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Abstract

In this paper, the direct and inverse nonstationary scattering problem for a hyperbolic system of n equations ($n > 3$) on a semi-axis is studied. The coefficients of the system are uniquely determined according to scattering operator. The solution of the problem is generalized for an arbitrary natural number n applying the properties of separation of factors by means of operator transformations to some elements of the scattering operator and its inverse as well as reducing the scattering problem given on a semi-axis to the scattering problem on the whole-axis.

Introduction

On the semi-axis $x \geq 0$ we consider the hyperbolic system of equations of the form

$$\sigma\frac{\partial\Psi(x,t)}{\partial t} - \frac{\partial\Psi(x,t)}{\partial x} = U(x,t)\Psi(x,t), \tag{1}$$

where $\Psi(x,t) = \{\Psi_1(x,t),\dots,\Psi_n(x,t)\}$ is the vector-valued function sought, $\sigma = diag(\xi_1,\xi_2,\dots,\xi_n)$ is a diagonal matrix of constants with $\xi_1 > \xi_2 > \dots > \xi_{n-1} > 0 > \xi_n$ and $U(x,t)$ is the matrix-valued potential with zero elements on the diagonal; that is $U_{ii}(x,t) = 0, (i = 1,2,\dots n)$. It is assumed that each of the elements $U_{ik}(x,t)(i,k = 1,2,\dots,n)$ satisfies

$$\int_{-\infty}^{\infty}\int_0^{\infty} |U_{ik}(x,t)|^2 dxdt < \infty. \tag{2}$$

For the system (1) defined on the semi-axis $(x \geq 0)$we have the case which consists of k number of incident waves $(\xi_1 > \xi_2 > \dots > \xi_k > 0)$, and (n-k) number of scattered waves $(0 > \xi_{k+1} > \dots > \xi_{n-1} > \xi_n)\,, k = 1,\dots,n-1$.

For $n = 2$ the direct and inverse nonstationary scattering problems for the hyperbolic system of equations (1) given on a semi-axis and the whole-axis were studied in [1] and the case where $n > 3$ were considered in [2] and [3].

In a hyberbolic system of equations given on a semi-axis, for the cases where,

$$\begin{aligned}
&n=3 \text{ and } (\xi_1>0>\xi_2>\xi_3),\\
&n=3 \text{ and } (\xi_1>\xi_2>0>\xi_3),\\
&n=4 \text{ and } (\xi_1>\xi_2>\xi_3>0>\xi_4),\\
&n=4 \text{ and } (\xi_1>0>\xi_2>\xi_3>\xi_4),
\end{aligned}$$

are studied in [4], [5], [6] and [7] respectively. However for the case $\xi_1>\xi_2>0>\xi_3>\xi_4$ the solution of inverse scattering problem could not be obtained. Scattering problem regarding nonstationary hyperbolic system of equations was studied in [8] and [9] in terms of operators. The inverse nonstationary problem for a hyperbolic system of equations ($n=2$) is applied to find a solution to the nonlinear equation of Davey A. and Stewartson [10].

If the potential $U(x,t)\equiv 0$ in the system (1), then the general solution is of the form

$$\Psi(x,t)=\{\varphi_1(t+\xi_1 x),\ldots,\varphi_n(t+\xi_n x)\}=T_x\varphi(t),$$

where $\varphi(x)=\{\varphi_1(x),\ldots,\varphi_n(x)\}$ is an arbitrary vector function and T_x is the shift operator. When the functions $\varphi(x)$ are nondifferentiable, the solution is understood in the generalized sense.

In this paper, we study the case of $(n-1)$ number of incident waves and one scattered wave $(0>\zeta_n)$. In the first part, scattering problem is defined and then the direct scattering problem is solved, that is, the scattering operator S has been obtained for the system (1) with the boundary conditions at $x=0$ and as $x\to\infty$. Second and third parts are preparation for the solution of the inverse problem and the properties of the elements of the operator S are studied. In the fourth part, the inverse scattering problem is solved for the system (1), i.e., the coefficents of the system (1) have been found according to scattering operator.

1. Scattering Problem

Let us consider the generalized solutions of the system (1) which are measurable with respect to x and t. These solutions belong to $L_2(-\infty,+\infty)$ as a function of t and L_2 norm is uniformly bounded with respect to x,i.e.,

$$\begin{aligned}
&\| \Psi(x,\cdot)\|_{L_2}=\left\{\int_{-\infty}^{+\infty}|\Psi(x,t)|^2dt\right\}^{\frac{1}{2}}<\infty,\\
&\sup_x \| \Psi(x,\cdot)\|_{L_2}<+\infty.
\end{aligned}$$

Every bounded solution $\Psi(x,t)$ of the system of equations (1) on the semi-axis $x \geq 0$ can be represented in the form

$$\begin{aligned}\Psi_i(x,t) &= a_i(t+\xi_i x) + \circ(1), \quad (i=1,2,\dots,n-1)\\ \Psi_n(x,t) &= b(t+\xi_n x) + \circ(1), \text{ as } x \to +\infty,\end{aligned}$$

where $a_i(s) \in L_2(-\infty,\infty)$ and $b(s) \in L_2(-\infty,\infty)$, $(i=1,2,\dots,n-1)$ define the profile of the incident waves and scattered waves respectively.

The problem of finding the solution of the system (1) according to incident waves $a_1(t),\dots,a_{n-1}(t)$ and the boundary conditions defined at $x=0$ is called the scattering problem.

In the following, we will examine $(n-1)$ number of scattering problems simultaneously:

A nonstationary $k-th\,(k=1,2,\dots,n-1)$ scattering problem for the system of equations (1) can be formulated as follows:

It is required to find a bounded solution $\Psi^k(x,t) = \left\{\Psi_1^k(x,t),\dots,\Psi_n^k(x,t)\right\}$ $(k = 1,2,\dots,n-1)$ of (1) such that the asymptotic relations

$$\Psi_i^k(x,t) = a_i(t+\xi_i x)+\circ(1) \text{ with } a_i(s) \in L_2(-\infty,\infty), i=1,2,\dots,n-1, \text{ as } x \to +\infty, \quad (3)$$

and $n-1$ different boundary conditions

$$\Psi_n^k(0,t) = \Psi_k^k(0,t), \quad (k=1,2,\dots,n-1) \quad (4)$$

are satisfied.

By considering $(n-1)$ problems together we have the scattering problem for the system (1) on the semi-axis.

Theorem 1. *If the coefficients of the system (1) satisfy the conditions (2), then there exists a unique solution of the scattering problem on the semi-axis for a given arbitrary incident wave vector* $a(t) = (a_1(t),\dots,a_{n-1}(t)) \in L_2((-\infty,\infty),E_{n-1})(E_{n-1} = EX\dots XE = E^{n-1}, E=(-\infty,\infty))$.

Proof. Every $k-th(k=1,2,\dots,n-1)$ scattering problem for the system of equations (1) is equivalent to seek solutions of the integral equations

$$\begin{aligned}\Psi_i^k(x,t) &= a_i(t+\xi_i x) + \int_x^{+\infty}\sum_{j=1}^n (U_{ij}\Psi_j^k)(y,t+\xi_i(x-y))dy,\\ \Psi_n^k(x,t) &= b_k(t+\xi_n x) + \int_x^{+\infty}\sum_{j=1}^n (U_{nj}\Psi_j^k)(y,t+\xi_n(x-y))dy, \quad (5)\\ &(i,k=1,2,\dots,n-1)\end{aligned}$$

where

$$b_k(t) = a_k(t) + \int_0^{+\infty} \sum_{j=1}^{n} \left[(U_{kj}\Psi_j^k)(y, t - \xi_k y) - (U_{nj}\Psi_j^k)(y, t - \xi_n y) \right] dy.$$

Existence and uniqueness of the solutions of the systems of equations (5) on the vector functions space, $\Psi(x,t) = \{\Psi_1(x,t), \ldots, \Psi_n(x,t)\}$, with the norm

$$\| \Psi \|_{E_n} = \max_{k=1,\ldots,n} \operatorname{ess\,sup}_x \| \Psi_k(x,\cdot) \|_{L_2}, \quad E_n = E \times \ldots\ldots \times E = E^n,$$

follow from the Volterra type equation in the variable t (Lemma 1, 2 in [3]). From (2) and (5) we conclude that when $x \to \infty$, $\Psi_n^k(x,t)$ can be represented asymptotically as

$$\Psi_n^k(x,t) = b_k(t + \xi_n x) + \text{o}(1), b_k(s) \in L_2(E), k = 1, \ldots, n-1. \tag{6}$$

In view of Theorem 1, (n-1) solutions $\Psi^k(x,t) = \left\{\Psi_1^k(x,t), \ldots, \Psi_n^k(x,t)\right\}$, $k = 1, \ldots, n-1$, which satisfy the system (1) together with the boundary conditions (4) correspond to a vector function $a(s) = (a_1(s), \ldots, a_{n-1}(s))$, which represents incident waves. These solutions correspond to the scattered wave $b_k(s) \in L_2(E)$, i.e.,

$$a(s) \to \Psi^k(x,t) \to b_k(s) \quad (k = 1, \ldots, n-1).$$

In other words,

$$a(s) = \begin{pmatrix} a_1(s) \\ \vdots \\ a_{n-1}(s) \end{pmatrix} \to \begin{pmatrix} \Psi^1(x,t) \\ \vdots \\ \Psi^{n-1}(x,t) \end{pmatrix} \to \begin{pmatrix} b_1(s) \\ \vdots \\ b_{n-1}(s) \end{pmatrix} = \hat{b}(s) \in L_2(E, E_{n-1}).$$

Hence an operator S is defined on $L_2(E, E_{n-1})$ mapping $a(s)$ onto $\hat{b}(s)$ i.e., $\hat{b}(s) = Sa(s)$. We call this operator the scattering operator on the semi-axis.

2. Integral Representation of Solutions

All possible solutions of equations (1) on the semi-axis can be represented as certain combinations of $a_1(t), \dots, a_{n-1}(t), b(t)$ and the values of solution at $x = 0$ (i.e., $\Psi_1(0,t), \dots, \Psi_n(0,t)$) we consider the following 2n vector functions

$$\begin{aligned}
h^1(t) &= \{\Psi_1(0,t), \Psi_2(0,t), \dots, \Psi_n(0,t)\}, \\
h^k(t) &= \{a_1(t), \dots, a_{k-1}(t), \Psi_k(0,t), \dots, \Psi_n(0,t)\} \\
&\quad (2 \le k \le n), \\
h^{n+1}(t) &= \{a_1(t), \dots, a_{n-1}(t), b(t)\}, \\
h^{n+k}(t) &= \{\Psi_1(0,t), \dots, \Psi_{k-1}(0,t), a_k(t), \dots, a_{n-1}(t), b(t)\} \ (2 \le k \le n-1), \\
h^{2n}(t) &= \{\Psi_1(0,t), \dots, \Psi_{n-1}(0,t), b(t)\}.
\end{aligned} \tag{7}$$

□

Lemma 1. *If the coefficients of system (1) satisfy conditions (2), then any possible solution has an integral representation,*

$$\Psi_i(x,t) = h_i^1(t+\xi_i x) + \int_{t+\xi_n x}^{t+\xi_1 x} \sum_{j=1}^{n} A_{ij}^1(x,t,s) h_j^1(s) ds, \tag{8}$$

$$\Psi_i(x,t) = h_i^2(t+\xi_i x) + \int_{-\infty}^{t+\xi_1 x} A_{i1}^2(x,t,s) h_1^2(s) ds + \int_{-\infty}^{t+\xi_2 x} \sum_{j=2}^{n} A_{ij}^2(x,t,s) h_j^2(s) ds, \tag{9}$$

$$\begin{aligned}
\Psi_i(x,t) = h_i^k(t+\xi_i x) &+ \int_{-\infty}^{\infty} \sum_{j=1}^{k-2} A_{ij}^k(x,t,s) h_j^k(s) ds + \int_{-\infty}^{t+\xi_{k-1} x} A_{i,k-1}^k(x,t,s) h_{k-1}^k(s) ds \\
&+ \int_{-\infty}^{t+\xi_k x} \sum_{j=k}^{n} A_{ij}^k(x,t,s) h_j^k(s) ds, \quad (3 \le k \le n)
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Psi_i(x,t) = h_i^{n+1}(t+\xi_i x) &+ \int_{t+\xi_1 x}^{\infty} A_{i1}^{n+1}(x,t,s) h_1^{n+1}(s) ds + \sum_{j=2}^{n-1} \int_{-\infty}^{\infty} A_{ij}^{n+1}(x,t,s) h_j^{n+1}(s) ds \\
&+ \int_{-\infty}^{t+\xi_n x} A_{in}^{n+1}(x,t,s) h_n^{n+1} ds,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\Psi_i(x,t) = h_i^{n+k}(t+\xi_i x) &+ \int_{t+\xi_{k-1} x}^{\infty} \sum_{j=1}^{k-1} A_{ij}^{n+k}(x,t,s) h_j^{n+k}(s) ds + \int_{t+\xi_k x}^{\infty} A_{ik}^{n+k}(x,t,s) h_k^{n+k}(s) ds \\
&+ \int_{-\infty}^{\infty} \sum_{j=k+1}^{n} A_{ij}^{n+k}(x,t,s) h_j^{n+k}(s) ds, \quad (2 \le k \le n-1)
\end{aligned} \tag{12}$$

$$\Psi_i(x,t) = h_i^{2n}(t+\xi_i x) + \int_{t+\xi_{n-1}x}^{\infty} \sum_{j=1}^{n-1} A_{ij}^{2n}(x,t,s) h_j^{2n}(s) ds + \int_{t+\xi_n x}^{\infty} A_{in}^{2n}(x,t,s) h_n^{2n}(s) ds. \quad (13)$$

$$(i = 1, \ldots, n)$$

The kernels of these transformations for a fixed x are square integrable in the variable t and s, i.e., they are Hilbert-Schmidt type kernels and also determined uniquely with respect to coefficients $U_{ij}(x,t)$ in system (1). The proof of this is similar to the proof given in [5].

3. The Properties of Scattering Operator

The integral representations (8)-(10) gives rise to relations among the coordinates of vectors $h^i(t)$. These relations can be obtained by the theory of Volterra integral operators.

Lemma 2. *For every $b_i(t) \in L_2(E)$, $i = 1, \ldots, n-1$,*

$$\Psi_p^p(0,t) - \Psi_q^p(0,t) = (I + A_{q+})(b_p(t) - b_q(t)), p, q \in \{1, \ldots, n-1\}, \quad (14)$$

where

$$A_{mn+}^{n+1} b(t) = \int_{-\infty}^{t} A_{mn}^{n+1}(0,t,s) b(s) ds, \quad A_{q+} = A_{n,n+}^{n+1} - A_{q,n+}^{n+1}.$$

Proof. In the representation (11), writing $h^{n+1}(t) = \{a_1(t), \ldots, a_{n-1}(t), b_p(t)\}$ and $h^{n+1}(t) = \{a_1(t), \ldots, a_{n-1}(t), b_q(t)\}$, which are suitable for the solution of p-th and q-th problem, and subtracting them from each other we obtain

$$\begin{aligned} \Psi_i^p(0,t) - \Psi_i^q(0,t) &= A_{in+}^{n+1}(b_p(t) - b_q(t)), i = 1, \ldots, n-1; \\ \Psi_p^p(0,t) - \Psi_q^q(0,t) &= \Psi_n^p(0,t) - \Psi_n^q(0,t) = \left(I + A_{nn+}^{n+1}\right)(b_p(t) - b_q(t)). \end{aligned}$$

For $i = q$ we get the equality (14). □

Lemma 3. *If $a_{k+1}(t) = \ldots = a_{n-1}(t) = b_1(t) = \ldots = b_{k-1}(t) = 0 \quad (2 \le k \le n-2, n \ge 4)$ then*

$$\Psi_1^i(0,t) = \ldots = \Psi_{k-1}^i(0,t) = B_{1k-}a_k(t), (i = 1, \ldots, k-1), \tag{15}$$
$$\Psi_k^i(0,t) = (I + B_{2k-})a_k(t), \tag{16}$$
$$a_k(t) = (I + B_{k-})^{-1}(I + A_{k+})b_k(t), \tag{17}$$

where

$$B_{1k-} = \left(I - \sum_{j=1}^{k-1} A_{nj-}^{n+k}\right)^{-1} A_{nk-}^{n+k},\ B_{2k-} = A_{kk-}^{n+k} + \sum_{j=1}^{k-1} A_{kj-}^{n+k} \left(I - \sum_{j=1}^{k-1} A_{nj-}^{n+k}\right)^{-1} A_{nk-}^{n+k},$$
$$B_{k-} = B_{2k-} - B_{1k-}.$$

Proof. Taking the Lemma conditions into consideration and substituting $x = 0$ for (12) we get

$$\begin{aligned} \Psi_k^1(0,t) &= (I + A_{kk+}^{n+k})a_k(t) + \sum_{j=1}^{k-1} A_{kj-}^{n+k}\Psi_j^1(0,t), \\ \Psi_n^1(0,t) &= A_{nk-}^{n+k}a_k(t) + \sum_{j=1}^{k-1} A_{nj-}^{n+k}\Psi_j^1(0,t), \quad (2 \le k \le n-2), \end{aligned} \tag{18}$$

for the first problem.
For $b_1(t) = \ldots = b_{k-1}(t)$ from Lemma 2 we find that

$$\Psi_1^i(0,t) = \Psi_2^i(0,t) = \ldots = \Psi_{k-1}^i(0,t), \quad (i = 1, \ldots, k-1).$$

Considering this together with $\Psi_n^1(0,t) = \Psi_1^1(0,t)$ and from the system (18) we obtain the equalities (15)-(17). □

Lemma 4. *Let $b(t) = b_1(t) = \ldots = b_{n-1}(t)$. Then the soltions of these (n-1) problems coinside. Let $\alpha(t) = \Psi_k^1(0,t) = \ldots = \Psi_k^{n-1}(0,t), (k = 1, 2 \ldots, n)$, then*

$$\alpha(t) = (I + N_-)b(t), \tag{19}$$
$$\alpha(t) = (I + N_+)a_1(t), \tag{20}$$

where

$$
\begin{aligned}
N_{+} &= (I-\sum_{j=2}^{n} A^{2}_{ij_{+}})^{-1}(I+A^{2}_{11+})-I,\\
N_{-} &= (I-A^{2n}_{nn-})^{-1}(I+\sum_{j=1}^{n-1} A^{2n}_{nj-})-I.
\end{aligned}
$$

Proof. Using $b_1(t)=\ldots=b_{n-1}(t)$ in Lemma 2 we find $\Psi^1_1(0,t)=\ldots=\Psi^1_n(0,t)$. Since the solution of Cauchy problem is unique, the solutions of these (n-1) problems are equal. The rest of the proof can be obtained from (9) and (13). □

Lemma 5. *If* $b_k(t)\neq 0(k\neq n-1), b_i(t)=0\,(i\neq k)$, *then*

$$
\begin{aligned}
\Psi^i_k(0,t) &= (I+C_{k-})(I+A_{k+})b_k(t), && (21)\\
\Psi^i_n(0,t) &= \Psi^i_{n-1}(0,t)=\Psi^i_i(0,t)=C_{k-}(I+A_{k+})b_k(t), && (22)\\
a_{n-1}(t) &= r_{k-}(I+A_{k+})b_k(t), \quad (k=1,\ldots,n-2). && (23)
\end{aligned}
$$

For $k=n-1$, *that is,* $b_{n-1}(t)\neq 0, b_1(t)=\ldots=b_{n-2}(t)=0$,

$$
\begin{aligned}
\Psi^i_n(0,t) &= \Psi^i_i(0,t)=C_{n-1-}(I+A_{n-1+}b_{n-1}(t) \quad (i=1,\ldots,n-2), && (24)\\
\Psi^i_{n-1}(0,t) &= (I+C_{n-1-})(I+A_{n-1+})b_{n-1}(t), && (25)\\
a_{n-1}(t) &= (I+B_{n-1-})(I+A_{n-1+})b_{n-1}(t), && (26)
\end{aligned}
$$

where

$$
\begin{aligned}
C_{k-} &= (I-C_{\hat{k-}})^{-1}-I,\\
C_{\hat{k-}} &= (I-\sum_{j=1,j\neq k}^{n-2} A^{n+k}_{n-j,j-})^{-1}A^{n+k}_{n-1,k-}+(I+A^{n+k}_{n-1,n-1-})\\
&\left[(I-\sum_{j=1,j\neq k}^{n-2} A^{n+k}_{n-1,j-})^{-1}A^{n+k}_{n-1,n-1-}-(I-\sum_{j=1,j\neq k}^{n-2} A^{n+k}_{nj-})^{-1}A^{n+k}_{n,n-1-})^{-1}\right]\\
&\left[(I-\sum_{j=1,j\neq k}^{n-2} A^{n+k}_{nj-})^{-1}A^{n+k}_{nk-}-(I-\sum_{j=1,j\neq k}^{n-2} A^{n+k}_{n-1,j-})^{-1}A^{n+k}_{n-1,k-}\right],
\end{aligned}
$$

$$
\begin{aligned}
C_{n-1-} &= (I - C_{\hat{n}-1-})^{-1} - I, \\
C_{\hat{n}-1-} &= (I - \sum_{j=1}^{n-2} A_{nj-}^{n+k})^{-1} A_{n,n-1-}^{n+k} \left[I + A_{n-1,n-1-}^{n+k} + \sum_{j=1}^{n-2} A_{n-1,j-}^{n+k} (I - \sum_{j=1}^{n-2} A_{nj-}^{n+k})^{-1} \right. \\
&\quad \left. A_{n,n-1-}^{n+k} \right], \\
r_{k-} &= (I + A_{n-1,n-1-}^{n+k})^{-1} \left[I - \sum_{j=1, j\neq k}^{n-2} A_{n-1,j-}^{n+k}) C_{\hat{k}-} - A_{n-1,k-}^{n+k} \right] (I - C_{\hat{k}-})^{-1}, \\
B_{n-1-} &= \left[I + A_{n-1,n-1-}^{n+k} + \sum_{j=1}^{n-2} A_{n-1,j-}^{n+k} (I - \sum_{j=1}^{n-2} A_{nj-}^{n+k})^{-1} A_{n,n-1-}^{n+k} \right]^{-1} \\
&\quad (I - C_{\hat{n}-1-})^{-1} - I.
\end{aligned}
$$

Proof. In the first case, from (12) we find that

$$
\Psi_{n-1}^{i}(0,t) = (I + A_{n-1,n-1-}^{n+k}) a_{n-1}(t) + \sum_{j=1, j\neq k}^{n-2} A_{n-1,j-}^{n+k} \Psi_{n-1}^{i}(0,t) + A_{n-1,k-}^{n+k} \Psi_{k}^{i}(0,t),
$$

$$
\Psi_{n}^{i}(0,t) = \Psi_{n-1}^{i}(0,t) = \sum_{j=1, j\neq k}^{n-2} A_{nj-}^{n+k} \Psi_{n-1}^{i}(0,t) + A_{nk-}^{n+k} \Psi_{k}^{i}(0,t) + A_{n,n-1}^{n+k} a_{n-1}(t). \quad (27)
$$

Solving the system we obtain either $\Psi_{n-1}^{i}(0,t) = C_{\hat{k}} \Psi_{k}^{i}(0,t)$ or

$$
\Psi_{k}^{i}(0,t) - \Psi_{n-1}^{i}(0,t) = \Psi_{k}^{i}(0,t) - \Psi_{n}^{i}(0,t) = \Psi_{k}^{i}(0,t) - \Psi_{i}^{i}(0,t) = (I - C_{\hat{k}}) \Psi_{k}^{i}(0,t). \quad (28)
$$

If we compare (14) and (27) we get (21) and (22). In a similar way, the proof of (23) can be obtained by using (14), (27) and 28. In the second case we obtain

$$
\begin{aligned}
\Psi_{n}^{i}(0,t) &= \left(I - \sum_{j=1}^{n-2} A_{nj-}^{n+k} \right)^{-1} A_{n,n-1-}^{n+k} a_{n-1}(t), \\
\Psi_{n-1}^{i}(0,t) &= \left[I + A_{n-1,n-1-}^{n+k} + \sum_{j=1}^{n-2} A_{n-1,j-}^{n+k} (I - \sum_{j=1}^{n-2} A_{nj-}^{n+k})^{-1} A_{n,n-1-}^{n+k} \right] a_{n-1}(t).
\end{aligned}
$$

If follows that $\Psi_{n}^{i}(0,t) = C_{\hat{n}-1-} \Psi_{n-1}^{i}(-0,t)$ or

$$
\Psi_{n-1}^{i}(0,t) - \Psi_{n}^{i}(0,t) = \left(I - C_{\hat{n}-1-} \right) \Psi_{n-1}^{i}(0,t). \quad (29)
$$

(14) and (29) result in (24) and (25). Equality (26) can be proved similarly. □

Lemma 6. *If* $a_1(t) = \ldots = a_{n-2}(t) = 0$, *then*

$$\Psi_i^i(0,t) = (I - A_{in+}^n)^{-1} A_{i,n-1+}^n a_{n-1}(t), \quad (i = 1, \ldots n-2), \tag{30}$$
$$\Psi_{n-1}^{n-1}(0,t) = (I - A_{n-1,n+}^n)^{-1}(I + A_{n-1,n-1+}^n) a_{n-1}(t). \tag{31}$$

Proof. In the representation (10) $(k = n)$ if we take $a_1(t) = \ldots = a_{n-2}(t) = 0$ for every problem we find

$$\Psi_i^i(0,t) = A_{i,n-1+}^n a_{n-1} + A_{in+}^n \Psi_i^i(0,t), \; i = 1, \ldots, n-2,$$
$$\Psi_{n-1}^{n-1}(0,t) = (I + A_{n-1,n-1+}^n) a_{n-1}(t) + A_{n-1,n+}^n \Psi_{n-1}^{n-1}(0,t).$$

Solving this system we get (30) and (31),

Now let us examine the properties of the operator S. □

Theorem 2. *If the coefficients of the system (1) satisfy the conditions (2), then there is an inverse of the scattering operator* $S^{-1} = \| \gamma_{i,j} \|_{i,j=1}^{n-1}$. *Then* $S = I + F$, $S^{-1} = I + J$, *where* F *and* J *are the Hilbert-Schmidt matrix integral operators. Also there is an inverse of the matrix operator* $\Delta_k = \| S_{i,j} \|_{i,j}^{k} = 1 \quad (1 \le k \le n-1)$. *The operators*

$$(\gamma_{1,1} + \ldots + \gamma_{1,n-1})^{-1}, \; \gamma_{n-1,n-1} = (\Delta_{n-1}^{-1})_{11}, \; S_{11}^{-1} = (\Delta_1^{-1})_{11}, \; (\Delta_k^{-1})_{kk} \; (2 \le k \le n-2)$$

are also left factorizable

$$(\gamma_{1,1} + \ldots + \gamma_{1,n-1})^{-1} = (I + N_-)^{-1}(I + N_+), \tag{32}$$
$$(\Delta_k^{-1})_{kk} = (I + B_{k-})^{-1}(I + A_{k+}), \; (1 \le k \le n-1). \tag{33}$$

Moreover, the operators

$$S_{n-1,n-1} - S_{k,n-1} \; (k = 1, \ldots, n-2), \; S_{k,1} \; (k = 2, \ldots, n-1),$$
$$\gamma_{1,k}(k = 1, \ldots, n-1), \; \gamma_{n-1,k} \; (k = 1, \ldots, n-2)$$

of the following form:

$$\begin{aligned} &S_{n-1,n-1} - S_{k,n-1} = I + G_{k+}, && (34)\\ &S_{k,1} = (I + A_{k+})^{-1} q_{k-} \ \ (q_{k-} = (A^{n+1}_{k1-} - A^{n+1}_{n1-}), && (35)\\ &\gamma_{1,k} = (I + N_+)^{-1} C_{k-}(I + A_{k+}) + \delta_{1k} I + t_{k+}, && (36)\\ &\gamma_{n-1,k} = r_{k-}(I + A_{k+}). && (37) \end{aligned}$$

Proof. In the representation (11) if we consider the boundary conditions (4) we arrive at the equalities

$$b_1(t) = (I + A_{1+})^{-1} \left[(I + A^{n+1}_{11-} - A^{n+1}_{n1-}) a_1 + \sum_{j=2}^{n-1} (A^{n+1}_{1j} - A^{n+1}_{nj}) a_j(t) \right],$$

$$b_k(t) = (I + A_{k+})^{-1} \Big[(I + A^{n+1}_{k1-} - A^{n+1}_{n1-}) a_1(t) + (I + A^{n+1}_{kk} - A^{n+1}_{nk}) a_k(t) +$$

$$\left. \sum_{j=2, j\neq k}^{n-1} (A^{n+1}_{kj} - A^{n+1}_{nj}) a_j(t) \right], \ \ (2 \le k \le n-1).$$

From this and using the definition of the operator S we find that

$$\begin{aligned} S_{1,1} &= (I + A_{1+})^{-1} (I + B_{1-}), \ S_{1,p} = (I + A_{1+})^{-1} (A^{n+1}_{1p} - A^{n+1}_{np}) \ (p = 2, \dots, n-1),\\ S_{k,1} &= (I + A_{k+})^{-1} (A^{n+1}_{kl-} - A^{n+1}_{n1-}), \ S_{k,k} = (I + A_{1+})^{-1} (I + A^{n+1}_{kk} - A^{n+1}_{nk}),\\ S_{k,p} &= (I + A_{k+})^{-1} (A^{n+1}_{kp} - A^{n+1}_{np}), \ p \neq k, \ (k = 2, \dots, n-1) \end{aligned}$$

where

$$B_{1-} = A^{n+1}_{11-} - A^{n+1}_{n1-}, \ A^{n+1}_{i1-} a_1(t) = \textstyle\int_t^\infty A_{i1}(0,t,s) a_1(s) ds \ (i = 1, 2, \dots, n),$$

$$A^{n+1}_{qm} a_m(t) = \textstyle\int_{-\infty}^\infty A^{n+1}_{qm}(0,t,s) a_m(s) ds \ (q = 1, 2, \dots, n; \ m = 2, 3, \dots, n-1).$$

So the factorization (33) and the property (35) are proved. We have shown that $S = I + F$, where F is a Hilbert-Schmidt operator. Let us show that the operator S^{-1} exists. Let $b_1(t) = b_2(t) = \ldots = b_{n-1}(t) = 0$, i.e., $Sa(t) = 0$. From the Lemmas 4 and 5 we find that $\alpha(t) = 0, a_1(t) = a_2(t) \ldots = a_{n-1}(t) = 0$. This shows that S^{-1} exists and $S^{-1} = I + J$, where J is a Hilbert-Schmidt operator. Existence of the inverse of the operator $\Delta_k (k = 2, \dots, n-2)$ can be proved in the same way. By means of Lemma 5 we obtain (37) and (33) $(k = n-1)$.

Using Lemma 3 and the definition of the operators $(\Delta_k^{-1})_{kk}$ yield (33). To show (36), assume that the properties of Lemma 5 are satisfied. Then for

$$b_1(t) \neq 0, b_2(t) = \ldots = b_{n-1}(t) = 0,$$

using (9) we find

$$\Psi_1(0,t) = (I + A^2_{11+})a_1(t) + \sum_{j=2}^{n} A^2_{ij+}\Psi_2(0,t),$$

or

$$a_1(t) = (I + A^2_{11+})^{-1}\left[\Psi_1(0,t) - \sum_{j=2}^{n} A^2_{1j+}\Psi_2(0,t)\right] =$$

$$(I + A^2_{11+})^{-1}(I + C_{1-} - \sum_{j=2}^{n} A^2_{1j+}C_{1-}),$$

$$(I + A_{1+})b_1(t) = \Big[(I + A^2_{11+})^{-1}(I + A_{1+}) + (I + A^2_{11+})^{-1}$$

$$\left.(I - \sum_{j=2}^{n} A^2_{1j+})C_{1-}(I + A_{1+})\right] b_1(t).$$

Using (20) together with the definition of S^{-1} lead to

$$\gamma_{1,1} = (I + N_+)^{-1}C_{1-}(I + A_{1+}) + (I + A^2_{11+})^{-1}(I + A_{1+}). \tag{38}$$

Similarly from (9), for $b_k(t) \neq 0,\ b_i(t) = 0,\ i \neq k,\ k = 2, \ldots, n-1$,

$$\gamma_{1,k} = (I + N_+)^{-1}C_{k-}(I + A_{k+}) - (I + A^2_{11+})^{-1}A^2_{1k+}(I + A_{1+}). \tag{39}$$

Taking $t_{1+} = (I + A^2_{11+})^{-1}(I + A_{1+}) - I$, $t_{k+} = -(I + A^2_{11+})^{-1}A^2_{1k+}(I + A_{1+})$, $k = 2, \ldots, n-1$ from (38) and (39) we obtain (36).

We will prove (32) and (34). Let $b_1(t) = b_2(t) = \ldots = b_{n-1}(t) = b(t)$. Then $a_1(t) = (\gamma_{1,1} + \gamma_{1,2} + \ldots + \gamma_{1,n-1})b(t)$ and using Lemma 4 $a_1(t) = (I + N_+)^{-1}(I + N_-)b(t)$, that is (32) is true.

Let $a_1(t) = \ldots = a_{n-2}(t) = 0$. From Lemma 6

$$\Psi^{n-1}_n(0,t) - \Psi^k_n(0,t) = \left[(I - A^n_{n-1,n+})^{-1}(I + A^n_{n-1,n-1+}) - (I - A^n_{k,n+})^{-1}A^n_{k,n-1+}\right] a_{n-1}(t).$$

At the same time, by using (11), we obtain that the following:

$$\Psi^{n-1}_n(0,t) - \Psi^k_n(0,t) = (I + A^{n+1}_{n,n+})(b_{n-1}(0,t) - b_k(0,t)).$$

By comparing the last two equalities above we find that

$$b_{n-1}(t)-b_k(t) = (I+A^{n+1}_{n,n+})^{-1}\left[(I-A^{n}_{n-1,n+})^{-1}(I+A^{n}_{n-1,n-1+})-(I-A^{n}_{k,n+})^{-1}A^{n}_{k,n-1+}\right]a_{n-1}(t$$

and therefore

$$S_{n-1,n-1}-S_{k,n-1} = I+G_{k+},$$

where

$$G_{k+} = (I+A^{n+1}_{n,n+})^{-1}\left[(I-A^{n}_{n-1,n+})^{-1}(I+A^{n}_{n-1,n-1+})-(I-A^{n}_{k,n+})A^{n}_{k,n-1+}-I\right].$$

□

4. Inverse Scattering Problem

Finding the coefficients of the system (1) from the given operator S is called an inverse scattering problem.

An inverse scattering problem on the semi-axis can be transformed into an inverse scattering problem on the whole-axis where the coefficients are zero for $x<0$.

For the system (1), let us define the operator $\prod$ as follows

$$\prod = \begin{pmatrix} a_1(t) \\ \vdots \\ a_{n-1}(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} \Psi_1(0,t) \\ \vdots \\ \Psi_{n-1}(0,t) \\ \Psi_n(0,t) \end{pmatrix}. \tag{40}$$

Theorem 3. *Let us assume that the coefficients of the system (1) satisfy the conditions in (2). Let S be an operator defined on the semi-axis. From (32) and (33) we can find the Volterra operator $N_-, N_+, C_{k-} (k=1,\dots,n-1)$. By means of these operators and using the elements of the scattering operator, the operator $\prod$ is obtained as*

$$\prod = \begin{bmatrix} I+C_{1-} & \cdots & C_{n-1-} & I+N_- \\ C_{1-} & \cdots & C_{n-1-} & I+N_- \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ C_{1-} & \cdots & I+C_{n-1-} & I+N_- \\ C_{1-} & \cdots & C_{n-1-} & I+N_- \end{bmatrix} \mathit{diag}\,(I+A_{1+},\dots,I+A_{n-1+},I)\begin{pmatrix} S & -I \\ 0 & I \end{pmatrix}, \tag{41}$$

where

$$C_{1-} = \left[(I+N_+)\gamma_{1,1}(1+A_{1+})^{-1} - I\right]_-, \tag{42}$$
$$C_{k-} = \left[(I+N_+\gamma_{1,k}(I+A_{k+})^{-1}\right]_-, \quad (k=2,\dots,n-1). \tag{43}$$

Proof. Let $\psi(x,t)$ be a solution of the i-th $(i=1,\dots,n-1)$ problem whenever $b_2(t)=\dots=b_{n-1}(t)=0$. Then, $a_1(t)=\gamma_{1,1}b_1(t),\dots,a_{n-1}(t)=\gamma_{n-1,1}b_1(t)$. According to Lemma 5 we find $\Psi_1(0,t)$ from formula (21) and we get $\Psi_2(0,t)=\dots=\Psi_n(0,t)$ from formula (22) by using $b_1(t)$. By considering these results in (40), we obtain that

$$\prod\begin{pmatrix}\gamma_{1,1}\\ \vdots\\ \gamma_{n-1,1}\\ 0\end{pmatrix} = \begin{pmatrix}(I+C_{1-})(I+A_{1+})\\ \vdots\\ C_{1-}(I+A_{1+})\\ C_{1-}(I+A_{1+})\end{pmatrix}, \tag{44}$$

by assuming that $b_1(t)=b_{k-1}(t)=b_{k+1}(t)=0$ $(k=2,\dots,n-2)$, $a_1(t)=\gamma_{1,k}b_k(t),\dots,$ $a_{n-2}(t)=\gamma_{n-2,k}b(t), a_{n-1}(t)=\gamma_{n-1,k}b_k(t)$ and by using Lemma 5 the following is obtained

$$\prod\begin{pmatrix}\gamma_{1,k}\\ \vdots\\ \gamma_{k,k}\\ \vdots\\ \gamma_{n-1,k}\\ 0\end{pmatrix} = \begin{pmatrix}C_{k-}(I+A_{k+})\\ \vdots\\ (I+C_{k-})(I+A_{k+})\\ \vdots\\ C_{k-}(I+A_{k+})\\ C_{k-}(I+A_{k+})\end{pmatrix}, k=2,\dots,n-1). \tag{45}$$

If $b_1(t)=\dots b_{n-1}(t)=b(t)$ then
$a_1(t)=(\gamma_{1,1}+\dots+\gamma_{1,n-1})b(t),\dots,a_{n-1}(t)=(\gamma_{n-1,1}+\dots+\gamma_{n-1,n-1})b(t)$ is found and from Lemma 4

$$\prod\begin{pmatrix}\gamma_{1,1}+\cdots+\gamma_{1,n-1}\\ \vdots\\ \gamma_{n-1,1}+\cdots+\gamma_{n-1,n-1}\\ I\end{pmatrix} = \begin{pmatrix}I+N_-\\ \vdots\\ I+N_-\\ I+N_-\end{pmatrix} \tag{46}$$

is obtained. If we consider (44)-(46) we obtain the following matrix equality

$$\prod \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{n-1,1} & \gamma_{1,1}\cdots+\gamma_{1,n-1} \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \gamma_{1,n-1} & \cdots & \gamma_{n-1,n-1} & \gamma_{n-1,1}+\cdots+\gamma_{n-1,n-1} \\ 0 & \cdots & 0 & I \end{pmatrix} = \tag{47}$$

$$\begin{pmatrix} I+C_{1-} & \cdots & C_{n-1-} & I+N_- \\ \cdot & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ C_{1-} & \cdots & I+C_{n-1-} & I+N_- \\ C_{1-} & \cdots & C_{n-1-} & I+N_- \end{pmatrix} diag(I+A_{1+},...,I+A_{n-1+},I).$$

Since

$$\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{n-1,1} & \gamma_{1,1}\cdots+\gamma_{1,n-1} \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \gamma_{1,n-1} & \cdots & \gamma_{n-1,n-1} & \gamma_{n-1,1}+\cdots+\gamma_{n-1,n-1} \\ 0 & \cdots & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} S_{11} & \cdots & S_{1,n-1}, & -I \\ \cdot & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdot & \cdot \\ S_{n-1,1} & \cdots & S_{n-1,n-1}, & -I \\ 0 & \cdots & 0 & I \end{pmatrix},$$

Then, we get (41) from (47)

(42) and (43) are obtained appropriately from (38) and (39).

The operator $\prod$ defined by (40) is closely related to scattering operator of systems of equation of hyperbolic type given on the plane (x,t). If we assume that coefficients of the system (1) are zero for $x<0$, then the operator $\prod$ is the same as scattering operator of the new system. The inverse scattering problem is solved on the whole-axis in the studies [2,3]. □

Theorem 4. *If the coefficients of system (1) satisfy conditions (2), then the coefficients* $U_{ij}(x,t)$ $(i,j=1,2,\ldots,n)$ *in the system (1) on semi-axis are uniquely determined by the scattering operator.*

Formulas (32), (33), (41), (42), (43) together with the solution on the whole axis provide an algorithm for the solution of inverse scattering problem given on semi-axis in system (1).

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HİPERBOLİK TİP DENKLEM SİSTEMİ İÇİN YARI-EKSENDE STASYONER OLMAYAN TERS SAÇILMA PROBLEMİ

Özet

Bu makalede, 1. mertebeden hiperbolik tip n denklem sistemi için ($n > 3$) yarı eksende düz ve ters saçılma problemi incelenmiştir. Denklem sisteminin katsayıları, saçılma operatörüne göre tek türlü olarak belirlenmiştir. Saçılma operatörünün ve tersinin bazı elemanlarının operatör dönüşümleri yardımı ile çarpanlara ayrılabilmesi özelliklerini kullanarak ve yarı eksendeki saçılma problemini tüm eksendeki saçılma problemine indirgeyerek; problemin çözümü keyfi n doğal sayısı için genelleştirilmiştir.

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