[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

Volume 21
Number 5 *EK*

Article 8

1-1-1997

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Tr. J. of Mathematics
21 (1997) , 73 – 75.

LOCALLY FINITE BARELY TRANSITIVE PERMUTATION GROUP WITH ALMOST NILPOTENT POINT STABILIZERS

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Abstract

We show that the groups mentioned in the title are solvable. Moreover, a point stabilizer H of the locally finite barely transitive group G is almost nilpotent, whenever the indices $|H : H \cap H^g|$ $(g \in G)$ have a finite upper bound.

Introduction

A group G is called *barely transitive*, if it has a faithful representation as a transitive permutation group on an infinite set Ω such that the orbits of its proper subgroups are finite. Obviously a group G is barely transitive if and only if it has a subgroup H of infinite index, which contains no non-trivial normal subgroup of G, and which satisfies $|K : K \cap H| < \infty$ for every proper subgroup K of G. Here the subgroup H is a point stabilizer. In this article we shall study locally finite barely transitive groups, which we shall call *LFBT-groups.*

It is an open question whether perfect LFBT-groups exist. Metabelian LFBT-groups have been constructed by B.Hartley in [3] and [4]. And the general structure of imperfect LFBT-groups is fairly well-understood: They are extensions of nilpotent p-groups of finite exponent by a quasicyclic p-group, where p is a prime ([4], Love's Theorem).

First investigations of the question, in how far the structure of a point stabilizer H influences the structure of the LFBT-group G, can be found in [7] and [5]: If H is almost locally solvable, then G is a p-group, whose proper normal subgroups are nilpotent of finite exponents. And if H is abelian, then G turns out to be metabelian. In the present article we shall generalize this latter result as follows.

Proposition 1. *Let H be a point stabilizer of the LFBT-group G. If H is almost nilpotent of class c, then G is solvable of length at most $c \cdot (c+1)$.*

Almost nilpotent point stabilizers show up for example in the following situation.

* AMS Subject Classification 20F50, 20B07

Proposition 2. *Let H be a point stabilizer of the LFBT-group G. If there exists a finite upper bound for the indices $|H : H \cap H^g|$ $(g \in G)$ then H is almost nilpotent.*

Acknowledgement

The author would like to thank Prof.J.Roseblade for suggesting Proposition 1, which improves [5], Corollary 1. He also thanks to the referee for the improvements in the exposition of the article.

Proofs of the Propositions

In the sequel, $\zeta_i(G)$ will denote the i-th term of the upper central series in the group G. We shall make use of the following two known results about LFBT-groups.

(1) ([6], Lemma 2.2) No LFBT-group contains a proper subgroup of finite index.

(2) ([7], Proposition 2) Every LFBT-group with solvable point stabilizers is the union of an ascending chain of proper normal subgroups.

Proof of Proposition 1. Let H_0 be a subgroup of finite index in H, which is nilpotent of class c. Then H_0 contains no non-trivial normal subgroup of G and satisfies $|K : K \cap H_0| < \infty$ for every proper subgroup K of G. Hence, from replacing H by H_0, we may assume that H itself is nilpotent of class c.

Consider any proper normal subgroup K of G. Then the core A of $K \cap H$ in K is a normal subgroup of finite index in K. As a subgroup of H, the group A is nilpotent of class at most c. Let B_i denote the i-th term of the upper FC-central series in K. Note that B_i is normal in G. Because of $\zeta_i(A) \leq B_i$, the quotient K/B_c must be finite. By (1) we have $[K, G] \leq B_c$.

Next, we consider the quotients $\Gamma_i = B_i/B_{i-1}$ for $1 \leq i \leq c$. Clearly the subgroup $(B_i \cap H)B_{i-1}/B_{i-1}$ is nilpotent of class at most c, and it has finite index in the FC-group Γ_i. Therefore the argument in the proof of [2], Lemma 3.10 shows, that $\zeta_c(\Gamma_i)$ has finite index in Γ_i. But from (1), the group G can only act trivially on the finite quotients $\Gamma_i/\zeta_c(\Gamma_i)$ and $(K/B_{c-1})/\zeta_c(\Gamma_c)$. Hence the quotients K/B_{c-1} and B_i/B_{i-1} $(1 \leq i \leq c-1)$ are nilpotent of class $\leq c+1$. In particular, the arbitrarily chosen proper normal subgroup K of G is solvable of length $\leq c \cdot (c+1)$. Because of (2), the whole group G inherits this property. $\square$

Proof of Proposition 2. By [1], Theorem 3, the finite bound on the indices $|H : H \cap H^g|$ $(g \in G)$ leads to a normal subgroup N of G such that both of the indices $|N : N \cap H|$ and $|H : H \cap N|$ are finite. In particular, $N \neq G$.

Consider any proper normal subgroup K of G. Then

$$|K : K \cap N| \leq |K : K \cap N \cap H| \leq |K : K \cap H| \cdot |K \cap H : K \cap H \cap N| < \infty.$$

From (1), the group G must centralize the finite quotient $K/K \cap N$, whence $[K, K] \leq N$. An application of (2) shows that $[G, G] \leq N$. From Love's Theorem, N is nilpotent then. In particular, H must be almost nilpotent. $\square$

Question. Let H be a point stabilizer of a LFBT-group G. Is it true that, if H is almost solvable then G is solvable?

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BİR NOKTA STABİLİZATÖRÜ HEMEN HEMEN NİLPOTENT OLAN YEREL SONLU YALIN GEÇİŞKEN PERMUTASYON GRUPLARI

Özet

Başlıkta adı geçen grupların çözülebilir gruplar olduğu gösterilmiştir. Yerel sonlu yalın geçişken permutasyon gruplarında bir nokta stabilizatörü H için $|H : H \cap H^g| (g \in G)$ indislerinin üstten sonlu bir sınırı olduğu zaman H nin hemen hemen nilpotent olduğu gösterilmiştir.

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Received 4.12.1995