Turkish Journal of Mathematics

Volume 21
Number 5 *EK*

Article 4

1-1-1997

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Q. DENG

M. Ş. YENİGÜL

N. ARGAÇ

Tr. J. of Mathematics
21 (1997) , 45 – 49.

ON IDEALS OF PRIME RINGS WITH (σ,τ)-DERIVATIONS

Q. Deng, M.Ş. Yenigül & N. Argaç

Abstract

Let R be a prime ring. Let σ,τ be two homomorphisms and d be a (σ,τ)-derivation of R. The purpose of this paper is to prove two results: (i) If char $R \neq 2$, U is a non-zero ideal of R, σ is subjective such that $\sigma(U)\neq 0$, τ is an automorphism and $[d(U),d(U)]_{\sigma,\tau}=0$, then $\sigma^2=\tau^2$ and $\sigma\tau=\tau\sigma$. (ii) Under the assumptions that either $\text{char}\, R=0$ or $\text{char}\, R>\max\{2,n\}, U$ is a non-zero right ideal, and σ,τ are automorphisms of R, suppose $[d(x),x^n]_{\sigma,\tau}\subseteq C_{\sigma,\tau}$ for all $x\in U$, then $\sigma=\tau$.

Introduction

Let R be a ring and σ,τ,α be any mappings from R to R. An additive mapping $d: R\to R$ is called (σ,τ)-derivation if $d(xy)=d(x)\sigma(y)+\tau(x)d(y)$ is fulfilled for all $x,y\in R$. We say that $d: R\to R$ is an α-derivation if $d(xy)=d(x)\alpha(y)+xd(y)$ for all $x,y\in R$.

We denote $[x,y]=xy-yx, [x,y]_{\sigma,\tau}=x\sigma(y)-\tau(y)x$ and $C_{\sigma,\tau}=\{c\in R : c\sigma(x)=\tau(x)c,$ for all $x\in R\}$. It is clear that $C_{1,1}=Z(R)$, where $Z(R)$ is the center of R. We define $I_a(x)=[a,x]$ to be an inner derivation of R.

The purpose of this paper is to prove two theorems, which are of independent interest and related to (σ,τ)-derivations. In Theorem 1 we remove the condition of the injectiveness imposed on σ in [1, Theorem 2], and in Theorem 2 we prove that $\sigma=\tau$ under the same hypothesises as in [2, Theorem].

Lemma 1. *Let R be a prime ring of char$R\neq 2$. Let U be a non-zero ideal of R,σ be a surjective homomorphism of R with $\sigma(U)\neq 0$ and τ be an automorphism of R. Suppose that R admits a non-zero (σ,τ)-derivation d such that $[d(u),a]_{\sigma,\tau}=0$ for all $u\in U$, a fixed $a\in R$, then $\sigma(a)\in Z(R)$.*

AMS Subject Classification (1991): 16A72, 16A68, 16A70
Key words and phrases: Prime ring, ideal, homomorphism, automorphism, (σ,τ)-derivation, α-derivation

Proof. Because $[d(uv), a]_{\sigma,\tau} = 0$ for all $u, v \in U$, we have $(d(u)\sigma(v) + \tau(u)d(v))\sigma(a) - \tau(a)(d(u)\sigma(v) + \tau(u)d(v)) = 0$. Since $[d(u), a]_{\sigma,\tau} = 0$ and $[d(v), a]_{\sigma,\tau} = 0$ we obtain

$$d(u)\sigma([v, a)] + \tau([u, a])d(v) = 0, \text{ for all } u, v \in U. \tag{1}$$

Replace v by va in (1) and using (1) we arrive at $\tau([u, a])\tau(U)d(a) = 0$. Since R is prime so are U and $\tau(U)$. Hence either $d(a) = 0$ or $a \in Z(U) \subseteq Z(R)$. If $d(a) = 0$ then $d([u, a]) = d(u)\sigma(a) - \tau(a)d(u) = [d(u), a]_{\sigma,\tau} = 0$. Therefore it follows that

$$d([U, a]) = 0 \tag{2}$$

Now replace v by vw in (1), where $w \in U$, then we have

$$d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) = 0 \text{ for all } u, v, w \in U \tag{3}$$

Taking $[w, a]$ instead of w in (3) and applying (2) we then get $d(u)\sigma(v)\sigma([[w, a], a]) = 0$ for all $u, v, w \in U$. Since $\sigma(U)$ is a non-zero ideal of R so either $d(U) = 0$ or $(\sigma[[w, a], a]) = 0$ for all $w \in U$. Then, by the fact that $d(U) \neq 0$, we have $\sigma([[w, a], a]) = 0$, that is, $I^2_{\sigma(a)}(\sigma(U)) = 0$. It follows then that $[\sigma(a), \sigma(U)] = 0$, which yields $\sigma(a) \in Z(R)$. If $a \in Z(R)$ again $\sigma(a) \in Z(R)$. □

Lemma 2. *If the same hypothesises in Lemma 1 are valid, then* $\sigma(a) = \tau(a)$ *and* $a \in Z(R)$.

Proof. From the hypothesises we have $[d(u), a]_{\sigma,\tau} = 0$ for all $u \in U$. This implies $d(u)\sigma(a) - \tau(a)d(u) = 0$ and by Lemma 1 we get $\sigma(a) \in Z(R)$. Therefore we arrive at $(\sigma(a) - \tau(a))d(u) = 0$ for all $u \in U$. From $(\sigma(a) - \tau(a))d(uv) = 0$ we get $(\sigma(a) - \tau(a))\tau(u)d(v) = 0$ for all $u, v \in U$. Since $\tau(u)$ is an ideal of prime ring R and $d \neq 0$, we obtain $\sigma(a) = \tau(a)$, consequently $\tau(a) \in Z(R)$ implies $a \in Z(R)$. □

Lemma 3. *Let* R *be a prime ring of* $charR \neq 2$ *and* U *be an ideal of* R. *Let* σ *be a surjective homomorphism such that* $\sigma(U) \neq 0$ *and* τ *be an automorphism of* R. *If* $d(U) \subseteq Z(R)$, *then* R *is commutative.*

Proof. Denote $W = \tau(U)$. Then we have $[x, d(uv)] = 0$ for all $u, v \in U, x \in R$. This implies $d(u)[x, \sigma(v)] + d(v)[x, \tau(u)] = 0$. Substituting $\sigma(v)$ for x we have $d(v)[\sigma(v), \tau(u)] = 0$, that is, $d(v)I_{\sigma(v)}(W) = 0$. Since W is an ideal of R, we get either $d(v) = 0$ or $\sigma(v) \in Z(R)$ for all $v \in U$. Let $K = \{v \in U : d(v) = 0)$ and $L = \{v \in U : \sigma(v) \in Z(R)\}$. Then $U = K \cup L$ and K, L are additive subgroups of R. By Brauers's trick it follows that $\sigma(v) \in Z(R)$ for all $v \in U$. Since σ is surjective and $\sigma(U)$ is a non-zero commutative ideal of R, one obtains R is commutative.

Now we are ready to give the proof of the following theorem. □

Theorem 1. *The R be a prime ring of $charR \neq 2$, and U a non-zero ideal of R. Let σ be a surjective homomorphism, τ an automorphism, and d a non-zero (σ,τ)-derivation of R. If $\sigma(U) \neq 0$ and $[d(U), d(U)]_{\sigma,\tau} = 0$ then $\sigma^2 = \tau^2$ and $\sigma\tau = \tau\sigma$. If $U = R$ and $d^2 \neq 0$, then $\sigma = \tau$.*

Proof. By Lemma 2 and Lemma 3, R is commutative and $\sigma(d(u)) = \tau(d(u))$ for all $u \in U$. We denote $f = \tau^{-1}\sigma$ then we can write $f(d(x)) = d(x)$ for all $x \in U$. From $f(d(xy)) = d(xy)$ one obtains □

$$\begin{aligned} f(d(x)\sigma(y) + \tau(x)d(y)) &= d(x)\sigma(y) + \tau(x)d(y) \\ d(x)f\sigma(y) + f\tau(x)d(y) &= d(x)\sigma(y) + \tau(x)d(y) \\ d(x)(f\sigma(y) - \sigma(y)) &= d(y)(\tau(x) - f\tau(x)) \end{aligned} \tag{4}$$

for all $x, y \in U$. Let $y = x$, then relation (4) gives $d(x)[f\sigma(x) + f\tau(x) - \sigma(x) - \tau(x)] = 0$ for all $x \in U$. We then gain

$$f\sigma(x) - \sigma(x) = \tau(x) - f\tau(x), \text{ for all } x \in U. \tag{5}$$

Replace y by yz in (4), where $z \in U$ and using (4) we arrive at

$$\begin{aligned} (d(y)\sigma(z) \ + \ \tau(y)d(z))(\tau(x) - f\tau(x)) = \\ \sigma(z)d(y)(\tau(x) - f\tau(x)) + \tau(y)d(z)(\tau(x) - f\tau(x)) = \\ \sigma(z)d(x)(f\sigma(y) - \sigma(y)) + \tau(y)d(x)(f\sigma(z) - \sigma(z)), \end{aligned}$$

that is, $(f\sigma(z) - \sigma(z))(f\sigma(y)) - \tau(y)) = 0$ for all $y, z \in U$. If there exists an element $z \in U$ such that $f\sigma(z) \neq \sigma(z)$ then we get $f\sigma(y) = \tau(y)$ for all $y \in U$ so that $\tau^{-1}f\sigma$ is identical on U. By [4, Lemma 3] we have $\tau^{-1}f\sigma = 1$, so $\sigma^2 = \tau^2$, and $f\sigma = \tau$. Using (5) we have $f\tau(x) = \sigma(x)$ for all $x \in U$ and $f\tau(rx) = \sigma(rx)$ gives $(f\tau(r) - \sigma(r))\sigma(x) = 0$ for all $r \in R, x \in U$. Therefore $f\tau(r) = \sigma(r)$ implies $f\tau = \sigma$, that is, $\sigma\tau = \tau\sigma$. If $f\sigma(z) = \sigma(z)$ all $z \in U$, then the endomorphism f is identical on the ideal $\sigma(U)$ so $f = I$. Hence we get $\sigma = \tau$ and $\sigma\tau = \tau\sigma$ again. Suppose that $U = R$, replacing y by $d(y)$ in (4) leads to

$$d(x)(f\sigma(d(y)) - \sigma(d(y))) = d^2(y)(\tau(x) - f\tau(x)) \tag{6}$$

for all $x, y \in R$. Since $\sigma d(y) = \tau d(y)$ for all $y \in R$ and $f\tau(x) = \tau^{-1}\sigma\tau(x) = \tau^{-1}\tau\sigma(x) = \sigma(x)$ for all $x \in R$, we can rewrite (6) in the form $d(x)(f\tau(d(y)) - \tau(d(y))) = d^2(y)(\tau(x) - \sigma(x))$. Since $f\tau(d(y)) - \tau(d(y)) = \sigma d(y) - \tau d(y) = 0$, one obtains $d^2(y)(\tau(x) - \sigma(x)) = 0$ for all $x, y \in R$. Using $d^2 \neq 0$ we conclude that $\sigma = \tau$. This completes the proof.

Corollary 1. *Let R be a prime ring of $charR \neq 2$. Let σ, τ be two automorphisms and d be a non-zero (σ,τ)-derivation of R. If $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, then $\sigma\tau = \tau\sigma$ and if $d^2 \neq 0$, then $\sigma = \tau$.*

Proof. R is commutative by [1, Theorem 3]. Since $[d(x),d(y)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$ for all $x,y \in R$, we have $d(x)\sigma(d(y))-\tau(d(y))d(x) \in C_{\sigma,\tau}$ hence $(\sigma(d(y))-\tau(d(y)))d(x) \in C_{\sigma,\tau}$. Then we get $[(\sigma(d(y)) - \tau(d(y)))d(x), z]_{\sigma,\tau} = 0$ for all $x,y,z \in R$. This implies that $(\sigma(d(y)) - \tau(d(y)))[d(x),z]_{\sigma,\tau} = 0$. If there exists $y \in R$ such that $(\sigma(d(y)) - \tau(d(y)) \neq 0$ then we arrive at $[d(x),z]_{\sigma,\tau} = 0$ for all $x,z \in R$. Therefore we obtain $[d(R),d(R)]_{\sigma,\tau} = 0$. If $\sigma(d(y)) = \tau(d(y))$ for all $y \in R$, then again we have $[d(x),d(y)]_{\sigma,\tau} = 0$ for all $x,y \in R$. Thus one obtains $\sigma\tau = \tau\sigma$, and furthermore if $d^2 \neq 0$, then $\sigma = \tau$ by Theorem 1. □

Corollary 2. *Let R be a prime ring of char$R \neq 2$. Let α be a surjective homomorphism and d be a non-zero α-derivation of R. If $[d(R),d(R)]_\alpha = 0$ and $\alpha d = d\alpha$ then $\alpha = 1$.*

Proof. In view of Theorem 1 it is enough to show that $d^2 \neq 0$. Suppose $d^2 = 0$. Thus $d^2(xy) = 0$ for all $x,y \in R$, which yields $d(x)d\alpha(y) = 0$ for all $x,y \in R$. Hence R is commutative by [3, Theorem 1]. Since $d \neq 0$, we have $d\alpha(R) = 0$ and also $d(R) = 0$. This is a contradiction. Hence the proof is complete. □

Corollary 3. *Let R be a prime ring of char$R \neq 2$. Let σ,τ be two automorphisms, and d a non-zero (σ,τ)-derivation of R. Let U be a non-zero ideal, and a non-zero fixed element of R. If $ad(U) \subseteq C_{\sigma,\tau}$ and $d(U) \neq 0$ then $\sigma = \tau$.*

Proof. R is commutative by [1, Lemma 5]. Therefore one obtains $ad(x)\sigma(y) = \sigma(y)ad(x)$ for all $x,y \in U$. The fact that $ad(x) \in C_{\sigma,\tau}$ gives $ad(x)\sigma(y) = \tau(y)ad(x)$ for all $x,y \in U$ so we have $a(\sigma(y) - \tau(y))d(x) = 0$ for all $x,y \in U$. $a \neq 0$ and $d(U) \neq 0$ implies $\sigma(y) = \tau(y)$ for all $y \in U$. Since R is commutative and U is a non-zero ideal of R, it follows that $\sigma = \tau$. □

Combining with Deng's result in [2, Theorem] we have the following theorem.

Theorem 2. *Let R be a prime ring of char$R = 0$ or of char$R > \max\{2,n\}$ with a positive integer n and let U be a non-zero right ideal of R. Let furthermore σ,τ be two automorphisms, and d a non-zero (σ,τ)-derivation of R. If $[d(x),x^n]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $x \in U$, then $\sigma = \tau$.*

Proof. R is commutative by [2, Theorem] and also we have $[d(x),x^n]_{\sigma,\tau} = 0$ by [2, Lemma 2]. Linearizing it and using [2, Lemma 1], we get $[d(y),x^n]_{\sigma,\tau} + [d(x),x^{n-1}y + x^{n-2}yx+\cdots+yx^{n-1}]_{\sigma,\tau} = 0$. Replacing y by x^2 in the last relation we have $n[d(x),x^{n+1}]_{\sigma,\tau} = 0, [d(x),x]_{\sigma,\tau}\sigma(x^n) = 0$, and $[d(x),x]_{\sigma,\tau} = 0$. Thus one obtains $d(x)(\sigma(x) - \tau(x)) = 0$ for all $x \in U$. Hence either $d(x) = 0$ or $\sigma(x) = \tau(x)$ for all $x \in U$. Let $S = \{x \in U : d(x) = 0\}$ and $T = \{x \in U : \sigma(x) = \tau(x)\}$. Then S and T are additive subgroups of U and also $U = S\ U\ T$. By Brauers's trick we get $U = T$. Since $d \neq 0$ and R is commutative it follows that $\sigma = \tau$. □

References

[1] Aydın, N and Kaya, K.: Some generalization in prime rings with (σ, τ)-derivations, Turkish J. of Math., 16, 169-171 (1992).

[2] Deng, Q.: On (σ, τ)-derivations of prime rings, Turkish J. of Math. 18, 130-133 (1994).

[3] Kaya, K.: On prime rings with weak α-derivations, Turkish J. of Math., 12, 46-51 (1988).

[4] Mayne, J.H.: Ideals and Centralizing mappings in prime rings, Proc. Amer. Math. Soc., 86, 211-212 (1982).

[5] Posner, E.C.: Derivations in prime rings, Proc. Amer. Math. Soc. 8, 1093-1100 (1957).

(σ, τ)-TÜREVLİ ASAL HALKALARIN İDEALLERİ ÜZERİNE

Özet

R bir halka, σ, τ R nin iki homomorfizması ve d R nin bir (σ, τ)-türevi olsun. Bu çalışmanın amacı aşağıdaki sonuçları ispatlamaktır: (i) $\mathrm{char}\, R \neq 2$, U R nin sıfırdan farklı bir ideali, σ, $\sigma(U) \neq 0$ olacak şekilde R nin bir örten homomorfizması ve $[d(U), d(U)]_{\sigma,\tau} = 0$ olsun. O zaman $\sigma^2 = \tau^2$ ve $\sigma\tau = \tau\sigma$ dur. (ii) $\mathrm{Char}\, R = 0$ ya da n pozitif bir tamsayı olmak üzere $\mathrm{char}\, R > \max\{2, n\}$ ve U R nin sıfırdan farklı bir sağ ideali, σ, τ R nin iki otomorfizması olsun. Her $x \in U$ için $[d(x), x^n]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$ ise o zaman $\sigma = \tau$ dur.

Q. DENG
Department of Mathematics
Soutwest China Teachers Univ.
Chongqing 630715, CHINA

M.Ş. YENİGÜL & N. ARGAÇ
Department of Mathematics
Science Faculty,
Ege University,
Bornova, 35100 İzmir-TURKEY

Received 5.2.1996