\varphi \in L^\infty(1/\omega)$ . It is not difficult to verified that  $P$  is a bounded projection with range  $X$ . For all  $f \in L^1(\omega), \varphi \in L^\infty(1/\omega)$  since

$$\begin{aligned} \langle g, P(f^* \varphi) \rangle &= \langle g, f^* \varphi \rangle - \langle g^* * f^* \varphi, E \rangle \\ &= \langle f^* * g, \varphi \rangle - \langle g^* * f^* \varphi, E \rangle \\ &= \langle f, f^* P\varphi \rangle \quad (g \in L^1(\omega)) \end{aligned}$$

we have  $P(f^* \varphi) = f^* P(\varphi)$  and it follows that  $X$  is topologically invariantly complemented in  $L^\infty(1/\omega)$ , as required. This completes the proof of the Theorem.

**Remark 6.** The involution  $\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$  (*resp*  $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ ) on  $L^\infty(1/\omega)$  (*resp*  $L^1(\omega)$ ) transforms every left invariant subspace of  $L^\infty(1/\omega)$  (*resp.* left ideals of  $L^1(\omega)$ ) into a right invariant subspace of  $L^\infty(1/\omega)$  (*resp.* right ideals of  $L^1(\omega)$ ). Furthermore,  $(\tilde{X})_\perp = (X_\perp)^*$  for  $X \subseteq L^\infty(1/\omega)$  and a left ideal  $I$  of  $L^1(\omega)$  has a bounded right a.i. if and only if  $I^*$  has a bounded left a.i. since the involution “\*” is continuous. Thus the Theorem 5 has an obvious version for right invariant subspaces of  $L^\infty(1/\omega)$ .

**Remark 7.** If we take  $\omega \equiv 1_G$  ( $1_G(x) = 1, x \in G$ ), then  $L^1(\omega) = L^1(G)$  and so, this study is clearly a generalization of the work [2].

We also would like to mention here that the equivalence (i)  $\implies$  (v) in the Theorem 5 is true for arbitrary Banach algebras (see, ([4], 4.1.4)

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**$L^\infty(1/\omega)$  UZAYININ INVARYANT TAMLANMIŞ ALT UZAYLARI VE  
 $L^1(\omega)$ ' NİN YAKLAŞIK BİRİME SAHİP İDEALLERİ**

**Özet**

$L^\infty(\omega)$  lokal kompakt  $G$  grubu üzerinde ağırlıklı grup cebiri ve  $X$  de  $L^\infty(1/\omega)$  nin zayıf  $*$ -topolojisine göre kapalı, sol ötelemeye göre invaryant olan bir alt uzay olsun. Bu çalışmada;

- (i)  $X, L^\infty(1/\omega)$  içinde topolojik olarak sol ötelemeye göre invaryant ve tamlanmış bir alt uzaydır;
- (ii)  $X, L^\infty(1/\omega)$  da sol ötelemeye göre invaryant ve tamlanmış uzaydır;
- (iii)  $L^1(\omega)$  içindeki her  $X_\perp$  sol idealinin yaklaşık birimi vardır;

önermelerinin denk olduklarını gösteriyoruz. Dolayısıyla  $X$  in  $L^\infty(1/\omega)$ 'da tamlanmış bir alt uzayı olması için gerek ve yeter şart  $X_\perp$  nin en az bir yaklaşık biriminin olmasıdır.

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