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CERTAIN CLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

We introduce a subclass $K^*_{n+p-1}(A,B)$ of analytic and p-valent functions with negative coefficients. Coefficient estimates, some properties, distortion theorems and closure theorems of functions belonging to the class $K^*_{n+p-1}(A,B)$ are determined. Also we obtain radii of close-to-convexity, starlikeness and convexity for the class $K^*_{n+p-1}(A,B)$. We also obtain class preserving integral operator of the form

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt, c>-p$$

for the class $K^*_{n+p-1}(A,B)$ Conversely when $F(z)\ \in K^*_{n+p-1}(A,B)$ radius of p-valence of $f(z)$ defined by the above equation is obtained.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$f(z)=z^P+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}(p\in\mathbb{N}=\{1,2,\cdots\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc $U=\{z:|z|<1\}$. Let $f(z)$ be in $S(p)$ and $g(z)$ be in $S(p)$. Then we denote by $f\ \star g(z)$ the Hadamard product of $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{p+k}z^{p+k}(p\in\mathbb{N}), \tag{1.2}$$

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then

$$f * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.3}$$

The $(n+p-1)-th$ order Ruscheweyh derivative $D^{n+p-1}f(z)$ of a function $f(z)$ of $S(p)$ is defined by

$$D^{n+p-1}f(z) = \frac{z^p (z^{n-1} f(z))^{n+p-1)}}{(n+p-1)!} \tag{1.4}$$

where n is any integer such that $n > -p$. It is easy to see that

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \tag{1.5}$$

$$= z^p + \sum_{k=1}^{\infty} \delta(n,k) a_{p+k} z^{p+k}, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+p+k-1}{n+p-1}. \tag{1.7}$$

Particularly, the symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z) \in S(1)$ by Al-Amiri [1].

Let $T(p)$ denote the subclass of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0; p \in \mathbb{N}). \tag{1.8}$$

Also let $K^*_{n+p-1}(A,B)$ denote the class of functions $f(z) \in T(p)$ such that

$$\left| \frac{2\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - 1\right)}{2B\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (A+B)} \right| < 1, (z \in U), \tag{1.9}$$

where $-1 \leq A < B \leq 1, 0 < B \leq 1$, and $n > -p$.

We note that:

(i) $K^*_{n+p-1}(-1,1) = K^*_{n+p-1}$ (Owa [3]);

(ii) $K^*_0((4\gamma-3)\beta, \beta) = T^*(\gamma,\beta)(0 \leq \gamma < 1, 0 < \beta \leq 1)$ (Gupta and Jain [2]);

(iii) $K^*_0(4\gamma-3,1) = T^*(\gamma)(0 \leq \gamma < 1)$ (Silverman [4]).

2. Coefficient Estimates

Theorem 1. *Let the function f(z) be defined by (1.8). Then f(z) is in the class* $K^*_{n+p-1}(A,B)$ *if and only if*

$$\sum_{k=1}^{\infty} D_k \delta(n,K) a_{p+k} \leq (B-A)(n+p), \tag{2.1}$$

where

$$D_k = [2k(B+1) + (B-A)(n+p)]. \tag{2.2}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we get

$$\begin{aligned}
&\left|2(D^{n+p}f(z) - D^{n+p-1}f(z))\right| - \left|2BD^{n+p}f(z) - (A+B)D^{n+p-1}f(z)\right| \\
&= \left|-2\sum_{k=1}^{\infty}\left(\frac{k}{n+p}\right)\delta(n,k)a_{p+k}z^{p+k}\right| - \Bigg|(B-A)z^P - \\
&\sum_{k=1}^{\infty}\left[2B\left(\frac{k}{n+p}\right) + (B-A)\right]\delta(n,k)a_{p+k}z^{p+k}\Bigg| \\
&\leq \sum_{k=1}^{\infty} 2\left(\frac{k}{n+p}\right)\delta(n,k)a_{p+k} - (B-A) + \sum_{k=1}^{\infty}\left[2B\left(\frac{k}{n+p}\right) + (B-A)\right]\delta(n,k)a_{p+k} \\
&= \sum_{k=1}^{\infty}\left[\frac{2k}{(n+p)}(B+1) + (B-A)\right]\delta(n,k)a_{p+k} - (B-A) \\
&\leq 0, \text{ by hypotheses.}
\end{aligned}$$

Hence by the maximum modulus theorem $f(z) \in K^*_{n+p-1}(A,B)$.
Conversely, suppose that

$$\left|\frac{2\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - 1\right)}{2B\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (A+B)}\right| = \left|\frac{-2\sum_{k=1}^{\infty}\left(\frac{k}{n+p}\right)\delta(n,k)a_{p+k}z^k}{(B-A) - \sum_{k=1}^{\infty}\left[2B\left(\frac{k}{n+p}\right) + (B-A)\right]\delta(n,k)a_{p+k}z^k}\right| \leq 1, z \in U. \tag{2.3}$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{\frac{-2\sum_{k=1}^{\infty}\left(\frac{k}{n+p}\right)\delta(n,k)a_{p+k}z^k}{(B-A)-\sum_{k=1}^{\infty}\left[2B\left(\frac{k}{n+p}\right)+(B-A)\right]\delta(n,k)a_{p+k}z^k}\right\} < 1. \tag{2.4}$$

Choose values of z on the real axis so that $\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we obtain

$$2\sum_{k=1}^{\infty}\left(\frac{k}{n+p}\right)\delta(n,k)a_{p+k} \leq (B-A)$$
$$-\sum_{k=1}^{\infty}\left[2B\left(\frac{k}{n+p}\right)+(B-A)\right]\delta(n,k)a_{p+k}.$$

This gives the required condition.

Finally, the function

$$f(z) = z^P - \frac{(B-A)(n+p)}{D_k\delta(n,k)}z^{p+k} \qquad (k \geq 1) \tag{2.5}$$

is an extremal function for the theorem. □

Corollary 1. *Let the function f(z) defined by (1.8) be in the class* $K^*_{n+p-1}(A,B)$. *Then*

$$a_{p+k} \leq \frac{(B-A)(n+p)}{D_k\delta(n,k)} \qquad (k \geq 1). \tag{2.6}$$

The result is sharp for the function f(z) given by (2.5).

3. Some Properties of the Class $K^*_{n+p-1}(A,B)$

Theorem 2. $K^*_{n+p}(A,B) \subset K^*_{n+p-1}(A,B)$ *for* $p \in \mathbb{N}, n > -p, -1 \leq A < B \leq 1$, *and* $0 < B \leq 1$.

Proof. Let the function $f(z)$ defined by (1.8) be in the class $K^*_{n+p}(A,B)$. Then, by Theorem 1, we have

$$\sum_{k=1}^{\infty}\left(\frac{2k(B+1)}{(n+p+1)}+(B-A)\right)\delta(n+1,k)a_{p+k} \leq (B-A) \tag{3.1}$$

and since

$$\Big(\frac{2k(B+1)}{(n+p)}+(B-A)\Big)\delta(n,k)\leq\Big(\frac{2k(B+1)}{(n+p+1)}+(B-A)\Big)\delta(n+1,k) \quad \text{for } k\geq 1, \tag{3.2}$$

we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\Big(\frac{2k(B+1)}{(n+p)}+(B-A)\Big)\delta(n,k)a_{p+k}\\ &\leq\sum_{k=1}^{\infty}\Big(\frac{2k(B+1)}{(n+p+1)}+(B-A)\Big)\delta(n+1,k)a_{p+k}\leq(B-A).\end{aligned} \tag{3.3}$$

The result follows from Theorem 1. □

Theorem 3. *Let* $-1\leq A_1\leq A_2<B_1\leq B_2\leq 1$ *and* $0<B_1\leq B_2\leq 1$. *Then we have*

$$K^*_{n+p-1}(A_1,B_2)\supseteq K^*_{n+p-1}(A_2,B_1).$$

Proof. Theorem 3 is an immediate consequence of the definitin of the class $K^*_{n+p-1}(A,B)$. □

4. Distortion Theorems

Theorem 4. *Let the function f(z) defined by (1.8) be in the class* $K^*_{n+p-1}(A,B)$. *Then we have*

$$|z|^p-\frac{(B-A)}{D_1}|z|^{p+1}\leq|f(z)|\leq|z|^P+\frac{(B-A)}{D_1}|z|^{p+1} \tag{4.1}$$

for $z\in U$. *The result is sharp.*

Proof. Since $f(z)\in K^*_{n+p-1}(A,B)$, in view of Theorem 1, we obtain

$$\begin{aligned}D_1\delta(n,1)\sum_{k=1}^{\infty}a_{p+k}&\leq\sum_{k=1}^{\infty}D_k\delta(n,K)a_{p+k}\\ &\leq(B-A)(n+p),\end{aligned} \tag{4.2}$$

which implies that

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{(B-A)}{D_1}. \tag{4.3}$$

Therefore we can show that

$$\begin{aligned} |f(z)| &\geq |z|^p - |z|^{p+1}\sum_{k=1}^{\infty} a_{p+k} \\ &\geq |z|^P - \frac{(B-A)}{D_1}|z|^{p+1} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} |f(z)| &\leq |z|^p + |z|^{p+1}\sum_{k=1}^{\infty} a_{p+k} \\ &\leq |z|^P + \frac{(B-A)}{D_1}|z|^{p+1} \end{aligned} \tag{4.5}$$

for $z \in U$. This completes the proof of Theorem 4. Finally, by taking the function

$$f(z) = z^P - \frac{(B-A)}{D_1}z^{p+1}, \tag{4.6}$$

we can show that the result of Theorem 4 is sharp. □

Corollary 2. *Let the function $f(z)$ defined by (1.8) be in the class $K^*_{n+p-1}(A,B)$. Then f(z) is included in a disc with its center at the origin and radius r_1 given by*

$$r_1 = \frac{D_1 + (B-A)}{D_1}. \tag{4.7}$$

Theorem 5 . *Let the function f(z) defined by (1.8) be in the class $K^*_{n+p-1}(A,B)$. Then we have*

$$p|z|^{p-1} - \frac{(B-A)(p+1)}{D_1}|z|^P \leq |f'(z)| \leq p|z|^{p-1} + \frac{(B-A)(p+1)}{D_1}|z|^P \tag{4.8}$$

for $z \in U$. The result is sharp.

Proof. In view of Theorem 1, we have

$$\begin{aligned}\frac{D_1\delta(n,1)}{(p+1)}\sum_{k=1}^{\infty}(p+k)a_{p+k} &\le \sum_{k=1}^{\infty}D_k\delta(n,K)a_{p+k}\\ &\le (B-A)(n+p)\end{aligned}\tag{4.9}$$

which implies that

$$\sum_{k=1}^{\infty}(p+k)a_{p+k} \le \frac{(B-A)(p+1)}{D_1}.\tag{4.10}$$

Hence, with the aid of (4.10), we have

$$\begin{aligned}|f'(z)| &\ge p|z|^{p-1} - |z|^p\sum_{k=1}^{\infty}(p+k)a_{p+k}\\ &\ge p|z|^{p-1} - \frac{(B-A)(p+1)}{D_1}|z|^p\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}|f'(z)| &\le p|z|^{p-1} + |z|^p\sum_{k=1}^{\infty}(p+k)a_{p+k}\\ &\le p|z|^{p-1} + \frac{(B-A)(p+1)}{D_1}|z|^p\end{aligned}\tag{4.12}$$

for $z \in U$. Further the results of Theorem 5 are sharp for the function $f(z)$ given by (4.6). □

Corollary 3. *Let the function $f(z)$ defined by (1.8) be in the class $K^*_{n+p-1}(A,B)$. Then $f'(z)$ is included in a disc with its center at the origin and radius r_2 given by*

$$r_2 = \frac{PD_1 + (B-A)(p+1)}{D_1}\tag{4.13}$$

5. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \cdots, m$, by

$$f_i(z) = z^P - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} (a_{p+k,i} \geq 0) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $K^*_{n+p-1}(A, B)$.

Theorem 6. *Let the functions $f_i(z)$ defined by (5.1) be in the class $K^*_{n+p-1}(A, B)$ for every $i = 1, 2, \cdots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geq 0) \tag{5.2}$$

is also in the same class $K^*_{n+p-1}(A, B)$, where

$$\sum_{i=1}^{m} c_i = 1. \tag{5.3}$$

Proof. By means of the definition of $h(z)$, we obtain

$$h(z) = z^P - \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{m} c_i a_{p+k,i} \Big) z^{p+k}. \tag{5.4}$$

Further, since $f_i(z)$ are in $K^*_{n+p-1}(A, B)$ for every $i = 1, 2, \cdots, m$, we get

$$\sum_{k=1}^{\infty} D_k \delta(n, k) a_{p+k,i} \leq (B - A)(n + p) \tag{5.5}$$

for every $i = 1, 2, \cdots, m$. Hence we can see that

$$\begin{aligned} &\sum_{k=1}^{\infty} D_k \delta(n, k) \Big(\sum_{i=1}^{m} c_i a_{p+k,i} \Big) \\ &= \sum_{i=1}^{m} c_i \Big(\sum_{k=1}^{\infty} D_k \delta(n, k) a_{p+k,i} \Big) \\ &\leq \Big(\sum_{i=1}^{m} c_i \Big) (B - A)(n + p) = (B - A)(n + p) \end{aligned} \tag{5.6}$$

with the aid of (5.5). This proves that the function $h(z)$ is in the class $K^*_{n+p-1}(A,B)$ by means of Theorem 1. Thus we have the theorem. □

Theorem 7. *Let* $f_p(z) = z^P$ *and*

$$f_{p+k}(z) = z^P - \frac{(B-A)(n+p)}{D_k\delta(n,k)} z^{p+k} \qquad (k \geq 1) \tag{5.7}$$

for $p \in \mathbb{N}, n > -p, -1 \leq A < B \leq 1$ *and* $0 < B \leq 1$. *Then f(z) is in the class* $K^*_{n+p-1}(A,B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{5.8}$$

where $\lambda_{p+k} \geq 0 (k \geq 0)$ *and* $\sum_{k=0}^{\infty} \lambda_{p+k} = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z) \\ &= z^P - \sum_{k=1}^{\infty} \frac{(B-A)(n+p)}{D_k\delta(n,k)} \lambda_{p+k} z^{p+k}. \end{aligned} \tag{5.9}$$

Then it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} D_k\delta(n,k) \frac{(B-A)(n+p)}{D_k\delta(n,k)} \lambda_{p+k} \\ &= (B-A)(n+p) \sum_{k=1}^{\infty} \lambda_{p+k} \\ &= (B-A)(n+p)(1-\lambda_p) \leq (B-A)(n+p). \end{aligned} \tag{5.10}$$

So by Theorem 1, $f(z) \in K^*_{n+p-1}(A,B)$.

Conversely, assume that the function $f(z)$ defined by (1.8) belongs to the class $K^*_{n+p-1}(A,B)$. Then

$$a_{p+k} \leq \frac{(B-A)(n+p)}{D_k\delta(n,k)} \qquad (k \geq 1). \tag{5.11}$$

Setting

$$\lambda_{p+k} = \frac{D_k\delta(n,k)}{(B-A)(n+p)}a_{p+k} \tag{5.12}$$

and

$$\lambda_p = 1 - \sum_{k=1}^{\infty}\lambda_{p+k}. \tag{5.13}$$

We can see that $f(z)$ can be expressed in the form (5.8). This completes the proof of Theorem 7. □

Corollary 4. *The extreme points of the class $K^*_{n+p-1}(A,B)$ are the functions $f_{p+k}(z)(k \geq 0)$ given by Theorem 7.*

6. Integral Operators

Theorem 8. *Let the function $f(z)$ defined by (1.8) in the class $K^*_{n+p-1}(A,B)$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{6.1}$$

also belongs to the class $K^*_{n+p-1}(A,B)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^P - \sum_{k=1}^{\infty}b_{p+k},$$

where

$$b_{p+k} = \Big(\frac{c+p}{c+p+k}\Big)a_{p+k}.$$

Therefore,

$$\sum_{k=1}^{\infty}D_k\delta(n,k)b_{p+k} = \sum_{k=1}^{\infty}D_k\delta(n,k)\Big(\frac{c+p}{c+p+k}\Big)a_{p+k}$$

$$\leq \sum_{k=1}^{\infty}D_k\delta(n,k)a_{p+k} \leq (B-A)(n+p),$$

since $f(z) \in K^*_{n+p-1}(A,B)$. Hence, by Theorem 1, $F(z) \in K^*_{n+p-1}(A,B)$.

Putting $c = 1-p$ in Theorem 8, we get the following. □

Corollary 5. *Let the function f(z) defined by (1.8) be in the class* $K^*_{n+p-1}(A,B)$ *and let F(z) be defined by*

$$F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^P} dt. \tag{6.2}$$

Then $F(z) \in K^*_{n+p-1}(A,B)$.

Theorem 9. *Let c be a real number such that* $c > -p$. *If* $F(z) \in K^*_{n+p-1}(A,B)$, *then the function* $f(z)$ *defined by (6.1) is p-valent in* $|z| < R^*_p$, *where*

$$R^*_p = \inf_k \left\{ \frac{p(c+p)D_k\delta(n,k)}{(p+k)(c+p+k)(B-A)(n+p)} \right\}^{\frac{1}{k}} \quad (k \geq 1). \tag{6.3}$$

The result is sharp.

Proof. Let $F(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0)$. It follows from (6.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{(c+p)} a_{p+k} z^{p+k}, (c > -p) \\ &= z^P - \sum_{k=1}^{\infty} \Big(\frac{c+p+k}{c+p} \Big) a_{p+k} z^{p+k}. \end{aligned}$$

To prove the result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \text{ for } |z| < R^\star_p.$$

Now

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| -\sum_{k=1}^{\infty} (p+k) \Big(\frac{c+p+k}{c+p} \Big) a_{p+k} z^k \right| \\ &\leq \sum_{k=1}^{\infty} (p+k) \Big(\frac{c+p+k}{c+p} \Big) a_{p+k} |z|^k. \end{aligned}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$ if

$$\sum_{k=1}^{\infty}\Big(\frac{p+k}{p}\Big)\Big(\frac{c+p+k}{c+p}\Big)a_{p+k}|z|^k \leq 1. \tag{6.4}$$

But Theorem 1 confirms that

$$\sum_{k=1}^{\infty}\frac{D_k\delta(n,k)}{(B-A)(n+p)}a_{p+k} \leq 1. \tag{6.5}$$

Thus (6.4) will be satisfied if

$$\Big(\frac{p+k}{p}\Big)\Big(\frac{c+p+k}{c+p}\Big)|z|^k \leq \frac{D_k\delta(n,k)}{(B-A)(n+p)} \qquad (k \geq 1),$$

or if

$$|z| \leq \left\{\frac{p(c+p)D_k\delta(n,k)}{(p+k)(c+p+k)(B-A)(n+p)}\right\}^{\frac{1}{k}} \quad (k \geq 1). \tag{6.6}$$

The required result follows now from (6.6). The result is sharp for the function

$$f(z) = z^p - \frac{D_k\delta(n,k)(c+p+k)}{(B-A)(n+p)(c+p)}z^{p+k} \qquad (k \geq 1). \tag{6.7}$$

□

7. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 10. *Let the function f(z) defined by (1.8) be in the class $K^*_{n+p-1}(A,B)$, then f(z) is p-valently close-to-convex of order $\alpha(0 \leq \alpha < p)$ in $|z| < r_1(A,B,n,p,\alpha)$ where*

$$r_1(A,B,n,p,\alpha) = \inf_k \left[\frac{(p-\alpha)D_k\delta(n,k)}{(p+k)(B-A)(n+p)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.1}$$

The result is sharp with the extremal function f(z) given by (2.5).

Proof. We must show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \alpha$ for $|z| < r_1(A,B,n,p,\alpha)$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \alpha$ if

$$\sum_{k=1}^{\infty} \frac{(p+k)}{(p-\alpha)} a_{p+k}|z|^k \leq 1. \tag{7.2}$$

Hence, by Theorem 1, (7.2) will be true if

$$\frac{(p+k)}{(p-\alpha)}|z|^k \leq \frac{D_k \delta(n,k)}{(B-A)(n+p)}$$

or if

$$|z| \leq \left[\frac{(p-\alpha) D_k \delta(n,k)}{(p+k)(B-A)(n+p)}\right]^{\frac{1}{k}}, (k \geq 1). \tag{7.3}$$

The theorem follows easily from (7.3). □

Theorem 11. *Let the function f(z) defined by (1.8) be in the class* $K^*_{n+p-1}(A,B)$, *then f(z) is p-valently starlike of order* α $(0 \leq \alpha < p)$ *in* $|z| < r_2(A,B,n,p,\alpha)$ *where*

$$r_2(A,B,n,p,\alpha) = \inf_k \left[\frac{(p-\alpha) D_k \delta(n,k)}{(p+k-\alpha)(B-A)(n+p)}\right]^{\frac{1}{k}} (k \geq 1). \tag{7.4}$$

The result is sharp with the extremal function f(z) given by (2.5).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \alpha$ for $|z| < r_2(A,B,n,p,\alpha)$. We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=1}^{\infty} k a_{p+k}|z|^k}{1 - \sum_{k=1}^{\infty} a_{p+k}|z|^k}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \alpha$ if

$$\sum_{k=1}^{\infty} \frac{(p+k-\alpha) a_{p+k}|z|^k}{(p-\alpha)} \leq 1. \tag{7.5}$$

Hence, by Theorem 1, (7.5) will be true if

$$\frac{(p+k-\alpha)|z|^k}{(p-\alpha)} \leq \frac{D_k \delta(n,k)}{(B-A)(n+p)}$$

or if

$$|z| \leq \left[\frac{(p-\alpha)D_k\delta(n,k)}{(p+k-\alpha)(B-A)(n+p)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.6}$$

The theorem follows easily from (7.6). □

Corollary 6. *Let the function f(z) defined by (1.8) be in the class $K^*_{n+p-1}(A,B)$, then f(z) is p-valently convex of order $\alpha(0 \leq \alpha < p)$ in $|z| < r_3(A,B,n,p,\alpha)$ where*

$$r_3(A,B,n,p,\alpha) = \inf_k \left[\frac{p(p-\alpha)D_k\delta(n,k)}{(p+k)(p+k-\alpha)(B-A)(n+p)}\right]^{\frac{1}{k}} \quad (k \geq 1). \tag{7.7}$$

The result is sharp with the extremal function f(z) given by (2.5).

8. Modified Hadamard Product

Let the functions $f_i(z)(i = 1,2)$ be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1}a_{p+k,2}z^{p+k}. \tag{8.1}$$

Theorem 12. *Let the function $f_1(z)$ defined by (5.1) be in the class $K^*_{n+p-1}(A,B)$ and the function $f_2(z)$ defined by (5.1) be in the class $K^*_{n+p-1}(C,E)(-1 \leq C \leq E \leq 1, 0 < E \leq 1)$. Then the modified Hadamard product $f_1 * f_2(z)$ belongs to the class*

$$K^*_{n+p-1}\left(1 - \frac{4(n+p)(B-A)(E-C)}{\delta(n,1)D_1G_1 - (n+p)^2(B-A)(E-C)}, 1\right) \tag{8.2}$$

where $D_k(k \geq 1)$ is defined by (2.2) and $G_k(k \geq 1)$ is given by

$$G_k = [2k(E+1) + (E-C)(n+p)]. \tag{8.3}$$

The result is sharp.

Proof. From Theorem 1, we have

$$\sum_{k=1}^{\infty} \frac{D_k\delta(n,k)}{(B-A)(n+p)} a_{p+k} \leq 1 \tag{8.4}$$

and

$$\sum_{k=1}^{\infty} \frac{G_k \delta(n,k)}{(E-C)(n+p)} a_{p+k} \leq 1. \tag{8.5}$$

We want to find the largest $\beta = \beta(n,p,A,B,C,E)$ such that

$$\sum_{k=1}^{\infty} \frac{[4k+(1-\beta)(n+p)]\delta(n,k)}{(1-\beta)(n+p)} a_{p+k,1}a_{p+k,2} \leq 1. \tag{8.6}$$

From (8.4) and (8.5) by means of Cauchy-Schawarz inequality we obtain

$$\sum_{k=1}^{\infty} \sqrt{\frac{D_k G_k}{(B-A)(E-C)} \frac{\delta(n,k)}{(n+p)}} \sqrt{a_{p+k,1}a_{p+k,2}} \leq 1. \tag{8.7}$$

Hence (8.6) will be satisfied if

$$\sqrt{a_{p+k,1}a_{p+k,2}} \leq \frac{(1-\beta)}{[4k+(1-\beta)(n+p)]} \sqrt{\frac{D_k G_k}{(B-A)(E-C)}} (k \geq 1). \tag{8.8}$$

From (8.7) it follows that

$$\sqrt{a_{p+k,1}a_{p+k,2}} \leq \frac{(n+p)}{\delta(n,k)} \sqrt{\frac{(B-A)(E-C)}{D_k G_k}} (k \geq 1). \tag{8.9}$$

Therefore (8.6) will be satisfied if

$$\frac{(n+p)}{\delta(n,k)} \sqrt{\frac{(B-A)(E-C)}{D_k G_k}} \leq \frac{(1-\beta)}{[4k+(1-\beta)(n+p)]} \sqrt{\frac{D_k G_k}{(B-A)(E-C)}} (k \geq 1). \tag{8.10}$$

that is, that

$$\beta \leq 1 - \frac{4k(n+p)(B-A)(E-C)}{\delta(n,k)D_kG_k - (n+p)^2(B-A)(E-C)}. \tag{8.11}$$

The right-hand side of (8.11) is an increasing function of k $(k \geq 1)$. Therefore, setting $k = 1$ in (8.11) we get

$$\beta \leq 1 - \frac{4(n+p)(B-A)(E-C)}{\delta(n,1)D_1G_1 - (n+p)^2(B-A)(E-C)}.$$

The result is sharp, with equality when

$$f_1(z) = z^P - \frac{(B-A)}{D_1} z^{p+1}$$

and

$$f_2(z) = z^P - \frac{(E-C)}{G_1} z^{p+1}.$$

□

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NEGATİV KATSAYILI BAZI ANALİTİK VE ÇOK KATLI FONKSİYON SINIFLARI

Özet

Analitik, p-katlı ve negativ katsayılı fonsiyonların bir $K^{\star}_{n+p-1}(A,B)$ altsınıfı tanımlanıp, bu sınıf için katsayı kestirmeleri bozulma teoremleri, kapanış teoremleri kanıtlanmıştır. Ayrıca bu sınıfın özellikleri incelenmiş ve elemanlarının yıldızlık, konvekslik, yaklaşık konvekslik çapları hesaplanmıştır.

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