On Univalent Functions with Three Preassigend Values

Y. AVCI

E. ZLOTKIEWICZ

Recommended Citation

AVCI, Y. and ZLOTKIEWICZ, E. (1997) "On Univalent Functions with Three Preassigend Values," *Turkish Journal of Mathematics*: Vol. 21: No. 5, Article 2. Available at: https://journals.tubitak.gov.tr/math/vol21/iss5/2

ON UNIVALENT FUNCTIONS WITH THREE PREASSIGEND VALUES*

Y. Avcı & E. Zlotkiewicz

Abstract

In this paper, univalent functions with three preassigend values are studied. The sharp bounds for the first coefficient are obtained. Moreover, the coefficient problem for a subclass of such functions is completely solved.

1. Introductory Remarks.

We initiate a study of functions that are univalent in the unit disc $D = \{z \in \mathbb{C} :| z |< 1\}$ and that satisfy there the conditions

$$f(0) = 0,\ f(a) = a,\ f(-a) = -a. \tag{1}$$

Let us denote the set of all such functions by $T(a)$. The class $T(a)$ is normal and compact in the topology of almost uniform convergence in D. It follows that the set of extreme points of $T(a)$ is non-empty. It is known [3, p.313 and 6, p.93] a function $f(z)$ is an exterme point of the class $S(i.e.$ the class of all univalent functions $f(z) = z + a_2 z^2 + \cdots)$ then necessarily the function $f(z)/z$ is univalent in D. In view of (1), we conclude that the sets of extreme points of $T(a)$ and S are disjoint. For the same reason, the extreme points of $T(a)$ do not have the monotonic modulus property [2]. It follows that solutions of linear extremal problems in $T(a)$ or in its compatc subclasses, essentially differ from those in S. This situation which is both a blessing and a curse makes the study of extremal problems for $T(a)$ difficult and challenging.

We conclude these remarks with an observation that the class $T(a)$ is much wider than the class of all odd univalent functions normalized by the conditions $f(0) = 0, f(a) = a$. To see this, one can easily verify that the functions given by the formulas

i) $\qquad f(z) = z + \rho(a^2 z^2 - z^4),\ (2a^2 + 4)\rho \le 1,$

ii) $\qquad g(z) = z + \varepsilon z \dfrac{z^2 - a^2}{(z - 2)(z - 3)},\ \varepsilon$ small enough,

---

are in $T(a)$. There are not any simple relationships between $T(a)$ and known classes of functions.

2. A Subclass of $T(a)$. Due to Alexander [1,5], it is known that if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k , z \in D$, satisfies the condition $\sum_{k=2}^{\infty} k \mid a_k \mid \le 1$ then f is univalent and $f(D)$ is a starlike domain. Such functions played a key role in the Ruscheweyh theory of neighborhoods [8]. We shall be now concerned with a subclass of $T(a)$ analogous to that of Alexander.

Let $A(a)$ denote the class of functions $f(z) = a_1 z + ...$ analytic in D and satisfying the conditions

$$(1°) \qquad \qquad \sum_{k=2}^{\infty} k \mid a_k \mid \le \mid a_1 \mid$$

$$(2°) \qquad \qquad f(a) = a , f(-a) = -a , 0 < a < 1.$$

In terms of the coefficients, conditions $(2°)$ can be written as

$$(3°) \qquad \qquad \sum_{k=1}^{\infty} a^{2k} a_{2k} = 0 , \qquad \sum_{k=1}^{\infty} a^{2k-2} a_{2k-1} = 1.$$

We want to solve some extremal problems in this class. Our solutions, although elementary, are not straightforward.

Theorem 1. *For f in $A(a)$. there holds*

$$\mid a_{2k-1} \mid \le \frac{1}{2k - 1 - a^{2k-2}} , k = 2, 3.... \qquad (2)$$

$$\mid a_{2k} \mid \le \frac{1}{2k + 2a^{2k-2}} , k = 2, 3, ... \qquad (3)$$

$$\frac{3}{3 + a^2} \le \mid a_1 \mid \le \frac{3}{3 - a^2} , \mid a_2 \mid \le \frac{a^2}{2(2 + a^2)}. \qquad (4)$$

All bounds are rendered sharp by functions

$$f_k(z) = \frac{2k + 1}{2k + 1 - a^{2k}} z - \frac{1}{2k + 1 - a^{2k}} z^{2k+1} , k \ge 1$$

$$g_k(z) = z - \frac{a^{2k-2}}{2(k + a^{2k-2})} z^2 + \frac{1}{2(k + a^{2k-2})} z^{2k} , k = 2, 3...$$

$$h_1(z) = \frac{3}{3 \mp a^2} z \mp \frac{1}{3 \mp a^2} z^3 , g_1(z) = z + \frac{a^2}{2(2 + a^2)} z^2 - \frac{1}{2(2 + a^2)} z^4$$

respectively

Proof. We begin with showing (2). Conditions (1°) and (3°) yield the inequality

$$\sum_{k=1}^{\infty} 2k \mid a_{2k} \mid \,\leq\mid 1 - \sum_{k=2}^{\infty} a^{2k-2} a_{2k-1} \mid - \sum_{k=2}^{\infty} (2k-1) \mid a_{2k-1} \mid$$

which leads to the condition

$$\sum_{k=1}^{\infty} 2k \mid a_{2k} \mid + \sum_{k=2}^{\infty} (2k-1-a^{2k-2}) \mid a_{2k-1} \mid \,\leq 1.$$

Since all terms are non-negative the result (2) follows.

The preceding inequality implies the condition

$$\sum_{k=1}^{\infty} 2k \mid a_{2k} \mid \,\leq 1 \tag{5}$$

with equality only if all odd coefficients, but a_1 of $f \in A(a)$ vanish. First, assume that there are exactly two, a_{2k}, a_{2l} say, even coefficients different from zero. Then (5), (3°) yield $(k < l)$:

$$\mid a_{2l} \mid \,\leq \frac{a^{2k}}{2ka^{2l} + 2la^{2k}} \leq \frac{a^2}{2a^{2l} + 2la^2}\,, \mid a_2 \mid \,\leq \frac{a^2}{2(2+a^2)}\,,$$

with equality for $k = 1$ and $l \geq 2$, respectively. Let us consider now the general case:

Let $l > 1$ be fixed. Since the class $A(a)$ is compact there exists a function $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} + \sum_{k=1}^{\infty} a_{2k} z^{2k}$ which renders sup $\mid a_{2l} \mid$. But functions $f_\theta(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} + e^{i\theta} = \sum_{k=1}^{\infty} a_{2k} z^{2k}$, $\theta-$ real, are also in $A(a)$, so there is no loss of generality in assuming $a_{2l} > 0$ for extremal functions.

In view of (3) and (5) we have $(k \neq l)$:

$$\sum_{k=1}^{\infty} 2k \mid a_{2k} \mid \,\leq 1 - 2la_{2l}\,,$$

and

$$\sum_{k=1}^{\infty} a^{2k-2} a_{2k} = -a^{2l-2} a_{2l}.$$

It follows that $(k \neq l)$:

$$\sum_{k=1}^{\infty} 2k \mid a_{2k} \mid + \sum_{k=1}^{\infty} 2a^{2k-2} a_{2k} \leq 1 - 2la_{2l} - a^{2l-2} 2a_{2l}.$$

For extremal functions there holds $a_{2l} \geq \frac{1}{2(l+a^{2l-2})}$ and one arrives at the condition $(k \neq l)$:

$$\sum_{k=2}^{\infty} (2k - 2a^{2k-2}) \mid a_{2k} \mid \leq \sum_{k=1}^{\infty} 2k \mid a_{2k} \mid + \sum_{k=1}^{\infty} 2a^{2k-2} a_{2k} \leq 0$$

this is possible only if $a_{2k} = 0$, $k \geq 2$, $k \neq l$. We conclude, that extremal functions of a_{2l} have all coefficients with even subscripts but a_2, a_{2l} equal zero. It takes us to the case we have already solved. Hence, the inequalities (3) follow.

The above argument is not applicable in case $l = 1$. Here the result follows with a trick suggested by considerations in the first step.

From (3) and (5) we obtain $(a_2 > 0)$:

$$2a^4 a_2 \leq a^4 - 4 \mid a^4 a_4 \mid - \sum_{k=3}^{\infty} 2ka^4 \mid a_{2k} \mid,$$

$$-a^4 a_4 = a^2 a_2 + \sum_{k=3}^{\infty} a^{2k} a_{2k}.$$

Now substitiution followed by an application of the triangle inequality yield the inequality

$$2a_2(a^2 + 2)a^2 \leq a^4 - a^4 \sum_{k=3}^{\infty} (2k - a^{2k-4}) \mid a_{2k} \mid$$

Since the last term on the right hand side is non-positive, the result follows. All coefficients a_{2k}, $k \geq 3$, vanish for the extremal case. This proves the inequality $\mid a_2 \mid \leq \frac{a^2}{2(2+a^2)}$ in (4).

It remains to justify the bounds for $\mid a_1 \mid$ in (4). It is evident that if $h(z) = z + b_2 z^2 + ...$ is analytic in D and satisfies the conditions

$$\sum_{k=1}^{\infty} k \mid b_k \mid \leq 1 \text{ and } h(a) + h(-a) = 0 \tag{6}$$

for a fixed a, $0 < a < 1$ then the function

$$f(z) = \frac{ah(z)}{h(a)} = a_1 z + ...$$

is in $A(a)$ and $a_1 = a/h(a)$. Now ,

$$\left| \frac{h(a)}{a} \right| \leq 1 + \sum_{k=1}^{\infty} a^{2k} \mid b_{2k+1} \mid \leq 1 + \frac{a^2}{3} \sum_{k=1}^{\infty} (2k+1) \mid b_{2k+1} \mid \leq 1 + \frac{a^2}{3},$$

and similarly

$$1 - \frac{a^2}{3} \leq \left| \frac{h(a)}{a} \right|.$$

This answers (4).

The functions $f_k(z), g_k(z)$ and $h_1(z)$ being extremal functions for the supremum of a linear functional over $A(a)$ are of course support points of $A(a)$. They are probably extreme points also. The class $A(a)$, unlike that defined by the Alexander condition only, is not a convex class. Its extreme points must satisfy the condition $\sum_{k=2}^{\infty} k \mid a_k \mid = \mid a_1 \mid$, but what are the forms of such functions remains an open problem.

We conclude this section by determining the Koebe disc for this class. $\qquad \square$

Theorem 2. *If $f \in A(a)$ then*

$$\bigcap_{f \in A(a)} f(D) \supset \{\omega : |\omega| < \frac{2}{3 + a^2}\}.$$

The constant $2/(3 + a^2)$ is best possible.

Proof. Let again $h(z) = z + b_2 z^2 + \dots$ belong to the class of functions that satisfy the conditions (6). Since the function $ah(z)/h(a)$ is in $A(a)$, it is sufficient to determine $\inf_{|z|=1} |h(z)|$. If h is in this class and

$$h(z) = z + \sum_{k=2}^{\infty} b_{2k-1} z^{2k-1} + \sum_{k=1}^{\infty} b_{2k} z^{2k}$$

so is

$$h_\theta(z) = z + \sum_{k=2}^{\infty} b_{2k-1} z^{2k-1} + e^{i\theta} \sum_{k=1}^{\infty} b_{2k} z^{2k}$$

for each real θ. Suppose that $\min_{|z|=1} |h(z)| = b > 0$ is rendered by a function for which $\sum_{k=1}^{\infty} b_{2k} e^{2ik\alpha} \neq 0$ for a given $\alpha, \alpha \in [0, 2\pi)$. Then for a properly chosen θ one gets $(|z| = 1)$:

19

$$|h_\theta(z)| \; \geq \; \left|1 + e^{i\theta}\sum_{k=1}^{\infty}b_{2k}z^{2ki\alpha}\right| - \sum_{k=2}^{\infty}|b_{2k+1}|$$

$$\geq 1 - \frac{1}{3} = \frac{2}{3}.$$

Ultimately for $f \in A(a)$

$$|f(z)| \geq \frac{2}{3}|a_1| \geq \frac{2}{3+a^2}\,,$$

the constant is rendered by $f(z) = \frac{3z}{3+a^2} + \frac{z^3}{3+a^2}$. The so-called Koebe set $\bigcap\limits_{f \in A(a)} f(D)$ contains also certain discs $|\omega \mp a| < R(a)$. But we have no information about the constant $R(a)$ or about the maximal set corvered by $f(D)$. □

3. The Class $T(a)$ We shall give here few results concerning extremal problems in the class $T(a)$. As it was noticed $T(a)$ is a compact class in the topology of uniform convergence on compact subsets, so extremal problems can be attacked by variational methods. In this direction it is important to know that the domain $f(D)$ for an extremal function f is dense in the whole plane. A result of this sort may be obtained by making use of variations given by the following

Lemma. *Suppose there exist a disc* $|\omega - \omega_0| \leq \rho\,, \rho > 0$ *cantained in the set* $\mathbb{C} \setminus f(D)$ *for a given* f *in* $T(a)$. *Then if* $\varepsilon > 0$ *is small enough and for every fixed complex number* A *the function*

$$f_\varepsilon(z) = f(z) + \varepsilon A f(z)\frac{f^2(z) - a^2}{(f(z) - \omega_0)^2}$$

is in $T(a)$.

Proof. The normalization condition for $f_\varepsilon(z)$ is obviously satisfied so it remains to show univalence. For, the function

$$g(\omega) = \omega + \varepsilon A\omega\frac{\omega^2 - a^2}{(\omega - \omega_0)^2}$$

is univalent in the domain exterior to the disc $|\omega - \omega_0| < \rho$, provided ε is small enough. Now, if f satisfies the assumption, then $f_\varepsilon = g o f$ is in $T(a)$.

Next we prove □

Theorem 3. *If $f(z) = A_1 z + A_2 z^2 + ...$ belongs to $T(a)$. then*

$$1 - a^2 \leq |A_1| \leq 1 + a^2$$

The results is best possible.

Proof. Let g be a univalent function in D taking zero to zero, then

$$f(z) = \frac{\frac{a}{2}\left\{\frac{1}{g(a)} - \frac{1}{g(-a)}\right\}}{\frac{1}{g(z)} - \frac{1}{2}\left\{\frac{1}{g(a)} + \frac{1}{g(-a)}\right\}} = A_1 z + \cdots$$

$\square$

is univalent, possibly meromorphic, and fulfills the conditions (1). The class of all such transformations contains $T(a)$. Now

$$A_1 = \frac{a}{2}\left\{\frac{1}{g(a)} - \frac{1}{g(-a)}\right\},$$

and the bounds for this quantity are known (see, Corollaries 6.5 and 6.6 in Jenkins [7], for example). It gives

$$1 - a^2 \leq |A_1| \leq 1 + a^2$$

and the rusult follows since the equalities are rendered by functions $(1 \mp a^2)z/(1 \mp z^2)$, respectively.

Remark. In Glousin's book [4,p.136], it is written that the variability region of the quantity $\log\left[\left(\frac{1}{f(z_1)} - \frac{1}{f(z_2)}\right) / \left(\frac{1}{z_1} - \frac{1}{z_2}\right)\right]$ is a disc whose center and the radius can not be expressed in terms of elementray functions. In the case $z_1 = -z_2 = a$ this disc has the diameter $[log(1-a^2)^{-1}, log(1+a^2)]$. It follows, that the variability region of A_1 in $T(a)$ is precisely this disc, since as one can prove, functions that correspond to the boundary of the disc are odd functions in $T(a)$.

Corollary. *Let $K(a)$ denote the shield like set defined by inequalities*

$$\frac{(1-a)^2}{1+a^2} \leq \left|1 \mp \frac{a}{\omega}\right| \leq \frac{1+a}{1-a}$$

then

$$\mathbb{C} \setminus \bigcap_{f \in T(a)} f(D) \supset K(a).$$

21

In particular, every ommited value $\omega, \omega \neq f(z), f \in T(a)$ satisfies the inequalities

$$|\omega| > \frac{\sqrt{a}(1-a)}{2} \quad \text{and} \quad |\omega \mp a| > \frac{\sqrt{a}(1-a)^3}{2(1+a^2)}.$$

Proof. Let $f \in T(a)$ and let $\omega \neq f(z)\, z \in D$ then the function

$$h(z) = \frac{1}{A_1} \frac{f(z)}{1 - \frac{f(z)}{\omega}}$$

is in the class S, so by the destortion theorem

$$\frac{|z|}{(1+|z|)^2} \leq |h(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Now, the result follows by the applying this at $z = \pm a$ and by making use of the theorem.

$\square$

References

[1] Alexander, J.W.: Functions which map the interior of the unit circle upon simple regions, Ann of Math., **17**. 12-22 (1915).

[2] Avci, Y.: Further results on the univalent functions with the monotonic modulus property, Ann. Polonici Math., **53**. 233-236 (1991).

[3] Duren, P.: Univalent Functions, New York-Berlin, Springer Verlag 1983.

[4] Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable, Translation of Mathematical Monographs Vol. 26, Providence, Amer.Soc. 1969.

[5] Goodman, A.W.: Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc. **8.** 598-601 (1957).

[6] Goodman, A.W.: Univalent Functions, Vol. I and II, Tampa, Florida, Mariner Publishing Company 1983.

[7] Jenkins, J.A.: Univalent Functions And Conformal Mapping, Berlin, Springer Verlag 1958.

[8] Ruscheweyh, St.: Neighborhoods of univalent functions, Proc. Amer. Math. Soc. **81**. 521-527 (1981).

BELİRLENMİŞ ÜÇ DEĞERİ ALAN YALINKAT FONKSİYONLAR ÜZERİNE.

Özet

Bu makalede, üç noktayı sabit bırakan yalınkat fonksiyonlar incelenmiş ve ilk katsayının kesin sınırları elde edilmiştir. Bundan başka, bir alt sınıf için katsayı problemi tamamen çözülmüştür.

Yusuf AVCI
Department of Mathematics
Istanbul University
34459 Vezneciler, Istanbul-TURKEY

Eligiusz ZLOTKIEWICZ
Department of Mathematics
Univ. Marie Curie-Sklodowska
20-031 Lublin-POLAND