Turkish Journal of Mathematics

Volume 20 | Number 4 Article 13

1-1-1996

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HADJIEV, D.; ÇALLIALP, F.; and EDEN, A. (1996) "ON A DIFFERENTIAL ANALOG OF THE PRIME-RADICAL AND PROPERTIES OF THE LATTICE OF RADICAL DIFFERENTIAL IDEALS IN ASSOCIATIVE DIFFERENTIAL RINGS," *Turkish Journal of Mathematics*: Vol. 20: No. 4, Article 13. Available at: https://journals.tubitak.gov.tr/math/vol20/iss4/13

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Tr. J. of Mathematics
20 (1996) , 571 – 582.
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ON A DIFFERENTIAL ANALOG OF THE PRIME-RADICAL AND PROPERTIES OF THE LATTICE OF RADICAL DIFFERENTIAL IDEALS IN ASSOCIATIVE DIFFERENTIAL RINGS

D. Hadjiev & F. Çallıalp

Abstract

In this paper we prove the following results: (1) For any assosiative differential ring with the unit we introduce a differential analog of the prime-radical and describe it; (2) any maximal differential ideal of a Ritt algebra is prime; (3) The lattice of radical differential ideals satisfies the condition of infinite $\cap$- distributivity.

0. Introduction

Let K be an associative differential ring with the unit. (i.e **d- ring**). Denote by $L_d(K)$ the set of all differential ideals (**d- ideals**) of K. Consider in $L_d(K)$ the relation of the inclusion of d- ideals and the operation of the multiplication of ideals. Then $L_d(K)$ is a complete lattice with the operation multiplication and it is an integral l-monoid. ([1], ch. XIV).

A d-ideal H is called **maximal** if $H \neq K, H \subseteq B \subseteq K, B \in L_d(K)$ implies that $H = B$ or $B = K$.

A d-ideal $H \in L_d(K), H \neq K$ will be called **d-prime** if $B.C \subseteq H, B, C \in L_d(K)$ implies that $B \subseteq H$ or $C \subseteq K$.

For $B \in L_d(K), B \neq K$, denote by $r_d(B)$ the intersection of all d-prime d-ideals containing B. A d-ideal B will be called **d-radical** if $B = r_d(B)$. Denote by $L_d(K)^r$ the lattice of d-radical d-ideals of K. We include K in $L_d(K)^r$ as the biggest element.

Our main results are the following:

i) For any d-ring K we introduce a differential analog N_d of the prime radical of a ring and describe N_d.

ii) Any maximal d-ideal of a Ritt algebra is a prime ideal.

iii) The lattice $L_d(K)^r$ satisfies the following condition:

$$A \wedge (\vee_{t\in T} B_t) = \vee_{t\in T}(A \wedge B_t)$$

for any $A, B_t \in L_d(K)^r, t \in T$. In particular, $L_d(K)^r$ is distributive.

The part of our results annonced in [2]. Further we use notions and notations of the book [4].

1. A Differential Analog of the Prime-Radical

Proposition 1.1 *For any d-ring K, there exists a maximal d- ideal of K.*

The proof of this proposition is standart.

Corollary 1.2 *For any d-ideal B of a d-ring K, there exists a maximal d-ideal C of K such that $B \subseteq C$.*

Proposition 1.3 *Any maximal d-ideal of a d-ring K is d-prime.*

Proof. First we note that $L_d(K)$ is an integral l-monoid. ([1], ch. XIV). Therefore

$$(B \vee C) \cdot A = B \cdot A \vee C \cdot A,\ A \cdot (B \vee C) = A \cdot B \vee A \cdot C,\ A \cdot K = A = K \cdot A \quad (1)$$

for any $A, B, C \in L_d(K)$.

Let H be a maximal d-ideal of K and $B \cdot C \subseteq H$ for some $B, C \in L_d(K)$. If $B \not\subseteq H$ then $B \vee H = K$. Using the properties (1), we obtain

$$C = K.C = (B \vee H).C = B.C \vee H.C \subseteq H. \quad (2)$$

Thus $B \subseteq H$ or $C \subseteq H$, that is H is d-prime. □

For an element x of a d-ring, denote by [x] the smallest d-ideal containing x.

Let $\mathbb{N}$ be the set of non-negative integers. Let K be a d-ring and d be a differentiation operation in K. For $a \in K$ and $n \in N$ put $a^{(0)} = a, a^{(n+1)} = da^{(n)}$.

Proposition 1.4 *For a d-ideal H of a d-ring, $H \neq K$, the following statements are equivalent:*
(1) H is d-prime.
(2) For $a, b \in K$ the condition $[a] \cdot [b] \subseteq H$ implies that $a \in H$ or $b \in H$.
(3) For $a, b \in K$ the conditions $a^{(m)} K b^{(n)} \subseteq H, \forall m, n \in \mathbb{N}$ imply that $a \in H$ or $b \in H$.
(4) For $a, b \in K$ the conditions $aKb^{(n)} \subseteq H, \forall n \in \mathbb{N}$ imply that $a \in H$ or $b \in H$.

Proof. (1) $\Longrightarrow$ (2) is obviously.

We prove that (2) $\Longrightarrow$ (1). Suppose that H satisfies the (2), $A, B \in L_d(K), A \cdot B \subseteq H$. If $A \not\subseteq H$ and $B \not\subseteq H$ then there exists $a \in A$ and $b \in B$ such that $a \notin H, b \notin H$. Thence $[a][b] \subseteq AB \subseteq H$. It is a contradiction. Thus (2) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (2).

Denote by $\sum$ the supremum of ideals of a ring K. We prove that (2) $\Longrightarrow$ (3). From the

$$[a] = \sum_{m \in \mathbb{N}} K a^{(m)} K, \quad [b] = \sum_{n \in \mathbb{N}} K b^{(n)} K$$

we obtain

$$[a][b] = \sum_{m,n \in \mathbb{N}} K a^{(m)} K b^{(n)} K.$$

Thence (2) $\Longleftrightarrow$ (3).

The implication (3) $\Longrightarrow$ (4) is obviously too.

Suppose $aKb^{(n)} \subseteq H$ for any $n \in \mathbb{N}$. We prove that $a^{(m)}Kb^{(n)} \subseteq H$ for any $m, n \in \mathbb{N}$.

Put $r = m + n$. If $r = 0$ then $aKb = a^{(0)}Kb^{(0)} \subseteq H$. Thence $(aKb)^{(1)}b \subseteq H$ and $a^{(1)}Kb = (aKb)^{(1)} - aKb^{(1)}b \subseteq H$.

Let us assume that $a^{(n)}Kb^{(m)} \subseteq H$ for any $m, n \in N, m + n \leq r$. Thence $a^{(1)}Kb^{(r)} = (aKb^{(r)})^{(1)} - aKb^{(r+1)} - aK^{(1)}b^{(r)} \subseteq H$. Further $a^{(2)}Kb^{(r-1)} = \left(a^{(1)}Kb^{(r-1)}\right)^{(1)} - a^{(1)}Kb^{(r-1)} \subseteq H$ etc. Thus (4) $\Longrightarrow$ (3). □

A non-empty set $S \subseteq K$ we call a **dm-system** if, for any $a, b \in S$, there exists $n \in \mathbb{N}$ and $r \in K$ such that $arb^{(n)} \in S$.

From the property (4) of d-prime d-ideals in the proposition 1.4 we obtain the following

Proposition 1.5 *A d-ideal H of a d-ring $K, H \neq K$, is d- prime iff $K \setminus H$ is a dm-system.*

A d-ideal B in a d-ring K we call **d-semiprime** if, for any d- ideal C of $K, C^2 \subseteq B$ implies that $C \subseteq B$.

A d-prime d-ideal is d-semiprime.

Proposition 1.6 *For any d-ideal B of the d-ring K, the following statements are equivalent:*
(1) B is d-semiprime.
(2) For $a \in K$, the condition $[a]^2 \subseteq B$ implies that $a \in B$.
(3) For $a \in K$, the conditions $a^{(m)}Ka^{(n)} \subseteq B, \forall \in \mathbb{N}$, imply that $a \in B$.
(4) For $a \in K$, the conditions $aKa^{(n)} \subseteq B, \forall n \in \mathbb{N}$, imply that $a \in B$.

The proof of this proposition is analoguosly to the proof of the proposition 1.4.

A set $S \subseteq K$ we call **dn-system** if for any $a \in S$, there exist $n \in \mathbb{N}$ and $r \in K$ such that $ara^{(n)} \in S$.

Remark The definitions of a dm-system and a dn-system are differential analog of the definitions of a m-system and a n-system in ([4], §10).

Proposition 1.7 *A d-ideal $B \subseteq K$ is d-semiprime iff $K \setminus B$ is a dn-system.*

The proof is obviously.

Let $x \in K$. Every sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$, where $x_0 = x, x_{n+1} \in [x_n]^2$, we call a **dn-sequence** of the element x.

Lemma 1.8 *Let S be a dn-system of a d-rign K and let $x \in S$. Then there exists a dn-sequence $\{x_0, x_2, \ldots, x_n, \ldots\} \subseteq S$ of the element x.*

Proof. Put $x_0 = x$. For S is a dn-system there exists $n \in N$ and $r_0 \in K$ such that $x_o r_o x_o^{(n)} \in S_0$. Put $x_1 = x_0 r_0 x_0^{(n)}$. Then $x_1 \in [x_0]^2$. Let us assume that there exists a set $\{x_0, \ldots, x_m\} \subset S$ such that $x_{i+1} \in [x_i]^2$ for any $i < m$. Then there exist $q \in \mathbb{N}$ and $r_m \in K$ such that $x_m r_m x_m^{(q)} \in S$. Put $x_{m+1} = x_m r_m x_m^{(q)}$. Then $x_{m+1} \in [x_m]^2$. □

Theorem 1.9 *For any d-ideal H of a d-ring K the following sets are equal.*
(1) the intersection of all d-prime d-ideals containing H;
(2) the set of $s \in K$ such that every dm-system containing s meets H;
(3) the set of $s \in K$ such that every dn-system containing s meets H.
(4) the set of $s \in K$ such that every dn-sequence of the element s meets H.

Proof. Denote by $D_i(H)$, the set defined in the condition (i) of the theorem, $i = 1, 2, 3, 4$. First we prove that $D_1(H) \supseteq D_2(H)$.

Let $s \in D_2(H)$ and let P be any d-prime d-ideal $\supseteq H$. Then $K \setminus P$ is a dm-system. If $s \in K \setminus P$ then $s \in H$ by the condition (2). It is a contradiction with $s \notin P, H \subseteq P$. Therefore $s \notin K \setminus P$. Then $s \in P$. Thence $s \in D_1(H)$. Thus $D_2(H) \subseteq D_1(H)$.

Now we prove that $D_3(H) \subseteq D_2(H)$. Let $s \in D_3(H)$ and let S be any dm-system containing s. For any dm-system is a dn-system, then from $s \in D_3(H)$ we obtain that S meets H. Thence $s \in D_2(H)$. Thus $D_3(H) \subseteq D_2(H)$.

Prove that $D_4(H) \subseteq D_3(H)$. Let $x \in D_4(H)$ and S be any dn-system containing x. By lemma 1.8 there exists a dn-sequence $\{x_0, x_1, \ldots, x_n, \ldots\} \subseteq S$ of the element x. For $x \in D_4(H)$ this dn-sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$ meets H. Therefore S meets H. Thence $D_4(H) \subset D_3(H)$.

Now we prove that $D_1(M) \subseteq D_4(H)$. Let $x \in D_1(H)$. Assume that x is not in $D_4(H)$. Then there exists a dn-sequence $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ of the element x such that $X \cap H = \emptyset$. Then $\sum$ is not empty as $H \in \sum$. We introduce in $\sum$ the partial order by the relation of the inclusion of d-ideals. Let $\{B_t, t \in T\}$ be a chain in $\sum$. We put

$$B = \cup_{t \in T} B_t.$$

Then B is a d-ideal and

$$B \cap X = (\cup_{\in T} B_T) \cap X = \cup_{t \in T}(B_t \cap X = \emptyset.$$

Therefore we can apply Zorn's lemma to the set $\sum$ so there exists a maximal element P of $\sum$. We are going to show that P is d-prime.

First, P is proper as $x \notin P$.

Let $A_1, A_2 \in L_d(K), A_1 \not\subseteq P, A_2 \not\subseteq P$, but $A_1 \cdot A_2 \subseteq P$. Then $P \vee A_1 \neq P$ and $P \vee A_2 \neq P$. By the maximality of P in $\sum$ we have $P \vee A_1 \notin \sum$ and $P \vee A_2 \notin \sum$. Therefore there exists natural numbers m and q such that $x_m \in P \vee A_1, x_q \in P \vee A_2$. Then

$$[x_m] \subseteq P \vee A_1, [x_q] \subseteq P \vee A_2.$$

Thence

$$x_{m+1} \in [x_m]^2 \subseteq P \vee A_1, \quad x_{q+1} \in [x_q]^2 \subseteq P \vee A_2.$$

Continuing in this manner we find that

$$x_{m+t} \in P \vee A_1, \; x_{q+t} \in P \vee A_2$$

for all natural numbers t. We put $n = max(m, q)$. Then

$$x_n \in P \vee A_1, \; x_n \in P \vee A_2.$$

Thence

$$x_{n+1} \in [x_n]^2 \subseteq (P \vee A_1)(P \vee A_2) \subseteq P \vee A_1 A_2.$$

But $x_{n+1} \notin P$. Thence $A_1 \cdot A_2 \notin P$. Therefore P is d-prime. Thus there exists a d-prime d-ideal P such that $x \notin P$ and $x \notin D_1(H)$. It is a contradiction. Thus $D_1(H) \subseteq D_4(H)$. □

For any d-ideal H of a d-ring K denote by $N_d(H)$ the set $D_1(H) = D_2(H) = D_3(H) = D_4(H)$ of the theorem 1.9.

Remark The equality $D_1(H) = D_2(H)$ is a differential analog of the theorem 10.7 in [5]. The equality $D_1(H) = D_4(H)$ is a differential analog of the proposition 1 in ([6], § 3.2).

Theorem 1.10 *For any d-ideal B of a d-ring K the following properties are equivalent:*
(1) B is a d-semiprime d-ideal.
(2) B is an intersection of d-prime d-ideals.
(3) $B = N_d(B)$.

Proof. (3) $\Longrightarrow$ (2). Let $B = N_d(B)$. Then by the theorem 1.9, B is an intersection of d-prime d-ideals.

(2) $\Longrightarrow$ (1) is clear as the intersection of any family of d- prime d-ideals is d-semiprime.

(1) $\Longrightarrow$ (3). Let B be a d-semiprime d-ideal. We prove that $N_d(B) \subseteq B$.

Let $x \notin B$. Then $S = K \setminus B$ is a dn-system containing x. By the lemma 1.8 there exists a dn-sequence $\{x_0, x_1, \ldots, x_n, \ldots\} \subseteq S$ of the element x. Then by the condition (4) of the theorem 1.9 $x \notin N_d(B)$. Thus $N_d(B) \subseteq B$. □

Remark This theorem is a differential analog of the theorem 10.11 in [5].

For any d-ring K we put $N_d = N_d(0)$.

An element x of a d-ring K will be called **d-nilpotent**, if $[x]^n = 0$ for some $n \in \mathbb{N}$. denote by N_d^0 the set of d- nilpotent elements of a d-ring K.

Denote by N^0 the set of nilpotents elemnts of a ring K. From the theorem 1.9 we obtain that $N_d^0 \subseteq N_d \subseteq N^0$.

Proposition 1.11 *For any d-ring $K, N_d(K/N_d) = 0$.*

The proof is standard.

1.12 *For any d-ring K the set N_d^0 is a d-ideal of K.*

Proof. Let $x \in N^0, a, b \in K$ and $[x]^n = 0$ for some n. From $0 \subseteq [axb] \subseteq [x]$, we have $0 \subseteq [axb]^n \subseteq [x]^n = 0$. Thence $axb \in N_d^0$.

Let $x, y \in N_d^0$. Then $[x]^m = [y]^n) = 0$ for some $m, n \in \mathbb{N}$ and

$$0 \subseteq [x+y] \subseteq [x] \vee [y].$$

Let $z \in [x] \vee [y]$. Working in $K/[y]$ and lifting to K, we obtain that $[z]^n \subseteq [y]$. Thence $[[z]^n]^m = 0$. Thus $x + y \in N_d^0$.

Let $x \in N_d^0$. Then $[x]^n = 0$ for some n and $d(x) \in [x]$. From $0 \subseteq [d(x)] \subseteq [x]$, we obtain $[d(x)]^n = 0$. Thus N_d^0 is a d-ideal of K. □

Theorem 1.13 *Assume that a d-ring K satisfies ascending chain condition for d-ideals. Then:*

(1) $N_d = N_d^0$
(2) N_d *is nilpotent.*

Proof. By the inclusion $N_d^0 \subseteq N_d$, we must prove that $N_d \subseteq N_d^0$.

Let x is not d-nilpotent. Then $[x]^n \neq 0$ for all natural numbers n. We are going to show that there exists a d-prime d-ideal P such that $[x] \nsubseteq P$. Then we obtain that $x \notin P$.

We prove a more general following □

Lemma *Assume that a d-ring K satisfies the ascending chain condition for d-ideals and A is not nilpotent d-ideal of K. Then there exists d-prime d-ideal P such that $A \nsubseteq P$.*

Proof. Denote by $\sum$ the set of d-ideals H of K such that

$$A^n \nsubseteq H$$

for all natureal numbers m. It is obviously that $[0] \in \sum$. Therefore $\sum$ is not empty.

Introduce in $\sum$ the usual partial order. Let $\{H_t, t \in T\}$ be a chain in $\sum$. Put

$$H = \cup_{t \in T} H_t.$$

Then there exist $t = t_0$ that $H = H_{t_0}$ as a d-ring K satisfies ascending chain condition for d-ideals. We can apply Zorn's lemma to the set $\sum$. Therefore there exists a maximal element P of $\sum$. First P is proper as $A \nsubseteq P$.

Let $a, b \in L_d(K), a \nsubseteq P, b \nsubseteq P$. The $P \vee a \neq P, P \vee b \neq P$. By the maximality of P, we have $P \vee a \notin \sum, P \vee b \notin \sum$. Therefore $A^m \subseteq P \vee a, A^n \subseteq P \vee b$ for some m, n. Thence $A^{m+n} \subseteq (P \vee a).(P \vee b) \subseteq P \vee ab$.

This means that $P \vee a.b \notin \sum$. Thence $a.b \nsubseteq P$. Therefore P is d-prime and $A \nsubseteq P$. The lemma is proved.

For $A = [x]$, by lemma there exists a d-prime d-ideal P such that $[x] \nsubseteq P$. Thence $x \notin P$. Thus $N_d^0 = N_d$.

If the d-ideal N_d is not nilpotent then by lemma there exists a d- prime d-ideal P such that $N_d \nsubseteq P$. But $N_d \subseteq P$ for all d-prime d-ideals P. It is a contradiction. Thus N_d is nilpotent. □

Now we consider a connection between d-prime d-ideals and prime ideals.

Note that there exists a d-ring K and its maximal d-ideal H such that H is not prime ideal.

Example Let k be a field of the characteristic 2 and e is a unit of k. Consider the k-algebra K with the following basis over $k : e, w, w^2 = 0, w \neq 0$. Then $K = \{x | x = y + zw, y, z \in k\}$.

Introduce the differentiation operation d on K in the following way:

If $x \in K$ and $x = y + zw, y, z \in k$, put $dx = d(y + zw) = z$. Then K is a d-ring. All the ideals of the ring K are following: $\{0\}, K, Kw$. The ideals $\{0\}$ and K are d-ideals. but the ideal Kw is not d-ideal as $dw = e \notin Kw$. Therefore d-ideal $\{0\}$ is a maximal d-ideal of the d- ring K and it is not prime.

Theorem 1.14 *Any maximal d-ideal of a Ritt algebra is a prime ideal.*

Proof. Let H be a maximal d-ideal of K and $r(H)$ be radical of H. (i.e. $r(H)$ is the intersection of all prime ideals of K containing H)

Let e be an unit of K. If $r(H) = K$ then $e = e^n \in H$ for some n. This is a contradiction with the $H \neq K$.

Therefore $r(H) \neq K$. By lemma 1.8 in [3], $r(H)$ is a d-ideal of a Ritt algebra K. Thence $r(H) = H$ by the maximality H in $L_d(K)$. Therefore any maximal d-ideal of a Ritt algebra K is a radical ideal of the ring K.

By the theorem 2.1 in [4], the d-ideal H is an intersection of some set $\{A_t, t \in T\}$ of prime d-ideals A_t of a Ritt algbera K:

$$H = \cap_{t \in T} A_t.$$

Then $H \subseteq A_t, \forall t \in T$. Thence $H = A_t, \forall t \in T$ as H is a maximal d-ideal of K. Thus H is a prime ideal. □

Proposition 1.15 *Let K be a Ritt algebra. Then the nil- radical N^0 of K is a d-ideal and it is an intersection of prime d- ideals of K.*

Proof. Consider the ideal $\{0\}$ of K. By the lemma 1.8 in [4] the radical of $\{0\}$ is a d-ideal of K. By the theorem 2.1 in [4], N^0 is an intersection of prime d-ideals of K. □

Proposition 1.16 *Let K be a Ritt algebra. Assume that K satisfies the ascending chain condition for ideals. Then $N_d^0 = N_d = N^0$.*

Proof. In this case the nil-radical N° of K is nilpotent. Therefore $(N^\circ)^n = 0$ for some n.

Let $x \in N^\circ$. By the proposition 1.15, N° is a d-ideal. Therefore $[x] \subseteq N^\circ$ and $[x]^n = 0$. Therefore $x \in N_d^0$. Thus $N_d^0 = N^0$. □

From this proposition, we obtain the following immediately

Corollary 1.17 *Let K be a noetherian Ritt algebra. Then any d-radical d-ideal of K is radical.*

Proposition 1.18 *Let K be a Ritt algebra. Assume that K satisfies the descending chain condition for ideals. Then any d-prime d-ideal of K is prime.*

Proof. By the corollary 1.17 any d-prime d-ideal H is an intersection of finite prime d-ideals:

$$H = H_1 \cap \ldots \cap H_n$$

Put $H_2' = H_2 \cap \ldots \cap H_n$. Then $H_1 H_2' \subseteq H_1 \cap H_2' = H$. Thence we obtain $H_1 \subseteq H$ or $H_2' \subseteq H$ (as H is d-prime). Let $H_1 \subseteq H$. If $H_1 \neq H$ then $H = H_1 \cap H_2' \subseteq H_1$. It is a contradiction. Therefore $H = H_1$ or $H = H_2'$. In the case $H = H_2'$ continuing in this manner we obtain that $H = H_i$ for some i. □

2. Properties of the lattice $L_d(K)^r$

Let $B \in L_d(K)$ and $B \neq K$. The corollary 1.2 and the proposition 1. 3 show that there exists a d-prime d-ideal C such that $B \subseteq C$. Therefore a d-radical $r_d(B)$ exists for any $B \in L_d(K)$, $B \neq K$.

Proposition 2.1 *For any $A, B \in L_d(K)$ the following properties are hold:*
(i) $A \subseteq r_d(A)$,
(ii) $r_d(A) = r_d(r_d(A))$,
(iii) if $A \subseteq B$ then $r_d(A) \subseteq r_d(B)$,

The proof is obviously.

Proposition 2.2 *The lattice $L_d(K)^r$ are complete.*

A proof follows from the proposition 2.1 and the corollary of the theorem 4 in ([1], ch V, § 1).

Denote the lattice operations on $L_d(K)$ by "$\cap$" and "$+$", on $L_d(K)^r$ by "$\wedge$" and "$\vee$".

Proposition 2.3 *For any $A, B \in L_d(K), C, D \in L_d(K)^r, C_t \in L_d(K)^r, t \in T$ the following statements are hold:*
(1) $r_d(A.B) = r_d(A \cap B) = r_d(A) \wedge r_d(B)$,
(2) $r_d(A + B) = r_d(r_d(A) + r_d(B))$,
(3) $C \wedge D = r_d(C + D)$,
(4) $\cap_{t \in T} C_t = \wedge_{t \in T} C_t$.

Theorem 2.4 *The lattice* $L_d(K)^r$ *satisfies the following condition:*

$$A \wedge (\vee_{t\in T} B_t) = \vee_{t\in T} (A \wedge B_t) \tag{3}$$

for any $A, B_t \in L_d(K)^r$.

In particular, the lattice $L_d(K)^r$ *is distributive.*

Proof. Let $A, B_t \in L_d(K)^r, t \in T$. Then

$$A.B_t \subseteq A.(\vee_{t\in T} B_t)$$

for all $t \in T$. Thence

$$\vee_{t\in T} (A.B_t) \subseteq A.(\vee_{t\in T} B_t). \tag{4}$$

Now we shall prove the inverse inequality.

If $V_{t\in T}(A.B_t) = (K)$ then from (3) we obtain $A.(V_{t\in T} B_t)) = K$.

Therefore the equality (2) is true in this case.

Let $\vee_{t\in T}(A.B_t) \neq K$. Then there exists a family $\{Q_\nu, \nu \in S\}$ of d-prime d-ideals such that

$$\vee_{t\in T}(A.B_t) = \cap_{\nu\in S} Q_\nu$$

Let Q be an element of the family $\{Q_\nu, \nu \in S\}$. Then:

$$A.B_t \subseteq Q$$

for any $t \in T$. Thence $A \subseteq Q$ or $B_t \subseteq Q$. In the case $A \subseteq Q$ we have

$$A.(\vee_{t\in T} B_t) \subseteq A \subseteq Q.$$

Let $A \not\subseteq Q$. Then $B_t \subseteq Q$ for all $t \in T$. Thence

$$\vee_{t\in T} B_t \subseteq Q$$

and

$$A.(\vee_{t\in T} B_t) \subseteq \vee_{t\in T} B_t \subseteq Q.$$

Therefore

$$A.(\vee_{t\in T} B_t) \subseteq Q$$

for any $Q \in \{Q_\nu, \nu \in S\}$.

Thence

$$A.(\vee_{t\in T}B_t) \subseteq \cap_{\nu\in S}Q_\nu = \vee_{t\in T}(A.B_t).$$

□

Remark 1. The distributivity of the lattice of radical ideals for commutative rings was obtained in [3].

Remark 2. Denote by $M_d(K)$ the set of maximal d-ideals of K. For $A \in L_d(K)$ denote by $R_d(A)$ the intersection of all maximal d-ideals containing A.

Denote by $L_d(K)^R$ the lattice of d-ideals $A \in L_d(K)$ such that $A = R_d(A)$. An analog of the theorem 2.4 is true for the lattice $L_d(K)^R$.

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ASOSYATİF DİFERENSİYEL HALKALARDA DİFERENSİYEL RADİKAL İDEALLER LATİSİNİN ÖZELLİKLERİ VE ASAL RADİKALİN DİFERENSİYEL ANALOJİSİ

Özet

Bu çalışmada elde edilen temel sonuçlar: (1) Birimli herhangi bir diferensiyel halkada, asal radikalin diferensiyel analojisini tanımlamak ve karakterize etmek, (2) Ritt cebirinin herhangi bir maksimal difrensiyel ideali asaldır, (3) Radikal diferensiyel idealler latisi sonsuz $\cap$-distribütif koşulunu sağlar.

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Received 22.3.1996