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ON SOME BOUNDS FOR THE SOLUTIONS OF THE SEMI-DISCRETIZED TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

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Abstract

We study the two-dimensional system of Time-Dependent Ginzburg-Landau Equations(TDGL) for modeling a thin film of superconductor subject to a uniform magnetic field. We discretize the TDGL for the space variables using bond variables and staggered grid partitioning technique. By investigating the temporal evolution of semi-discrete Helmholtz enery functional and that of Semi-discretized TDGL, we provide bounds for some observable physical quantities of interest such as superelectron density, supercurrent density, charge density, electric field, and induced magnetic field.

1. Introduction

Both steady and time-dependent Ginzburg-Landau equations are being extensively studied for modeling superconductors of various characteristics as a result of the recent discovery of High T_c superconductivity. The new era marked by the discovery of A. Bednorz and K. A. Muller in 1986 attracts more attention from various scientific communities as the new compounds with higher transition temperatures are being reported.

Theoretical and numerical studies about superconductivity, in general, and Ginzburg-Landau model, in particular, falls far behind the physical experimental studies. We refer to [1],[2],[3],[4],[6], and [7] for recent numerical and theoretical studies on steady and time-dependent Ginzburg-Landau model. In what follows, we mention recent studies on the mathematical aspect of the TDGL relevant to the subject of this paper.

Du[5] illustrated that the original TDGL with the prescribed boundary conditions are not well-posed and identified possible choices of extra conditions one can impose, better known as *gauge fixing*, to have a well-posed problem. In a so-called zero potential gauge, he gave a proof of global existence and uniqueness of strong solutions in two and

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three dimesional bounded domains and provided some bounds for order parameter, induced magnetic field and charge density over a *finite time interval.* Kwong and Kaper [8] considered two dimensional steady GL and defined a new gauge in which the equations for the two nonzero components of the vector potential are only weakly coupled through the order parameter. Also, in[8] discrete GL equations with quasi-periodic boundary conditions are derived using staggered grid discretization, a technique well known in numerical fluid mechanics, and some properties of solutions of the steady GL are investigated.

In this paper, we consider the two dimensional TDGL for modelling a finite size rectangular thin film superconductor subject to a uniform applied field normal to the plane of the film. The problem then leads to the use of natural boundary conditions. Among the gauge choices discussed in [5], the zero potential gauge (see Section 3.1) is used. This choice of gauge eliminates the scalar electric potential from the system, thus leads to a convenient form for the framework at which the TDGL is investigated in this study. Another reason is that the TDGL can be viewed as a gradient flow with respect to Helmholtz energy functional in this gauge, see Section 3.2.

We discretize the equations only for the space variables by using the nonstandard discretization technique employed in [8] and use bond variables so that the Semi-Discrete TDGL will also be gauge invariant. The bond variables don't alter the boundary conditions for the vector potential, but the conditions for the order parameter had to be modified as discussed in Section 2.3. Throughout the paper, the resulting Semi-Discrete TDGL will be referred to as SDTDGL.

We prove that the semi-discrete Helmholtz energy functional is a decreasing function with respect to the time variable t. To the best of the author's knowledge, the analog of this result did not appear in the literature for the continuous energy functional. We extend the well known property of order parameter[8] for steady GL to SDTDGL. A similar result is presented in [5] for TDGL where the bound holds almost everywhere(outside of a set of measure zero). Furthermore, we use the boundedness of energy functional and that of order parameter in connection with the natural boundary conditions to prove a series of new bounds for some physical quantities of interest. The continuous analogs of two of these bounds are presented in [5] over a *finite time interval.* We also provide a bound in l_2 norm for the derivative of order parameter with respect to time variable t. As far as we know, this bound and those of supercurrent density and electric field have no analogs for TDGL yet. Unlike the ones in [5], the bounds presented in this study hold $\forall t > 0$.

The paper is organized as follows: In Section 2.1 we introduce bond variables into the Helmholtz energy functional. In Section 2.2 we discretize the functional for the space variables. In Section 3.1 we introduce the zero potential gauge, and formulate SDTDGL in Section 3.2. Finally, in Section 3.3 we show that the semi-discrete Helmholtz energy functional is a decreasing function of time and present some useful bounds for various physical quantities of interest. The bounds depends only on initial data, physical parameters, and size of the domain.

2. Discrete Energy Functional

2.1. Energy Functional with Bonds Variables

For the purpose of this study, we only present the nondimensionalized form of energy functional. The various scales used to non-dimensionalize the functional (or TDGL) can be found in [2]. We note that to obtain the values of variables corresponding to the physical domain, one has to scale back the variables. Helmholtz energy functional (modulo constant) in nondimensionalized form can be given as [2]

$$\mathcal{G}(\psi, \mathbf{A}) = \int_{\Omega} \left(-|\psi|^2 + \frac{1}{2}|\psi|^4 + \left| \left(\frac{\nabla}{k} - i\mathbf{A} \right) \psi \right|^2 + |\nabla \times \mathbf{A} - \overline{\mathbf{H}}|^2 \right) d\Omega, \tag{2.1}$$

where the order parameter Ψ is assumed to be *complex* scalar-valued function such that $|\Psi|^2$ represents the local superelectron density, the vector potential $\mathbf{A}$ is a real three-dimensional vector-valued function, $\overline{\mathbf{H}}$ is the *applied* magnetic field, k is the Ginzburg-Landau parameter, and Ω is the domain. For the problem considered in this study, the vector potential has only two non-zero components, namely

$$\mathbf{A} = (A, B, 0)^T$$

and

$$\overline{\mathbf{H}} = (0, 0, H)^T$$

where $H > 0$ is the strength of the applied magnetic field and Ω is taken to be a rectangle of size $L_x \times L_y$ for any positive real numbers L_x, L_y.

We are interested in solutions of TDGL, namely, minimizers of the functional (2.1), with the following so-called *natural boundary conditions*,

$$(\nabla \times \mathbf{A}) \times \mathbf{n} = \overline{\mathbf{H}} \times \mathbf{n} \tag{2.2}$$

and

$$\left(\frac{\nabla}{k} - i\mathbf{A} \right) \Psi.\mathbf{n} = 0, \tag{2.3}$$

on Γ, where Γ denotes the boundary of Ω and $\mathbf{n}$ the unit outer normal vector to Γ. Here, the first equation expresses the continuity of the normal component of the magnetic field, and the second expresses the fact that the normal component of the supercurrent vanishes, that is, no current flows through the boundary of the material.

Next we introduce the *bond* variables into the functional (2.1). These are introduced (see [8]) to preserve the gauge invariance property of the discrete form of both the energy functional and that of the corresponding system.

Let us rewrite (2.1) in a slightly different form

$$\mathcal{G}(\Psi, A, B) = \int_\Omega \left(-|\Psi|^2 + \frac{1}{2}|\Psi|^4\right) dxdy$$
$$+\int_\Omega \left(|\frac{1}{k}(\partial_x - kiA)\Psi|^2 + |\frac{1}{k}(\partial_y - kiB)\Psi|^2 + (\partial_x B - \partial_y A - H)^2\right) dxdy \qquad (2.4)$$

The bond variables are (see, [8])

$$W(x,y) = e^{ik\int^x A(\zeta,y)d\zeta}, \quad V(x,y) = e^{ik\int^y B(x,\eta)d\eta}.$$

We note that

$$|\partial_x(W^*\Psi)| = |(\partial_x - kiA)\Psi|,$$
$$|\partial_y(V^*\Psi)| = |(\partial_y - kiB)\Psi|. \qquad (2.5)$$

Using (2.5), the functional (2.4) can be rewritten as

$$\mathcal{G}(\Psi, A, B) = \int_\Omega (-|\Psi|^2 + \frac{1}{2}|\Psi|^4)dxdy$$
$$+\int_\Omega \left(|\frac{1}{k}\partial_x(W^*\Psi)|^2 + |\frac{1}{k}\partial_y(V^*\Psi)|^2 + (\partial_x B - \partial_y A - H)^2\right) dxdy. \qquad (2.6)$$

The bond variables don't alter the boundary conditions for A or B. However, the boundary conditions for the order parameter Ψ change and become

$$(W^*\Psi)_x, \; (V^*\Psi)_y.\mathbf{n} = 0, \qquad (2.7)$$

2.2. Discrete Energy Functional

We use $(N_x+2)\times(N_y+2)$ grid points, where the points $(0,j)$, (N_x+1,j), $(i,0)$, and (i,N_y+1), for $i = 1,\ldots,N_x$, and $j = 1,\ldots,N_y$ are used as fictitious boundary points, half a mesh size off the domain Ω. The mesh size ir the x- and y- directions then becomes

$$h_x = \frac{L_x}{N_x}, \quad h_y = \frac{L_y}{N_y},$$

where L_x and L_y are the dimensions of the corresponding physical domain in the x- and y- directions, respectively.

We partition the computational domain in four different ways, giving rise to a staggered grid in Ω. This is a practice employed in [8]. The various evaluation points are shown in Figure 1. As a result of this setup, the normal component of the induced magnetic field, $H_n := B_x - A_y$, is evaluated at the center of $\Omega_{\underset{ij}{\nearrow}}$.

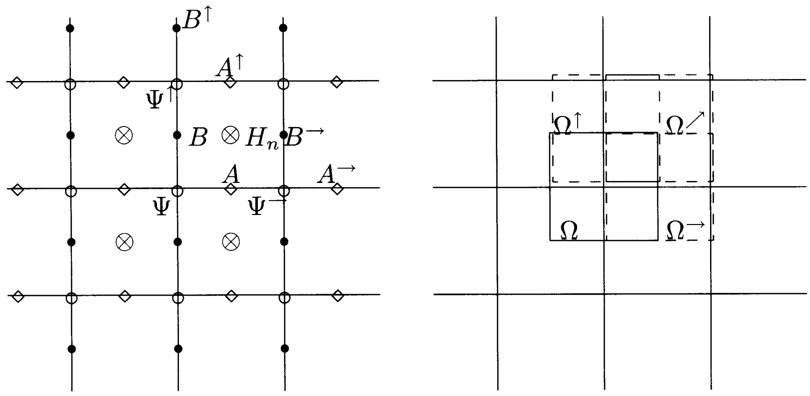

Figure 1. (Left) Evaluation points for $\Psi(\circ), A(\diamond)$, $B(\bullet)$, $H_n(\otimes)$. (Right) The domains Ω (Solid Frame), $\Omega^{\rightarrow}, \Omega^{\uparrow}, \Omega^{\nearrow}$ (Dashed Frames).

Approximating integrals in (2.6) by the value of the function at the center of the cell times the area of the cell, and the first order partial derivatives by centered differences, we obtain the following discrete form of the energy functional

$$\mathcal{G}_d(\Psi, A, B) = \sum_{grid}(-|\Psi|^2 + \frac{1}{2}|\Psi|^4)h_x h_y \tag{2.8}$$
$$+\sum_{grid}\left(\left|\frac{\Psi^{\rightarrow} - W\Psi}{kh_x}\right|^2 + \left|\frac{\Psi^{\uparrow} - V\Psi}{kh_y}\right|^2 + \left|\frac{B^{\rightarrow} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H\right|^2\right) h_x h_y.$$

The arrows are used to represent the neighboring points in the appropriate direction.

Here, we mention that (see [8]) the discrete functional (2.8) is a second-order approximation to the continuous functional (2.1) because the derivatives are approximated by the central differences and the integrals are approximated by the midpoint rule; both of these approximations are of second order.

This functional is invariant under gauge transformation discussed in Section 3.1.

3. Formulation and Properties of SDTDGL

3.1. Zero Potential Gauge

TDGL, in its general form, is not well-posed since it lacks uniqueness. Various valid gauge choices that can be used to have a well-posed problem is discussed in [5]. The original TDGL is a nonlinear system, $G(\Psi, \mathbf{A}, \Phi)$, for the complex-valued order parameter Ψ, vector-valued function $\mathbf{A}$ and scalar electric potential Φ. An important property of this system is that of gauge invariance. That is, given a function χ,

$$G(\Psi, \mathbf{A}, \Phi) = G(\bar{\Psi}, \bar{\mathbf{A}}, \bar{\Phi}),$$

where

$$\bar{\Psi} = \Psi e^{i\chi}, \qquad \bar{\mathbf{A}} = \mathbf{A} + \nabla\chi \qquad \text{and} \qquad \bar{\Phi} = \Phi - \frac{\partial \chi}{\partial t}.$$

Given a solution $(\Psi, \mathbf{A}, \Phi)$, the zero potential gauge is defined as[5]

$$\frac{\partial \chi}{\partial t} = \Phi$$

and at $t = 0, \Delta\chi = -div\mathbf{A}$ in Ω with $\nabla\chi.\mathbf{n} = -\mathbf{A}.\mathbf{n}$ on Γ, boundary of Ω. With this choice of gauge, electric potential is eliminated from the system. In this gauge SDTDGL can be viewed as a *gradient flow* with the energy functional.

3.2. SDTDGL as a Gradient Flow

The time-dependent version of the GL in zero potential gauge are analogous to that of a gradient flow in which the direction of steepest descent of a function at any point is opposite to the direction of the gradient vector at that point. In our case, as the free energy tends to a minimum, the variation of (Ψ, A, B) with respect to t should be in the opposite direction of the gradient of the energy functional. Writing the complex-valued order parameter Ψ as $\Psi = \Psi_1 + i\Psi_2$ and introducing a multiplicative factor $1/2$ for convenience, the argument above in the zero potential gauge can be formulated as

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\left(\frac{\partial \mathcal{G}_d}{\partial \Psi_1} + i\frac{\partial \mathcal{G}_d}{\partial \Psi_2}\right) = -\frac{\partial \mathcal{G}_d}{\partial \Psi^*}, \tag{3.1}$$

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\frac{\partial \mathcal{G}_d}{\partial A}, \quad \frac{\partial B}{\partial t} = -\frac{1}{2}\frac{\partial \mathcal{G}_d}{\partial B}. \tag{3.2}$$

Computing the variation of the energy functional with respect to Ψ^*, A, B, we can rewrite the above system explicitly as

$$\frac{\partial \Psi}{\partial t} = \frac{h_x h_y}{k^2}\left(\frac{e^{i\overleftarrow{A}kh_x}\overleftarrow{\Psi} - 2\Psi + e^{-iAkh_x}\overrightarrow{\Psi}}{h_x^2} + \frac{e^{iB^{\downarrow}kh_y}\Psi^{\downarrow} - 2\Psi + e^{-iBkh_y}\Psi^{\uparrow}}{h_y^2}\right) + h_x h_y N(\Psi), \tag{3.3}$$

$$\frac{\partial A}{\partial t} = -h_x\left(\frac{\vec{B} - B + B^{\downarrow} - B^{\searrow}}{h_x} - \frac{A^{\uparrow} - 2A + A^{\downarrow}}{h_y}\right) - \frac{h_y}{k}N(A, \Psi), \tag{3.4}$$

$$\frac{\partial B}{\partial t} = -h_y\left(\frac{\overleftarrow{A} - A + A^{\uparrow} - A^{\nwarrow}}{h_y} - \frac{\vec{B} - 2B + \overleftarrow{B}}{h_x}\right) - \frac{h_x}{k}N(B, \Psi), \tag{3.5}$$

where

$$N(\Psi) = (1 - |\Psi|^2)\Psi, \tag{3.6}$$
$$N(A, \Psi) = |\Psi^* \vec{\Psi}| \sin(h_x k A - arg(\Psi^* \vec{\Psi})), \tag{3.7}$$
$$N(B, \Psi) = |\Psi^* \Psi^{\uparrow}| \sin(h_y k B - arg(\Psi^* \Psi^{\uparrow})). \tag{3.8}$$

We assume that the material is initially in perfect superconducting state, with zero external magnetic field. An external field is then suddenly turned on. This assumption serves only for the purpose of providing specific bounds for the quantities of interest. Certainly, the results hold for any type of initial conditions and the bounds depend only on the initial conditions, some physical parameters, and the size of the domain. The initial conditions then become

$$\begin{aligned} \Psi_1(x, y, 0) &= 1, \\ \Psi_2(x, y, 0) &= 0, \\ A(x, y, 0) &= B(x, y, 0) = 0 \qquad \forall (x, y) \in \Omega. \end{aligned} \tag{3.9}$$

The solution corresponding to these initial conditions simulates how the magnetic field penetrates the material.

3.3. Some Results about the Solution to SDTDGL

We first note that the discrete form of the natural boundary conditions (2.2),(2.7), and (??) can be written in the following form.

$$\Psi = \Psi^{\downarrow} e^{ikh_y B^{\downarrow}}, \qquad A = A^{\downarrow} - (H - \frac{\vec{B} - B}{h_x})h_y \qquad (top), \tag{3.10}$$

$$\Psi = \Psi^{\uparrow} e^{-ikh_y B}, \quad A = A^{\uparrow} + (H - \frac{\vec{B} - B}{h_x})h_y \qquad (bottom), \tag{3.11}$$

$$\Psi = \vec{\Psi} e^{-ikh_x A}, \quad B = \vec{B} - (H + \frac{A^{\uparrow} - A}{h_y})h_x \qquad (left), \tag{3.12}$$

$$\Psi = \overleftarrow{\Psi} e^{ikh_x \overleftarrow{A}}, \qquad B = \overleftarrow{B} + (H + \frac{A^{\uparrow} - A}{h_y})h_x \qquad (right). \tag{3.13}$$

We start with the following useful lemma.

Lemma 3.1 *Let $(\Psi(t), A(t), B(t))$ be the solution of SDTDGL under a uniform magnetic field, i.e, $H = c > 0$. Then the following identities hold:*

$$\sum_{grid} (\frac{\vec{\Psi} - e^{ikh_x A}\Psi}{h_x^2}) \frac{\partial \vec{\Psi}^*}{\partial t} = \sum_{grid} (\frac{\Psi - e^{ikh_x \overleftarrow{A}} \overleftarrow{\Psi}}{h_x^2}) \frac{\partial \Psi^*}{\partial t}, \tag{3.14}$$

$$\sum_{grid}\left(\frac{\Psi^{\uparrow} - e^{ikh_yB}\Psi}{h_y^2}\right)\frac{\partial\Psi^{*\uparrow}}{\partial t} = \sum_{grid}\left(\frac{\Psi - e^{ikh_yB^{\downarrow}}\Psi^{\downarrow}}{h_x^2}\right)\frac{\partial\Psi^{*}}{\partial t}, \tag{3.15}$$

$$\sum_{grid}\left(\frac{A^{\uparrow} - A}{h_y} - \frac{\vec{B} - B}{h_x}\right)\frac{\partial A^{\uparrow}}{\partial t} = \sum_{grid}\left(\frac{A - A^{\downarrow}}{h_y} - \frac{B^{\searrow} - B^{\downarrow}}{h_x}\right)\frac{\partial A}{\partial t}, \tag{3.16}$$

$$\sum_{grid}\left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y}\right)\frac{\partial \vec{B}}{\partial t} = \sum_{grid}\left(\frac{B - \overleftarrow{B}}{h_x} - \frac{A^{\nwarrow} - \overleftarrow{A}}{h_y}\right)\frac{\partial B}{\partial t}. \tag{3.17}$$

Here we give only the proof of (3.14); the proofs of the remaining identities will follow similarly.

Proof. Using the boundary conditions (3.12) and (3.13), we proceed as follows

$$\begin{aligned}
\sum_{grid}\left(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2}\right)\frac{\partial\vec{\Psi}^{*}}{\partial t} &= \sum_{\substack{1\le i\le N_x\\ 1\le j\le N_y}}\left(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2}\right)\frac{\partial\vec{\Psi}^{*}}{\partial t}\\
&= \sum_{\substack{i=0\\ 1\le j\le N_y}}\left(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2}\right)\frac{\partial\vec{\Psi}^{*}}{\partial t} + \sum_{\substack{1\le i\le N_x\\ 1\le j\le N_y}}\left(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2}\right)\frac{\partial\vec{\Psi}^{*}}{\partial t}\\
&= \sum_{\substack{1\le i\le N_x\\ 1\le j\le N_y}}\left(\frac{\Psi - e^{ikh_x\overleftarrow{A}}\,\overleftarrow{\Psi}}{h_x^2}\right)\frac{\partial\Psi^{*}}{\partial t} + \sum_{\substack{i=N_x\\ 1\le j\le N_y}}\left(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2}\right)\frac{\partial\vec{\Psi}^{*}}{\partial t}\\
&= \sum_{\substack{1\le i\le N_x\\ 1\le j\le N_y}}\left(\frac{\Psi - e^{ikh_x\overleftarrow{A}}\,\overleftarrow{\Psi}}{h_x^2}\right)\frac{\partial\Psi^{*}}{\partial t}\\
&= \sum_{grid}\left(\frac{\Psi - e^{ikh_x\overleftarrow{A}}\,\overleftarrow{\Psi}}{h_x^2}\right)\frac{\partial\Psi^{*}}{\partial t}.
\end{aligned}$$

□

Next, we present a result concerning the time evolution of the energy functional

Theorem 3.1 *Let's consider the energy functional $\mathcal{G}_d$ (2.8) of the solution of the time-dependent GL equations, discretized with respect to the spatial variables. Then*

1. *if H is constant then $\mathcal{G}_d$ is decreasing as a function of $t \ge 0$ if and only if not all the partial derivatives $\frac{\partial\Psi}{\partial t}$, $\frac{\partial A}{\partial t}$, and $\frac{\partial B}{\partial t}$ are identically zero on the grid points of the domain Ω.*

2. *If, in particular, the initial conditions (3.9) are used, we also have* $-\frac{1}{2}L_xL_y \leq \mathcal{G}_d(t) \leq (H^2 - \frac{1}{2})L_xL_y \quad \forall t \geq 0$*, where* L_x *and* L_y *are the dimensions of the physical domain in the* x*- and* y*- directions, respectively.*

Proof. 1. $\Longrightarrow$ Assume that not all the partial derivatives in the statement of the theorem are identically zero. We now differentiate $\mathcal{G}_d$ with respect to t. A lengthy computation gives

$$
\begin{aligned}
\frac{1}{2h_xh_y}\frac{\partial \mathcal{G}_d}{\partial t} &= \sum_{grid}(-1+|\Psi|^2)\Psi\frac{\partial \Psi^*}{\partial t} \\
&+ \frac{\Re}{k^2}\left[\left((\frac{\Psi - e^{-ikh_xA}\vec{\Psi}}{h_x^2}) + (\frac{\Psi - e^{-ikh_yB}\Psi^{\uparrow}}{h_y^2})\right)\frac{\partial \Psi^*}{\partial t}\right] \\
&+ \frac{\Re}{k^2}\left[(\frac{\vec{\Psi} - e^{ikh_xA}\Psi}{h_x^2})\frac{\partial \vec{\Psi^*}}{\partial t} + (\frac{\Psi^{\uparrow} - e^{ikh_yB}\Psi}{h_y^2})\frac{\partial {\Psi^*}^{\uparrow}}{\partial t}\right] \\
&+ \frac{1}{kh_x}|\vec{\Psi}\Psi^*|\sin(h_xkA - arg(\Psi^*\vec{\Psi}))\frac{\partial A}{\partial t} \qquad (3.18)\\
&+ \frac{1}{kh_y}|\Psi^{\uparrow}\Psi^*|\sin(h_ykB - arg(\Psi^*\Psi^{\uparrow}))\frac{\partial B}{\partial t} \\
&+ \frac{1}{h_y}\left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y}\right)\frac{\partial A}{\partial t} + \frac{1}{h_x}\left(\frac{A^{\uparrow}-A}{h_y} - \frac{\vec{B}-B}{h_x}\right)\frac{\partial B}{\partial t} \\
&+ \frac{1}{h_y}\left(\frac{A^{\uparrow}-A}{h_y} - \frac{\vec{B}-B}{h_x}\right)\frac{\partial A^{\uparrow}}{\partial t} + \frac{1}{h_x}\left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y}\right)\frac{\partial \vec{B}}{\partial t}.
\end{aligned}
$$

We use the notation $\Re(c)$ to denote the real part of a complex number c. Using (3.14) - (3.17) in (3.18) we get

$$
\begin{aligned}
&\frac{\partial \mathcal{G}_d}{\partial t} = 2h_xh_y\sum_{grid}(-1+|\Psi|^2)\Psi\frac{\partial \Psi^*}{\partial t} \qquad (3.19)\\
&-\frac{\Re}{k^2}\left(((\frac{\overleftarrow{\Psi}\, e^{ikh_x\overleftarrow{A}} - 2\Psi + e^{-ikh_xA}\vec{\Psi}}{h_x^2}) + (\frac{\Psi^{\uparrow}e^{-ikh_yB} - 2\Psi + e^{ikh_yB^{\downarrow}}\Psi^{\downarrow}}{h_y^2}))\ \frac{\partial \Psi^*}{\partial t}\right) \\
&+\frac{1}{h_y}(\frac{\vec{B} - B - B^{\searrow} + B^{\downarrow}}{h_x} - \frac{A^{\uparrow} - 2A + A^{\downarrow}}{h_y})\frac{\partial A}{\partial t} \\
&+\frac{1}{kh_x}|\vec{\Psi}\Psi^*|\sin(h_xkA - arg(\Psi^*\vec{\Psi}))\frac{\partial A}{\partial t} \\
&+\frac{1}{h_x}(\frac{A^{\uparrow} - A - A^{\nwarrow} + \overleftarrow{A}}{h_y} - \frac{\vec{B} - 2B + \overleftarrow{B}}{h_x})\frac{\partial B}{\partial t}
\end{aligned}
$$

$$+\frac{1}{kh_y}|\Psi^{\uparrow}\Psi^*|\sin(h_y kB - arg(\Psi^*\Psi^{\uparrow}))\frac{\partial B}{\partial t}.$$

Finally, using (3.3), (3.4) and (3.5) in (3.19) we get

$$\frac{\partial \mathcal{G}_d}{\partial t} = -2h_x h_y \sum_{grid}\left[\left|\frac{\partial \Psi}{\partial t}\right|^2 + \left(\frac{\partial A}{\partial t}\right)^2 + \left(\frac{\partial B}{\partial t}\right)^2\right] < 0.$$

($\Longleftarrow$) : Trivial

2. This follows easily from the initial conditions

$$\Psi_1(x_i, y_j, 0) = 1,$$
$$\Psi_2(x_i, y_j, 0) = A(x_i, y_j, 0) = B(x_i, y_j, 0) = 0,$$

for $i = 1, \dots, N_x$, $\quad j = 1, \dots, N_y$ and part 1 of this theorem. □

We now state our result for the order parameter Ψ, as stated before square of its magnitude, $|\Psi|^2$, represents *local superelectron density.* This result is true for the SDTDGL and holds for all $\Omega \times (0, \infty)$ unlike its analog presented for TDGL in [5] which holds over $\Omega \times (0, T)a.e.$ for a given $T > 0$.

Theorem 3.2 *Let* $(\Psi(t), A(t), B(t))$ *be the solution of the SDTDGL. Then* $|\Psi(t)| \leq 1 \ \forall t \in R^+$. *Here* $\Psi(t) = \Psi(x_i, y_j, t), 1 \leq i \leq N_x, 1 \leq j \leq N_y$.

Proof. Assume that the conclusion of the theorem is false. Let $t_0 \in R^+$ be the first point such that $|\Psi(t_0)| > 1$, and let P be a grid point where $\Psi(t_0)$ reaches a maximum.

Since $(\Psi(t), \mathbf{A}(t))$ and $(\Psi(t)e^{i\chi}, \mathbf{A}(t) + \nabla\chi)$ determine the same electromagnetic state of the material for any differentiable function χ defined on Ω, we can choose χ that will render Ψ real at all the grid points. More precisely, following [8], we define

$$\chi = \begin{cases} -arg(\Psi(t)) & \text{if } \Psi(t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\zeta(t) \;=\; \Psi(t)e^{i\chi} = |\Psi(t)|e^{iarg(\Psi(t))}e^{-iarg(\Psi(t))} = |\Psi(t)|$$

at all the grid points. Using the assumption above and taking the real part of equation (3.3) under this gauge transformation we get

$$\begin{aligned} 0 \leq \left.\frac{\partial \zeta}{\partial t}\right|_{(P,t_0)} &= \frac{1}{k^2 h_x^2}\left(\Re(e^{iAkh_x})\overleftarrow{\zeta} - 2\zeta + \Re(e^{-iAkh_x})\vec{\zeta}\right) \\ &+ \frac{1}{k^2 h_y^2}\left(\Re(e^{iBkh_y})\zeta^{\downarrow} - 2\zeta + \Re(e^{-iBkh_y})\zeta^{\uparrow}\right) \end{aligned}$$

$$
\begin{aligned}
&+ (1-\zeta^2)\zeta \\
&\leq \frac{1}{k^2h_x^2}(\overleftarrow{\zeta} - \zeta + \vec{\zeta} - \zeta) \\
&+ \frac{1}{k^2h_y^2}(\zeta^{\downarrow} - \zeta + \zeta^{\uparrow} - \zeta) \\
&+ (1-\zeta^2)\zeta \\
&< 0,
\end{aligned}
$$

□

which leads to a contradiction.

Some bounds in l_2 norm for the observable physical quantities: electric field, supercurrent density and magnetic field are provided in Theorem 3.3. Also, a counterpart of the bound for $\nabla \times \mathbf{A}$ is presented[5] for the corresponding TDGL, but again this bound holds over a finite interval for the time variable. As far as we know, the other two bounds given in the following theorem have no counterparts for TDGL.

First we note that in zero potential gauge, the electric field $\mathbf{E}$, supercurrent density $\mathbf{J}$, and the induced magnetic field $\mathbf{H}$ are defined as

$$
\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{J} = \nabla \times (\nabla \times \mathbf{A}), \quad \mathbf{H} = \nabla \times \mathbf{A}. \tag{3.20}
$$

For the problem considered in this study, we have

$$
\mathbf{E}(t) = \left(\frac{\partial A}{\partial t}, \frac{\partial B}{\partial t}, 0\right), \tag{3.21}
$$

$$
\mathbf{J}(t) = (A_{yy} - B_{xy}, B_{xx} - A_{yx}, 0), \tag{3.22}
$$

$$
\mathbf{H}(t) = (0, 0, B_x - A_y). \tag{3.23}
$$

Also, charge density is given by $\nabla \cdot \mathbf{E}(\mathrm{t})$. First, we give the following lemma.

Lemma 3.2 *Let $A(t)$, and $B(t)$ be the x-, and y- components of the vector potential $\mathbf{A}$ in SDTDGL with constant $H > 0$ and natural boundary conditions. Then the following identities hold:*

$$
\sum_{grid}\left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H\right)^2 = \sum_{grid}\left(\frac{B^{\searrow} - B^{\downarrow}}{h_x} - \frac{A - A^{\downarrow}}{h_y} - H\right)^2, \tag{3.24}
$$

$$
\sum_{grid}\left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H\right)^2 = \sum_{grid}\left(\frac{B - B^{\leftarrow}}{h_x} - \frac{A^{\nwarrow} - A^{\leftarrow}}{h_y} - H\right)^2. \tag{3.25}
$$

Proof. We give only the proof of (3.24). The proof of (3.25) is similar. Note that in the staggered grid setup, the boundary conditions for A and B in (3.12), (3.13) can be written as

$$\frac{B(i+1,N_y)-B(i,N_y)}{h_x}-\frac{A(i,N_y)-A(i,N_y-1)}{h_y}=H, \tag{3.26}$$

$$\frac{B(i+1,0)-B(i,0)}{h_x}-\frac{A(i,1)-A(i,0)}{h_y}=H, \quad i=1,\dots,N_x. \tag{3.27}$$

Then by (3.26) and (3.27), we have

$$\begin{aligned}
&\sum_{grid}\left(\frac{\vec{B}-B}{h_x}-\frac{A^{\uparrow}-A}{h_y}-H\right)^2 \\
&=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{B(i+1,j)-B(i,j)}{h_x}-\frac{A(i,j+1)-A(i,j)}{h_y}-H\right)^2 \\
&=\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{B(i+1,j-1)-B(i,j-1)}{h_x}-\frac{A(i,j)-A(i,j-1)}{h_y}-H\right)^2 \\
&+\sum_{i=1}^{N_x}\left(\frac{B(i+1,N_y)-B(i,N_y)}{h_x}-\frac{A(i,N_y)-A(i,N_y-1)}{h_y}-H\right)^2 \\
&-\sum_{i=1}^{N_x}\left(\frac{B(i+1,0)-B(i,0)}{h_x}-\frac{A(i,1)-A(i,0)}{h_y}-H\right)^2 \\
&=\sum_{grid}\left(\frac{B^{\searrow}-B^{\downarrow}}{h_x}-\frac{A-A^{\downarrow}}{h_y}-H\right)^2 .
\end{aligned}$$

□

Theorem 3.3 *Let* $\mathbf{A}(t)=(A(t),B(t),0)$ *be the vector potential satisfying SDTDGL with the associated initial and the natural boundary conditions. Then there exist non-negative constants* c_1, c_2 *and* c_3*, depending only on the initial conditions, the parameters* k, H*, and the size of* Ω *such that*

(i) $\| \mathbf{E}(t) \|_2 \le c_1$,

(ii) $\| \mathbf{J}(t) \|_2 \le c_2$,

(iii) $\| \mathbf{H}(t) \|_2 \le c_3 \quad \forall t \ge 0.$

Proof. (i) Note that as a result of both parts of Theorem 3.1., we have

$$\sum_{grid}\left|\frac{\vec{\Psi}-W\Psi}{h_x}\right|^2 \le k^2H^2N_{xy} , \tag{3.28}$$

$$\sum_{grid} |\frac{\Psi^{\uparrow} - V\Psi}{h_y}|^2 \leq k^2 H^2 N_{xy}, \tag{3.29}$$

$$\sum_{grid} \left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H \right)^2 \leq H^2 N_{xy}, \forall t \geq 0, \tag{3.30}$$

where $N_{xy} = N_x \times N_y$. Here we point out that the square root of the expressions on the left hand side of (3.28) and (3.29) can be thought of as l_2 norms of the *generalized derivative* of Ψ with respect to x- and y-, respectively. In this sense, these inequalities by themselves are important. By using Lemma 3.2.,

$$\begin{aligned} \sum_{grid} \left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H \right)^2 &= \sum_{grid} \left(\frac{B^{\searrow} - B^{\downarrow}}{h_x} - \frac{A - A^{\downarrow}}{h_y} - H \right)^2 \\ &= \sum_{grid} \left((-1).(\frac{B^{\searrow} - B^{\downarrow}}{h_x} - \frac{A - A^{\downarrow}}{h_y} - H) \right)^2 \\ &= \sum_{grid} \left(\frac{B^{\downarrow} - B^{\searrow}}{h_x} - \frac{A^{\downarrow} - A}{h_y} + H \right)^2 \\ &\leq H^2 N_{xy} \quad \forall t \geq 0. \end{aligned} \tag{3.31}$$

Then adding up the first and the last summation in (3.31), we have

$$\sum_{grid} \left(\frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y} - H \right)^2 + \sum_{grid} \left(\frac{B^{\downarrow} - B^{\searrow}}{h_x} - \frac{A^{\downarrow} - A}{h_y} + H \right)^2 \leq 2H^2 N_{xy}, \tag{3.32}$$

$\forall t \geq 0$. On the other hand, using the inequality

$$(a+b)^2 \leq 2(a^2 + b^2) \quad \forall a, b \in R \tag{3.33}$$

we get

$$\sum_{grid} \left(\frac{\vec{B} - B + B^{\downarrow} - B^{\searrow}}{h_x} - \frac{A^{\uparrow} - 2A + A^{\downarrow}}{h_y} \right)^2 \leq 4H^2 N_{xy} \quad \forall t \geq 0. \tag{3.34}$$

Similarly, using the identity (3.25) and following the same procedures as above, we get

$$\sum_{grid} \left(\frac{\overleftarrow{A} - A + A^{\uparrow} - A^{\nwarrow}}{h_y} - \frac{\vec{B} - 2B + \overleftarrow{B}}{h_x} \right)^2 \leq 4H^2 N_{xy} \quad \forall t \geq 0. \tag{3.35}$$

Now, we square both sides of (3.4) and (3.5) and add them up over the grid points. Using the inequality (3.33) and Theorem 3.2., we have

$$\begin{aligned}\sum_{grid}\left(\frac{\partial A}{\partial t}\right)^2 &\le 2h_x^2\sum_{grid}\left(\frac{\vec{B}-B+B^{\downarrow}-B^{\searrow}}{h_x}-\frac{A^{\uparrow}-2A+A^{\downarrow}}{h_y}\right)^2\\ &+2(\frac{h_y}{k})^2\sum_{grid}\left(|\Psi^*\vec{\Psi}|\sin(h_xkA-arg(\Psi^*\vec{\Psi}))\right)^2\\ &\le 2N_{xy}(4H^2h_x^2+\frac{h_y^2}{k^2})=c_{11}\quad \forall t\ge 0.\end{aligned}$$

In particular, if $L_x=L_y=L$ and $N_x=N_y=N$ then c_{11} becomes

$$c_{11}=2L^2(4H^2+\frac{1}{k^2}). \tag{3.36}$$

Likewise, the use of (3.35) together with a similar procedure yields

$$\sum_{grid}\left(\frac{\partial B}{\partial t}\right)^2\le c_{12}, \tag{3.37}$$

where c_{12} is obtained by changing the role of h_x and h_y in c_{11} . In the particular case mentioned above, c_{12} becomes equal to c_{11}. Finally, we have

$$\| \mathbf{E}(t)\|_2=\left(\sum_{grid}(\frac{\partial A}{\partial t})^2+(\frac{\partial B}{\partial t})^2\right)^{1/2}\le c_1=(c_{11}c_{12})^{1/2}\quad\forall t\ge 0. \tag{3.38}$$

(ii) Combining (3.34) and (3.35), we have

$$\begin{aligned}\| \mathbf{J}(t)\|_2^2 &= \sum_{grid}\left[\frac{\vec{B}-B+B^{\downarrow}-B^{\searrow}}{h_x}-\frac{A^{\uparrow}-2A+A^{\downarrow}}{h_y}\right]^2\\ &+\left[\frac{\overleftarrow{A}-A+A^{\uparrow}-A^{\nwarrow}}{h_y}-\frac{\vec{B}-2B+\overleftarrow{B}}{h_x}\right]^2\le 8H^2N_{xy}\\ &\Longrightarrow\| \mathbf{J}(t)\|_2\le 2\sqrt{2}H(N_{xy})^{1/2}=c_2\quad\forall t\ge 0.\end{aligned}$$

(iii) First we observe that Theorem 3.1. (ii) together with the Hölder's inequality imply that

$$\sum_{grid}\left(\frac{\vec{B}-B}{h_x}-\frac{A^{\uparrow}-A}{h_y}-H\right)$$

$$\leq \left(\sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} - H \right)^2 \right)^{1/2} (N_{xy})^{1/2}$$
$$\leq (H^2 N_{xy})^{1/2} (N_{xy})^{1/2}$$
$$= N_{xy} H$$

or,

$$\frac{1}{N_x N_y} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} \right) \leq 2H \quad \forall t \geq 0, \tag{3.39}$$

i.e, the average magnetic field is less than or equal to twice the strength of the applied field at any given time. This, of course, is an expected result from the physical point of view.

From Theorem (ii), we have

$$\sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} - H \right)^2$$
$$= \sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} \right)^2 - 2H \sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} \right) + \sum_{grid} H^2$$
$$\leq H^2 N_{xy}.$$

This and (3.39) give

$$\sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} \right)^2 \leq 2H \sum_{grid} \left(\frac{\vec{B}-B}{h_x} - \frac{A^{\uparrow}-A}{h_y} \right) \leq 4H^2 N_{xy}$$

$$\Longrightarrow \quad \| \mathbf{H}(t) \|_2 = \| \nabla \times \mathbf{A}(t) \|_2 \leq c_3 = 2H\sqrt{N_{xy}}, \quad \forall t \geq 0.$$

□

Next, we provide a bound for the charge density and show that $\nabla \cdot \mathbf{J}(t) = 0 \quad \forall t \geq 0$ for SDTDGL. The analog of the latter for the TDGL is obviously true.

Theorem 3.4 *Under the same hypothesis as Theorem 3.3,*

(i) $\nabla \cdot \mathbf{J}(t) = 0 \quad \forall t \geq 0$.

(ii) There exists a nonnegative constant c *such that* $|\nabla \cdot \mathbf{E}(t)| \leq c \quad \forall t \geq 0$.

Proof. (i) We first note that the last two SDTDGL, (3.4) and (3.5), can be written as follows:

$$\frac{1}{h_x h_y}\frac{\partial A}{\partial t} = \frac{hh^{\downarrow} - hh}{h_y} + \frac{1}{kh_x}N(A, \Psi)\,, \tag{3.40}$$

$$\frac{1}{h_x h_y}\frac{\partial B}{\partial t} = \frac{hh - \overleftarrow{hh}}{h_x} + \frac{1}{kh_y}N(B, \Psi), \tag{3.41}$$

$$\tag{3.42}$$

where,

$$hh = \frac{\vec{B} - B}{h_x} - \frac{A^{\uparrow} - A}{h_y}. \tag{3.43}$$

Then

$$\mathbf{J}(t) = (J_1, J_2, 0) = \left(\frac{hh^{\downarrow} - hh}{h_y}, \frac{hh - \overleftarrow{hh}}{h_x}, 0\right). \tag{3.44}$$

Thus, approximating the spatial derivatives by backward differences we have

$$\nabla \cdot \mathbf{J}(t) = \frac{J_1 - \overleftarrow{J_1}}{h_x} + \frac{J_2 - J_2^{\downarrow}}{h_y} = 0, \quad \forall t \geq 0. \tag{3.45}$$

(ii) We differentiate (3.40) with respect to x and (3.41) with respect to y. Then, we approximate the spatial derivatives by backward differences and add up the resulting equations. Using part (i) of this theorem, we obtain

$$\begin{aligned} |\nabla \cdot \mathbf{E}(t)| &= \left|\frac{h_y}{k}\left(\frac{N(A, \Psi) - N(\overleftarrow{A}, \Psi)}{h_x}\right) + \frac{h_x}{k}\left(\frac{N(B, \Psi) - N(B, \Psi)^{\downarrow}}{h_y}\right)\right| \\ &\leq \frac{2}{k}\left(\frac{h_y}{h_x} + \frac{h_x}{h_y}\right) = c. \end{aligned}$$

□

Also, we note that the *average supercurrent density*, i.e,

$$\begin{aligned} \mathbf{J}(t)_{aver} &= \frac{1}{N_{xy}} \sum_{grid} \mathbf{J}(t) \\ &= \left(\frac{1}{N_{xy}} \sum J_1(t), \frac{1}{N_{xy}} \sum J_2(t), 0\right) = \mathbf{0}. \end{aligned}$$

Finally, we provide a bound in l_2 norm for the time derivative of the order parameter $\Psi(t)$. Of course, one can easily derive a bound for this quantity using only Theorem 3.2 and (3.3). But by using Theorem 3.1 (ii), Theorem 3.2, (3.3), and natural boundary conditions, we can give a smaller bound.

Theorem 3.5 *Let* $(\Psi(t), A(t), B(t))$ *be the solution of the SDTDGL. Then there exists a nonnegative constant* c *such that*

$$\| \, |\frac{\partial \Psi}{\partial t}| \, \|_2 \leq c, \quad \forall t \geq 0.$$

Proof. Let us consider the following inequality

$$\sum \left(|\frac{\vec{\Psi} - W\Psi}{kh_x}|^2 + |\frac{\Psi^{\uparrow} - V\Psi}{kh_y}|^2 \right) \leq H^2 N_{xy} \tag{3.46}$$

$$\tag{3.47}$$

and the first SDTDGL, i.e.,

$$\frac{1}{h_x h_y} \frac{\partial \Psi}{\partial t} = \frac{1}{k^2} \left(\frac{\overleftarrow{W} \overleftarrow{\Psi} - 2\Psi + W^* \vec{\Psi}}{h_x^2} + \frac{V^{\downarrow} \Psi^{\downarrow} - 2\Psi + V^* \Psi^{\uparrow}}{h_y^2} \right) + N(\Psi). \tag{3.48}$$

The inequality (3.46) and the definition of W imply that

$$\sum |\frac{\vec{\Psi} - W\Psi}{kh_x}|^2 = \sum |W^*|^2 |\frac{\vec{\Psi} - W\Psi}{kh_x}|^2 = \sum |\frac{W^* \vec{\Psi} - \Psi}{kh_x}|^2 \leq H^2 N_{xy}. \tag{3.49}$$

Now, let us consider the boundary conditions for Ψ, i.e,

$$\Psi(0,j) W(0,j) - \Psi(1,j) = 0, \tag{3.50}$$

$$\Psi(N_x + 1, j) - \Psi(Nx, j) W(N_x, j) = 0, \tag{3.51}$$

$$\Psi(i,0) V(i,0) - \Psi(i,1) = 0, \tag{3.52}$$

$$\Psi(i, N_y + 1) - \Psi(i, N_y) V(i, N_y) = 0, \tag{3.53}$$

for $i = 1 \ldots N_x, \quad j = 1 \ldots N_y$.

By using (3.50) and (3.51), we have

$$\sum_{grid} |\frac{\vec{\Psi} - W\Psi}{kh_x}|^2 = \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} |\frac{\Psi(i+1,j) - W(i,j)\Psi(i,j)}{kh_x}|^2$$

$$= \sum_{j=1}^{N_y} \sum_{i=1}^{N_x} |\frac{\Psi(i,j) - W(i-1,j)\Psi(i-1,j)}{kh_x}|^2$$

$$+ \sum_{\substack{j=1 \\ i=N_x}}^{N_y} |\frac{\Psi(i+1,j) - W(i,j)\Psi(i,j)}{kh_x}|^2 - \sum_{\substack{j=1 \\ i=0}}^{N_y} |\frac{\Psi(i+1,j) - W(i,j)\Psi(i,j)}{kh_x}|^2$$

$$= \sum_{grid} |\frac{\Psi - \overleftarrow{W} \overleftarrow{\Psi}}{kh_x}|^2. \tag{3.54}$$

Then by using (3.33), (3.49), and (3.54) we have

$$\sum_{grid} |\frac{\overleftarrow{W}\overleftarrow{\Psi} - 2\Psi + W^*\vec{\Psi}}{kh_x}|^2 \leq 2\sum_{grid}\left(|\frac{W^*\vec{\Psi} - \Psi}{kh_x}|^2 + |\frac{-\Psi + \overleftarrow{W}\overleftarrow{\Psi}}{kh_x}|^2\right) \leq 4H^2 N_{xy}. \quad (3.55)$$

In a similar way, from (3.52), (3.53), and (3.33), we have

$$\sum_{grid}\left|\frac{V^{\downarrow}\Psi^{\downarrow} - 2\Psi + V^*\Psi^{\uparrow}}{kh_y}\right|^2 \leq 4H^2 N_{xy}. \quad (3.56)$$

Now, we take a close look at the equation (3.48). By Theorem 3.2, the nonlinear term,

$$N(\Psi) = (1 - |\Psi|^2)\Psi \quad (3.57)$$

is bounded in absolute value for all $t \geq 0$, more specifically

$$|N(\Psi)| \leq \frac{2}{3\sqrt{3}} \quad \forall t \geq 0. \quad (3.58)$$

Then using (3.58), (3.48), and the inequality

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2) \quad for \quad a, b, c \in R \quad (3.59)$$

we have

$$\left|\frac{\partial\Psi}{\partial t}\right|^2 \leq \frac{3}{k^2}\left(h_y^2\left|\frac{\overleftarrow{W}\overleftarrow{\Psi} - 2\Psi + W^*\vec{\Psi}}{kh_x}\right|^2 + h_x^2\left|\frac{V^{\downarrow}\Psi^{\downarrow} - 2\Psi + V^*\Psi^{\uparrow}}{kh_y}\right|^2\right) + \frac{4}{9}(h_x h_y)^2. \quad (3.60)$$

Summing up (3.60) over the grid points and using (3.55) and (3.56), we have

$$\sum_{grid}\left|\frac{\partial\Psi}{\partial t}\right|^2 \leq N_{xy}\left(\frac{12H^2}{k^2}(h_x^2 + h_y^2) + \frac{4}{9}(h_x h_y)^2\right). \quad (3.61)$$

In particular, if $L_x = L_y = L$ and $N_x = N_y = N$, so that $h_x = h_y := \Delta$, then the inequality (3.61) after taking square roots of both sides reduces to

$$\|\left|\frac{\partial\Psi}{\partial t}\right|\|_2 \leq 2L\sqrt{\frac{6H^2}{k^2} + \frac{\Delta^2}{9}} \quad \forall t \geq 0. \quad (3.62)$$

□

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ZAMANA BAĞIMLI YARI-AYRIK GINZBURG-LANDAU DENKLEMLERİNİN ÇÖZÜMÜ İÇİN BAZI SINIRLAR

Özet

Bu çalışmada düzgün bir magnetik alan etkisi altında bulunan ince film şeklindeki bir süperiletkeni modelleyen iki boyutlu zamana bağımlı Ginzburg-Landau denklem sistemi incelenmektedir. Bu sistem, bağ değiskenleri ve çapraz ızgara tekniği ile yer değişkenleri için ayrıklaştırılmış ve daha sonra yarı ayrık Helmholtz enerji fonksiyoneli ile SDTDGL sisteminin zamana göre değişimi incelenerek, süperakım, süperelektron ve yük yoğunluğu ile magnetik alan ve elektrik alanı gibi gözlenebilir fiziksel nicelikler için sınırlar elde edilmiştir.

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