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ON HIGH ORDER RIESZ TRANSFORMATIONS GENERATED BY A GENERALIZED SHIFT OPERATOR

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Abstract

In this paper, we determine high order Riesz transformations by using generalized shift operators and giving some of their properties

1. Introduction

Let us consider Riesz Transformation as

$$(R_j f)(x) = \lim_{\epsilon \to 0} c_n \int\limits_{|x-y|\geq\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy \qquad (j = 1, 2, ..., n)$$

where $c_n = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$. In this transformations the difference $x - y$ can be regarded as an ordinary shift operation.

In 1987, Riesz transformations have been determined by considering generalized shift operator by I. Aliev [1].

In this study, we determine the high order Riesz transformations generated by generalized shift operators that are ordinary shift according to first $n-2$ variables and are also R^+ shift according to the last two variable [5].

2. Background and Notation

Let $\mathbf{R}_n^{++} = \{x = (x_1, x_2, ... x_{n-1}, x_n) : x_{n-1} \geq 0, \ x_n \geq 0\}$. Let the space of testing functions $\mathcal{Z}_+(\mathbf{R}_n^+) = \mathcal{Z}_+$ be class of all C^∞ functions φ on $\mathbf{R}_n^+$ (i.e. the partial derivatives of φ exist and are continuous) such that

$$\sup_{x \in \mathbf{R}_n^+} |x^\beta (D^\alpha \varphi)(x)| < \infty$$

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[2] This paper is dedicated to Ord.Prof.Dr. *C ARF*

for all n-tuples $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ of nonnegative integers, $D^\alpha = \partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2} \cdot \partial x_n^{\alpha_n}$. The dual space of $\mathcal{Z}_+$ is denoted by $\mathcal{Z}'_+$. We define $\mathcal{L}_{p,\nu_1,\nu_2}$ space

$$\mathcal{L}_{p,\nu_1,\nu_2} = \left\{ f(x) : ||f(x)||_{p,\nu_1,\nu_2} = \left(\int_{R_n^{++}} |f(x)|^p x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx \right)^{\frac{1}{p}} < \infty \right\}$$

where $\nu_1, \nu_2 > 0$ and $1 \leq p < \infty$. The principal value of $f(x)$ is

$$(v.p\ f, \varphi) = \lim_{\epsilon \to 0} \int_{\substack{0 < \varepsilon < |x| \\ 0 \leq x_n < \infty \\ 0 \leq x_{n-1} < \infty}} f(x)\varphi(x) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx, \qquad \varphi \in \mathcal{Z}_+.$$

Let Δ_B be the Laplacean-Bessel operators,

$$\Delta_B = \sum_{k=1}^{n-2} \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2\nu_1}{y}\frac{\partial}{\partial y} + \frac{2\nu_2}{z}\frac{\partial}{\partial z}, \qquad \nu_1, \nu_2 > 0$$

If

$$\frac{d^2u}{dr^2} + \frac{2\nu_1}{r}\frac{du}{dr} + \lambda^2 u = 0 \quad u(0) = 1, \quad u'(0) = 0 \quad (\lambda > 0) \tag{1}$$

and

$$\frac{d^2u}{dr^2} + \frac{2\nu_2}{r}\frac{du}{dr} + \lambda^2 u = 0 \quad u(0) = 1, \quad u'(0) = 1 \quad (\lambda > 0) \tag{2}$$

then solutions of equations (1) and (2) are denoted by $j_{\nu_1-\frac{1}{2}}(\lambda r)$ and $j_{\nu_2-\frac{1}{2}}(\lambda r)$ respectively. The Fourier-Bassel transformations can be defined by

$$[F_B\varphi](y) \int_{R_n^{++}} \varphi(x) e^{-i<x'',y''>} j_{\nu_1-\frac{1}{2}}(x_{n-1}y_{n-1}) x_{n-1}^{2\nu_1} j_{\nu_2-\frac{1}{2}}(x_n y_n) x_n^{2\nu_2} dx,$$

$$\varphi \in \mathcal{Z}_+.$$

and its invers transformations can be given by

$$[F_B^{-1}\varphi](y) = [F_B\varphi](-y), \qquad \varphi \in \mathcal{Z}_+.$$

where,
$< x'', y'' >= x_1y_1 + x_2y_2 + \cdots + x_{n-2}y_{n-2}$ and

$c_\nu = (2\pi)^{-\frac{n-2}{2}} 2^{-\nu_1-\nu_2+1}\Gamma(\nu_1+\frac{1}{2})^{-1}\cdot\Gamma(\nu_2+\frac{1}{2})^{-1}$
The generalized shift operator is denoted by T_x^y [6], [2],

$$T_x^y\varphi(x) = c_\nu \int_0^\pi \int_0^\pi \varphi\big[x''-y'' = \sqrt{x_{n-1}^2+y_{n-1}^2-2x_{n-1}y_{n-1}\cos\alpha_1},$$
$$\sqrt{x_n^2+y_n^2-2x_ny_n\cos\alpha_2}\big]\sin^{2\nu_1-1}\alpha\sin^{2\nu_2-1}\alpha d\alpha_1 d\alpha_2$$

where $x''y'' \in R_{n-2}$ and $c_\nu = \dfrac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\dfrac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}$.

3. High Order Riesz Transformations Generated By A Generalized Shift Operator

In this section, we consider generalized shift operator which has ordinary shift according to first $n-2$ terms and are also R^+-shift according to the last two terms. We study relations between the Fourier-Bessel operator and this generalized shift operator. We give the Fourier-Bessel transformation of homogeneous polinomial which holds Laplacean-Bessel equations. Finally, we define Riesz transformations related to the shift operators and so we show that this Riesz transformations holds the condition of classical Riesz transformation [5].

Theorem 3.1. *If $P_k(x) = P_k(x_1, x_2, ..., x_{n-1}^2, x_n^2)$, is a homogeneous polynomial with order k which holds $\Delta_B P_k(x) = 0$ Laplecean Bessel equations, then*

$$\big[P_k(x)e^{-|x|^2}\big]\hat{}(y) = 2^{-(k+\nu_1+\nu_2+\frac{n}{2})} i^k P_k(y) e^{\frac{-|y|^2}{4}}$$

where $\hat{}$ denotes Fourier-Bassel transformations.

Proof. By. [3] and [7], [8],

if $x'', y'' \in R_{n-2}$ and $\alpha > 0$, then

$$(2\pi)^{\frac{2-n}{2}} \int_{R_{n-2}} e^{-\alpha|x''|^2 - i(x''\cdot y'')} dx'' = \left(\frac{\pi}{\alpha}\right)^{\frac{n-2}{2}} e^{\frac{-|y''|^2}{4\alpha}} \tag{3}$$

If $\nu > -1, \alpha > 0$ and $J_\nu(br)$ is Bessel function, then

$$\int_0^\infty e^{-\alpha r^2} r^{\nu+1} J_\nu(br)dr = \frac{b^\nu}{(2\alpha)^{\nu+1}} e^{\frac{-b^2}{4\alpha}} \tag{4}$$

Since $J_\nu(r) = [2^\nu\Gamma(\nu+1]^{-1} r^\nu j_\nu(r)$, by (3) and (4) we have

$$F_B\big(e^{-\alpha|x|^2}\big)(y) = e^{\frac{-|y|^2}{4\alpha}}(2\alpha)^{-\frac{n+2\nu_1+2\nu_2}{2}}, \qquad y \in R_n^{++}$$

Hence

$$\int\limits_{R_n^{++}} e^{|x|^2+2i(x''\cdot y'')} \quad j_{\nu_1-\frac{1}{2}}(x_{n-1}2y_{n-1})x_{n-1}^{2\nu_1}\cdot j_{\nu_2-\frac{1}{2}}(x_n\,2y_n)x_n^{2\nu_2}dx$$
$$= \frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}\pi^{\frac{n-2}{2}}e^{-|y|^2} \tag{5}$$

If we apply differential operator

$$P_k(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},\ldots,\frac{\partial}{\partial t_{n-2}},Bt_{n-1},Bt_n)$$

to (5), then we obtain

$$\int\limits_{R_n^{++}} P_k(x)e^{|x|^2+2i(x''\cdot t'')} \quad j_{\nu_1-\frac{1}{2}}(x_{n-1}2t_{n-1})x_{n-1}^{2\nu_1}\cdot j_{\nu_2-\frac{1}{2}}(x_n 2t_n)x_n^{2\nu_2}dx$$
$$= Q(t)\frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}\pi^{\frac{n-2}{2}}e^{-|t|^2}$$

where $Q(t)$ is a polynomial and

$$B_{t_k} = \frac{\partial^2}{\partial t_k^2}+\frac{2\nu}{t_k}\frac{\partial}{\partial t_k} \qquad k=n-1,n.$$

Using the following formula [7].

$$j_{\nu-\frac{1}{2}}(r) = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma\frac{1}{2})}\int\limits_0^\pi e^{i\,r\cos\alpha}(\sin\alpha)^{2\nu-1}d\alpha$$

we have

$$Q(t) = 2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})]^{-1}\frac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\frac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}\int\limits_{R_n^{++}} P_k(x)e^{|t|^2-|t|^2+2i(x''.t'')}$$
$$\cdot\left(\int\limits_0^\pi e^{2ix_nt_n\cos\varphi}(\sin\varphi)^{2\nu-1}d\varphi\int\limits_0^\pi e^{2ix_{n-1}t_{n-1}\cos\alpha}(\sin\alpha)^{2\nu-1}d\alpha\right)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx$$
$$= 2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\int\limits_{R_n^{++}} P_k(x)x_{n-2}^{2\nu_1}x_n^{2\nu_2}dx$$
$$\frac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\frac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}\int\limits_0^\pi\int\limits_0^\pi e^{-|x''-it''|^2}e^{-(x_n^2+i^2t_n^2-2ix_nt_n\cos\varphi)}(\sin\varphi)^{2\nu_1-1}$$

$$e^{-(x_{n-1}^2+i^2t_{n-1}^2-2ix_{n-1}t_{n-1}\cos\alpha)}(\sin\alpha)^{2\nu_2-1}d\varphi d\alpha$$

By the properties of T_x^y, we obtain

$$Q(t)=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\cdot\int\limits_{R_n^{++}}P_k(x)\left[T_{x_n,x_{n-1}}^{-it}e^{-|x'|^2}\right]x_n^{2\nu_2}x_{n-1}^{2\nu_1}dx$$

$$Q(-it)=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\int\limits_{R_n^{++}}\left[T_{x_n,x_{n-1}}^{-t}P_k(x)\right]e^{-|x'|^2}x_n^{2\nu_2}x_{n-1}^{2\nu_1}dx$$

If $x=r\theta$ $(0<r<\infty,\theta\in S^+=\{|x|=1,\ x_{n-1},x_n\geq 0\})$, then

$$Q(-it)=c_{\nu^*}\int\limits_0^\infty r^{2\nu_1+2\nu_2+n-1}\left(\int\limits_{S+}T_{r\theta,r\theta'}^{-t}P_k(r\theta,r\theta')\theta_n^{2\nu_2}\theta_{n-1}^{2\nu_1}\right)e^{-r^2}dr \qquad (6)$$

where $c_{\nu^*}=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}$.
Applying the mean value theorem for $\Delta_B u=0$, [4] to (6), we have

$$Q(-it)=\frac{2}{\Gamma(\nu_1+\nu_2+\frac{n}{2})}P_k(t)\int\limits_0^\infty r^{2\nu_1+2\nu_2+n-1}e^{-r^2}dr=P_k(t)$$

and

$$\int\limits_{R_n^{++}}P_k(x)e^{|x|^2+2i(x''.t'')}j_{\nu_1-\frac{1}{2}}(x_{n-1}2t_{n-1})x_{n-1}^{2\nu_1}j_{\nu_2-\frac{1}{2}}(x_n2t_n)x_n^{2\nu_2}dx$$
$$=P_k(it)e^{-|t|^2}\pi^{\frac{n-2}{2}}\frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}$$

Since P_k are homogeneous, we have

$$\left[P_k(x)e^{-|x|^2}\right]^{\wedge}=2^{-(k+\nu_1+\nu_2+\frac{n}{2})}i^kP_k(y)e^{\frac{-|y|^2}{4}}$$

and so the theorem is proved □

Now we theorem is proved.

Lemma 3.2. *If*

$$\int\limits_{S^+}f(\theta_1,\theta_2,...,\theta_{n-1},\theta_n)\theta_{n-1}^{2\nu_1}\theta_n^{2\nu_2}dS^+=0$$

then

$$\frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}} \to v.p\,\frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2}}$$

for $\epsilon \to 0$, where

$$(v.pf.\varphi) = \lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0\le x_n<\infty \\ 0\le x_{n-1}<\infty}} f(x)\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx \quad \varphi \in Z_+$$

Proof. The proof follows immediatly from the representation.

$$\int\limits_{R_n^+} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx$$

$$= \int\limits_{|x|\le 1} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}[\varphi(x)-\varphi(0)]x_{n-1}^{2\nu_1}x_n^{2\nu_1}dx$$

$$+ \int\limits_{|x|>1} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx.$$

By considering Theorem 3.1 and Lemma 3.2, we have □

Theorem 3.3. *Let P_k be homogeneous polynomial with order k, then*

$$\left[v.p\frac{P_k}{|x|^{k+n+2\nu_1+2\nu_2}}\right]^{\hat{}}(y) = 2^{\frac{n+2\nu_1+2\nu_2}{2}}i^k\frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+n+2\nu_1+2\nu_2}{2})}\frac{P_k(y)}{|y|^k}$$

where $\hat{}$ denotes Fourier-Bassel transformations.

Proof. Let us consider

$$F_B[f(\alpha x)](t) = \alpha^{-n-2\nu_1-2\nu_2}F_B[f(x)](\frac{t}{\alpha}) \tag{7}$$

By (7) and Theorem 3.1 we have

$$[P_k(x)e^{-\alpha|x|^2}]^{\hat{}}(y) = (2\alpha)^{-(k+\nu_1+\nu_2+\frac{n}{2})}i^k e^{\frac{-|y|^k}{4\alpha}}P_k(y)$$

If $\varphi \in \mathcal{Z}_+$, then

$$\int\limits_{R_n^+} P_k(x) e^{\frac{-\alpha|x|^2}{4\alpha}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx \;=\; (2\alpha)^{-(k+\nu_1+\nu_2+\frac{n}{2})} i^k \cdot \int\limits_{R_n^+} P_k(x) e^{-\alpha|x|^2} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx \tag{8}$$

If we apply $\alpha^{\frac{k+n+2\nu_1+2\nu_2-\epsilon-2}{2}}$ to (8), integrate with respect to α from 0 to ∞ and use

$$\int\limits_0^\infty e^{-\alpha|x|^2} \alpha^{\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}-1} d\alpha = \Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right) |x|^{-(k+n+2\nu_1+2\nu_2-\epsilon)}$$

then we have

$$\Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right) \int\limits_{R_n^+} \frac{P_k(x)}{|x|^{-(k+n+2\nu_1+2\nu_2-\epsilon)}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx$$

Hence

$$2^{-\frac{2\nu_1+2\nu_2+n}{2}+\epsilon} \Gamma\left(\frac{k+\varepsilon}{2}\right) i^k \int\limits_{R_n^+} \frac{P_k(x)}{|x|^{k+\epsilon}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx$$

Therefore

$$\left[\frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2-\epsilon}}\right]\hat{}(y) = 2^{-\frac{2\nu_1+2\nu_2+n}{2}+\epsilon} i^k \cdot \frac{\Gamma\left(\frac{k+\epsilon}{2}\right)}{\Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right)} \cdot \frac{P_k(y)}{|y|^{k+\epsilon}}$$

By Lemma 3.2, we have

$$\left[p.v \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2-\epsilon}}\right]\hat{}(y) = 2^{\frac{-2\nu_1+2\nu_2+n}{2}} i^k \cdot \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n+2\nu_1+2\nu_2}{2}\right)} \cdot \frac{P_k(y)}{|y|^k}$$

Thus, the proof is complete. □

Now we give the Riesz transformations generated by generalized shift operators what we call Riesz-Bessel transformations.

Definition 3.4. *Let T_x^y be generalized shift operator. Then*

$$\begin{aligned} (R_B^{(k)} f)(\xi) &= c_k(n,\nu_1,\nu_2)\left(p.v \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2}} \star f\right)(\xi) \\ &\equiv c_k(n,\nu_1,\nu_2) \lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty}} \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx \end{aligned}$$

is called Riesz-Bessel transformation with order k *for* $f(x) \in \mathcal{Z}_+$, *where*

$$c_k(n,\nu_1,\nu_2)2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{k+n+2\nu_1+2\nu_2}{2}\right)\left[\Gamma\left(\frac{k}{2}\right)\right]^{-1} \quad k=1,2,...$$

Let $P_k(x)$ be homogeneous polynomial with order k and $\Delta_B P_k = 0$. By Theorem 3.3, we have

$$\left(R_B^{(k)}f\right)^{\hat{}}(\xi) = i^k \frac{P_k(\xi)}{|\xi|^k} f^{\hat{}}(\xi) \tag{9}$$

We note that in (9) $i^k P_k |\xi|^{-k}$ is a factor corresponding to the transformation $R_B^{(k)}$.

Now consider Riesz-Bessel transformations with order one. Note that the number of these transformations is $n-2$ and $(j=1,2,...,n-2)$

$$(R_{B_i}f)(\xi) = c_1(n,\nu_1,\nu_2)\lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty \\ 0<x_{n-1}<\infty}} \frac{x_j}{|x|^{1+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx \tag{10}$$

where,

$$c_1(n,\nu_1,\nu_2) = 2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{1+n+2\nu_1+2\nu_2}{2}\right)\frac{1}{\sqrt{\pi}}$$

We have also two Riesz-Bessel transformations with order 2 such that

$$(R_{B_i}^2 f)(\xi) = c_2(n,\nu_1,\nu_2)\lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty \\ 0<x_{n-1}<\infty}} \frac{P_2(x)}{|x|^{2+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx \tag{11}$$

where,

$$c_2(n,\nu_1,\nu_2) = 2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{2+n+2\nu_1+2\nu_2}{2}\right)$$

and corresponding polynomial to P_k are

$$P_2(x) = \frac{2\nu_1+2\nu_2+2}{\frac{n}{2}+\nu_1+\nu_2}|x|^2 - 2x_{n-1}^2 - 2x_n^2$$

and

$$P_2'(x) = -\frac{2\nu_1+2\nu_2+2}{n+2\nu_1+2\nu_2}|x|^2 + x_{n-1}^2 + x_n^2$$

It can be easily shown that $\Delta_B P_2 = 0$ and $\Delta_B P_2' = 0$. Then, by (9) we have

$$(R_{B_n}f)^{\hat{}}(\xi) = \left(\frac{2\nu_1+2\nu_2+2}{\frac{n}{2}+\nu_1+\nu_2} - \frac{2\xi_{n-1}^2}{|\xi|^2} - \frac{2\xi_n^2}{|\xi|^2}\right) f^{\hat{}}(\xi)$$

and

$$(R_{B_{n-1}}f)^{\hat{}}(\xi) = \left(-\frac{2\nu_1+2\nu_2+2}{n+2\nu_1+2\nu_2} + \frac{\xi_{n-1}^2}{|\xi|^2} + \frac{\xi_n^2}{|\xi|^2}\right) f^{\hat{}}(\xi)$$

We note that

(i)

$$\sum_{j=1}^{n-2}(R_{B_j})^2 + R_{B_{n-1}} + R_{B_n} = \left(-1 + \frac{2\nu_1+2\nu_2+2}{2\nu_1+2\nu_2+n}\right) E$$

wehere E is the identity operator in $L_{p,\nu}(R_n^+)$.

(ii) If $1 \le j,k \le n-2$ then

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_{B_j} + R_{B_k}\Delta_B f, \qquad f \in \mathcal{Z}_+$$

(iii)

$$[(B_{x_n} + B_{x_{n-1}})f](y) = \left[\frac{2\nu_1+2\nu_2+2}{2\nu_1+2\nu_2+n} - R_{B_n} - R_{B_{n-1}}\right]\Delta_B f$$

where $B_{x_n} = \dfrac{\partial^2}{\partial x_n^2} + \dfrac{2\nu}{x_n}\dfrac{\partial}{\partial x_n}$.

Remark By using the same method, this proof also can be given in general for this transformations generated by generalized shift operators that are ordinary shift operator according to m variables and are also R^+-shift operators with respect to k variables. Here $m+k=n$.

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References

[1] Aliev, I. A.: On Riesz Transformations Generated by a Generalized Shift Operators, *Ivestiya Acad of Sciences of Azerbaydian,* **1**, 7-13, (1987).

[2] Aliev, I.A., Gadjiev, A.D.: The Weighted Estimates for Multidimetional Singular Integrals Generated by The Generalized Shift Opetaros, *Matematika Sbornik Rossiyskaya Acad. Nauk,* **183**, 9, 45-66, (1992).

[3] Gray, A., G.B. Mathews.: A. Treatise on Bessel Functions and Their Applications to Physics, Mac. and Co. Lim. St., London, 1931.

[4] Kipriyanova, N.I.: *Differential Equations,* Vo. 121, No: 11, 1985.

[5] Levitan, B.M.: *Uspehi Mat. Nauk.,* **6**, 2(42, 102, 143-163, 1967.

[6] Neri, U.: Singular Integrals, *Lecture Notes in Mathematics,* Springer Verlag, Berlin-New York, 1971.

[7] Stein, E.M.: Singular Integrals, *Lecture Notes in Mathematics,* Springer Verlag, Berlin-New York, 1971.

[8] Stein, E.M.: Singular Integrals Differential Properties of Functions, Princeton Uni. Press, Princeton, New Jersey, 1970.

[9] Stein, E.M., Weis, G.: Introduction to Fourier Analysis on Euclidean Spaces, Princeton Uni. Press, Princeton New Jersey, 1971.

GENELLEŞTİRİLMİŞ ÖTELEME OPERATÖRÜ İLE ELDE EDİLEN YÜKSEK MERTEBELİ RİESZ DÖNÜŞÜMLERİ

Özet

Bu çalışmada Genelleştirilmiş Öteleme Operatörü yardımıyla yüksek mertebeli Riesz dönüşümleri ve bu dönüşümlerin bazı özellikleri verilmiştir.

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