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# Complex Conjugation Equivariant Topology of Complex Surfaces

*Sergey Finashin*

## 1. Introduction

### 1.1. Quotients of complex surfaces by complex conjugation

A series of remarkable discoveries in four-dimensional topology due to gauge-theoretical methods and the magic Seiberg–Witten invariants changed completely the face of the subject and raised many new interesting questions. However, its most famous and fundamental problem, Smooth Poincaré Conjecture in dimension 4, seems to be far from resolution yet. In a broad sense, the difficulty is to deal with the closed simply connected 4-manifolds whose Seiberg–Witten invariants *vanish*, for instance if  $b_2^\pm$  are even or the adjunction formula is violated. It is interesting to discover exotic smooth structures on such manifolds, and no less important to develop the methods for constructing diffeomorphisms. There are particular examples which may potentially bring a new type of exotic structures and therefore look challenging to study; we consider here one class of such examples, the quotients of complex algebraic surfaces defined over  $\mathbb{R}$  by the complex conjugation.

By a real variety (real curve, real surface etc.) we mean a pair,  $(X, \text{conj}_X)$ , where  $X$  is a complex analytic variety and  $\text{conj}_X : X \rightarrow X$  an anti-holomorphic involution called *the real structure* or *the complex conjugation*. Given a non-singular algebraic variety over  $\mathbb{R}$ , we consider the set of its complex points with the natural complex conjugation as the corresponding real variety. The fixed point set of  $\text{conj}_X$  is called *the real part* of  $X$  and denoted by  $X_{\mathbb{R}}$ . Put  $Y = X/\text{conj}_X$  and identify in the notation  $X_{\mathbb{R}}$  with its image  $q(X_{\mathbb{R}})$  under the quotient map  $q : X \rightarrow Y$ ;  $Y$  is a closed 4-manifold, the map  $q$  is a 2-fold covering branched along  $X_{\mathbb{R}}$ , thus  $Y$  inherits from  $X$  an orientation and a smooth structure making  $q$  smooth and orientation-preserving.

There are no examples known to the author in which  $Y$  has an exotic differential structure. In the simplest examples  $Y$  either is not simply connected or splits into a connected sum of copies of  $\mathbb{C}P^2$ ,  $\overline{\mathbb{C}P}^2$  and  $S^2 \times S^2$ . If such a splitting exists ( $Y \cong S^4$  is also allowed as a sum of 0 copies), then  $X$  will be called CDQ-surface (Completely Decomposable Quotient). Note that  $Y$  is not simply connected if  $X_{\mathbb{R}} = \emptyset$ , and is simply connected provided  $X$  is simply connected and  $X_{\mathbb{R}} \neq \emptyset$ .

The existence of non CDQ algebraic surfaces with simply connected quotients was a problem mentioned for instance in the recent Kirby's list [12], Problem 4.104. Examples answering it were constructed in [10]; slightly earlier R. Gompf used a quite different construction to give such examples (with  $Y$  being spin and having non-zero signature) for the "softer" symplectic version of this problem.

## 1.2. History of the subject

It can be easily seen and has long been known as a folklore that  $\mathbf{CP}^2/\text{conj}$  splits into a union of two balls (V. Rokhlin mentioned that L. Pontrjagin told him about it at the end of the 30's). This implies that  $\mathbf{CP}^2/\text{conj}$  is homeomorphic to  $S^4$ , and since these balls are readily seen to be smooth, Cerf's theorem implies that is diffeomorphic to  $S^4$ . The beauty and fundamental importance of this fact attracted the attention of remarkable mathematicians, and various interesting versions of its proof appeared [13], [19] and still appear (cf. [3]) in literature (for the application of this homeomorphism in the topology of real algebraic curves see [2]).

A similar interest was not expressed towards the other surfaces and only 10 years after [13], [19], the topology of  $Y$  for quadric and cubic surfaces  $X \subset \mathbf{CP}^3$  was described in [16].  $Y$  was shown to be diffeomorphic to  $S^4$  or to a connected sum of  $\overline{\mathbf{CP}}^2$ , unless  $X$  is a quadric with  $X_{\mathbb{R}} = \emptyset$ . The topology of quotients  $Y$  attracted more attention after [9], where the effect for  $Y$  of logarithmic transforms on  $X$  was studied and  $X_{\mathbb{R}} \subset Y$  for Dolgachev surfaces  $X$  were shown to be exotic knottings in  $S^4$ .

S. Donaldson [5] noticed that the quotients  $Y$  for real K3 surfaces  $X$  (in particular for quartics in  $\mathbf{CP}^3$ ) are diffeomorphic to rational surfaces (i.e., to  $\mathbf{CP}^2 \#_n \overline{\mathbf{CP}}^2$  or  $S^2 \times S^2$ ) provided  $X_{\mathbb{R}} \neq \emptyset$ , and thus  $X$  are CDQ. In fact, he noticed that by varying the complex structure on a K3 surface  $X$  one can make the involution  $\text{conj}$  *holomorphic*, which supplies  $Y$  with a complex structure and makes  $Y$  an actual rational surface. This fancy idea reminds the arguments of Hitchin [11] in the case  $X_{\mathbb{R}} = \emptyset$  and is based on Yau's solution to the Calabi conjecture, which yields the existence of a  $\text{conj}$ -invariant Einstein metric on  $X$ . Donaldson suggested in [5] the problem to find out if the quotients  $Y$  for real surfaces  $X$  can produce new examples of exotic 4-manifolds.

Next, S. Akbulut [1] found a family of CDQ double planes,  $X \rightarrow \mathbf{CP}^2$ , and thus gave examples of CDQ surfaces of a general type. The branching loci in Akbulut's examples were real curves of degree  $2n$  with real part  $A_{\mathbb{R}}$  consisting of  $n$  ovals ordered by inclusion in  $\mathbf{RP}^2$ . (This construction gave some more CDQ examples, since the modifications of  $Y$  after deformations passing a node were known [14].) He also proved, together with G. Mikhalkin and S. Wang, that Seiberg–Witten invariants for quotients  $Y$  vanish in many cases and conjectured that such a vanishing is a general property. In fact, if  $X_{\mathbb{R}}$  contains an oriented component of genus  $\geq 2$ , then the vanishing of Seiberg–Witten invariants for  $Y$  is an immediate corollary of the adjunction formula. In some other cases vanishing is not so trivial a result: for instance, S. Wang proved it for all  $(X, \text{conj}_X)$  with  $X_{\mathbb{R}} = \emptyset$ , which contrasts with the irreducibility of  $Y$  in this case.

The author has modified Akbulut's approach to prove CDQ property for double planes whose branching loci are obtained from  $2m$ ,  $m \geq 1$ , real lines in generic position by a small perturbation [6]. Similarly, double quadrics branched along real curves of bidegree  $(2m, 2n)$ ,  $m, n \geq 1$ , obtained by a perturbation of a union of generatrices are CDQ. Moreover, for any CDQ surface  $X$  and a real even pencil on  $X$  with a null-dimensional base point set, the double covering over  $X$  branched along a properly chosen divisor from this pencil is also CDQ [6]. For two families of surfaces  $X$  (in addition to K3 surfaces) CDQ property was set up for *all* choices of real structures,  $\text{conj}_X$ , with simply connected  $Y$ : for Rational and Enriques surfaces [7]. For Enriques surfaces, the proof uses a proper modification of Hitchin–Donaldson trick, a variation of the complex structure on the covering K3 (such modification was found independently in [4]).

### 1.3. The results

In this paper, following closely the line of [8], I prove that the real complete intersections in  $\mathbf{CP}^{n+2}$  of an arbitrary multi-degree  $(d_1, \dots, d_n)$  constructed in the most natural way (recursively by the method of a small perturbation of transversally intersecting surfaces) are CDQ. This yields the existence of CDQ surfaces among complete intersections of an arbitrary multi-degree. Note for instance that the existence of CDQ hyper-surfaces in  $\mathbf{CP}^3$  was a problem beginning from degree 5.

The proof of Equivariant Deformation Theorem in the last section, may also be interesting for its own sake (it was only sketched in [8]).

### 1.4. Outline of the proof

The idea of Theorem 2.3 is inspired by the almost complete decomposability theorem [18] for complete intersections. We start with the known case  $X = \mathbf{CP}^2$  and proceed by induction, applying Small Perturbation Theorem 2.1 for the induction step. It can be summarized as follows.

Assume  $X_0^{(1)}$  and  $X_0^{(2)}$  are nonsingular hyper-surfaces in a real 3-fold  $(V, \text{conj}_V)$  which are real and intersect each other transversally along  $A = X_0^{(1)} \cap X_0^{(2)}$ . Assume that  $A$  is connected,  $A_{\mathbb{R}} \neq \emptyset$ ,  $X_0^{(1)}$ ,  $X_0^{(2)}$  are CDQ and the divisor  $X_0 = X_0^{(1)} \cup X_0^{(2)}$  can be regularly perturbed. Then surface  $X_\varepsilon \subset V$  obtained by a small real perturbation of  $X_0$  is also CDQ.

To prove this we blow up  $V$  along the curves  $X_0^{(i)} \cap X_\varepsilon$ ,  $i = 1, 2$  to get a fibered real 3-fold  $W$  and then use the Equivariant Deformation Theorem (Theorem 3.1) to deduce that the quotient  $Y_\varepsilon$  of  $X_\varepsilon$  is an amalgamated connected sum of the quotients  $Y^{(i)}$  of  $X_0^{(i)}$ . Then by virtue of the Laudénbach–Poénaru theorem, such a connected sum is diffeomorphic to  $Y^{(1)} \# Y^{(2)} \# R$ , where either  $R = \#_k(S^2 \times S^2)$  or  $R = \#_k(\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2)$ , which implies that  $X_\varepsilon$  is CDQ if  $X_0^{(i)}$  are CDQ (such a usage of Laudénbach–Poénaru theorem may remind the usage of Cerf's theorem in the proof of  $\mathbf{CP}^2/\text{conj} \cong S^4$ .)

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## 2. CDQ Property for Complete Intersections

### 2.1. CDQ-bundles and Small Perturbation Theorem

Let  $(V, \text{conj}_V)$  be a real variety. A holomorphic line bundle  $p : L \rightarrow V$  will be called a *real bundle* if it is supplied with an anti-linear involution  $\text{conj}_L : L \rightarrow L$ , which commutes with  $\text{conj}_V$  and  $p$ . A section  $f : V \rightarrow L$  is called real if  $\text{conj}_L \circ f = f \circ \text{conj}_V$ . The zero divisor of a real section  $f$  is a real subvariety of  $V$  (possibly singular).

Assume now that  $(V, \text{conj}_V)$  is a real nonsingular 3-fold,  $L_i \rightarrow V$ ,  $i = 1, 2$ , are real linear bundles and  $f_i : V \rightarrow L_i$  are real sections whose zero divisors  $X_0^{(i)}$  are nonsingular and intersect each other transversally. Assume further that  $f : V \rightarrow L$  is a real section of  $L = L_1 \otimes L_2$  with the zero divisor  $X$  intersecting transversally surfaces  $X_0^{(i)}$ ,  $i = 1, 2$ , and curve  $A = X_0^{(1)} \cap X_0^{(2)}$ . Consider the section  $f_\varepsilon : V \rightarrow L_0$ ,  $f_\varepsilon = f_1 \otimes f_2 + \varepsilon f$ ,  $\varepsilon \in \mathbb{R}$ , and denote by  $X_\varepsilon$  its zero divisor, which is nonsingular for a sufficiently small  $\varepsilon \neq 0$ , as can easily be seen.

We discuss the proof of the following Small Perturbation Theorem in section 2.3 and use it first to deduce our main result, Theorem 2.3.

**Theorem 2.1.** *Suppose  $X_0^{(1)}$  and  $X_0^{(2)}$  are CDQ-surfaces,  $A$  is connected,  $A_{\mathbb{R}} \neq \emptyset$  and  $\varepsilon > 0$ . Then  $X_\varepsilon$  is a CDQ-surface provided that  $\varepsilon$  is small enough.*

A real linear bundle  $L$  will be called a *CDQ-bundle* if it is very ample and admits a real section with a nonsingular CDQ zero divisor.

**Theorem 2.2.** *If  $L$  is a CDQ-bundle then its multiples  $L^{\otimes d}$ ,  $d \geq 1$ , are CDQ-bundles as well.*

*Proof.* Let  $X_0^{(1)}$  be a CDQ-divisor of  $L$ . We prove by induction on  $d$  that there exists a CDQ-divisor,  $X_0^{(d)}$ , of  $L^{\otimes d}$  which intersects  $X_0^{(1)}$  transversally along a curve having a nonempty real part. This claim is trivial for  $d = 1$ , since we can perturb  $X_0^{(1)}$  so that the result will intersect  $X_0^{(1)}$  transversally and contain a given real point of it.

Suppose that  $X_0^{(d)}$  satisfies the induction assumption. By the Lefschetz Theorem,  $A$  is connected. A generic real section of  $L^{\otimes(d+1)}$  has zero divisor  $X$  transversal to  $X_0^{(1)}$ ,  $X_0^{(d)}$  and to  $X_0^{(1)} \cap X_0^{(d)}$ , hence, we can apply Theorem 2.1 and get a CDQ-divisor  $X_\varepsilon$  by a perturbation of  $X_0^{(1)} \cup X_0^{(d)}$  via  $X$ . We can also choose  $X$  containing a real point of  $X_0^{(1)}$ , since  $L^{\otimes(d+1)}$  is very ample. Then, for a sufficiently small  $\varepsilon > 0$ ,  $X_\varepsilon$  intersects  $X_0^{(1)}$  transversally and  $X_\varepsilon \cap X_0^{(1)} = X \cap X_0^{(1)}$  has a nonempty real part.  $\square$

## 2.2. Existence of CDQ complete intersections

**Theorem 2.3.** *For arbitrary integers  $n, d_1, \dots, d_n \geq 1$  there exists a CDQ-surface  $X \subset \mathbb{C}P^{n+2}$  which is a complete intersection of multi-degree  $(d_1, \dots, d_n)$ .*

*Proof.* Again we carry out the proof by induction on  $n$ . The diffeomorphism  $\mathbb{C}P^2/\text{conj} \cong S^4$  means that  $\mathcal{O}_{\mathbb{C}P^3}(1)$  is a CDQ-bundle, which gives the base of induction. By Theorem 2.2,  $\mathcal{O}_{\mathbb{C}P^3}(d)$ , for any  $d \geq 1$ , is a CDQ-bundle as well. Assume now that we are given a complete intersection of real hypersurfaces,  $X = H_1 \cap \dots \cap H_n \subset \mathbb{C}P^{n+2}$ , of multi-degree  $(d_1, \dots, d_n)$  and that  $X$  is a CDQ-surface. Choose hypersurfaces  $H'_i \subset \mathbb{C}P^{n+3}$ ,  $i = 1, \dots, n$ , so that  $H_i = H'_i \cap \mathbb{C}P^{n+2}$ , and the intersection  $V = H'_1 \cap \dots \cap H'_n$  is transversal. Then the bundle  $L \rightarrow V$  induced from  $\mathcal{O}_{\mathbb{C}P^{n+3}}(1)$  is a CDQ-bundle, since  $X$  is its zero divisor. By Theorem 2.2,  $L^{\otimes d}$  is also a CDQ-bundle, hence, there exists a CDQ-complete intersection of multi-degree  $(d_1, \dots, d_n, d)$ .  $\square$

## 2.3. Proof of Small Perturbation Theorem

Denote by  $N^{(i)}$  a  $\text{conj}_V$ -invariant compact tubular neighborhood of  $A$  in  $X_0^{(i)}$  and put  $M^{(i)} = N^{(i)}/\text{conj}_V$ ,  $B = A/\text{conj}_V$ ,  $Y^{(i)} = X_0^{(i)}/\text{conj}_V$  and  $Y_\varepsilon = X_\varepsilon/\text{conj}_V$ . It can be easily seen that  $M^{(i)}$  is a regular neighborhood of  $B$  in  $Y^{(i)}$ . Let  $2k$  denote the number of imaginary points in  $A \cap X$ .

**Theorem 2.4.** *There exists a diffeomorphism  $\tilde{\varphi} : \partial M^{(1)} \rightarrow \partial M^{(2)}$ , such that  $Y_\varepsilon \cong (\text{Cl}(Y^{(1)} - M^{(1)}) \cup_{\tilde{\varphi}} \text{Cl}(Y^{(2)} - M^{(2)})) \#_k \overline{\mathbb{C}P}^2$ .*

Let us first derive Theorem 2.1 from the above theorem.

*Proof.* Since  $A$  is connected and has a nonempty real part,  $B$  is a compact connected surface with a nonempty boundary, hence,  $M^{(i)}$  are handlebodies with one 0-handle and several 1-handles embedded into  $Y^{(i)}$ . It is well known that if we glue a pair of simply connected 4-manifolds,  $Y^{(i)}$ ,  $i = 1, 2$ , along the boundary of such handlebodies the result is diffeomorphic to  $Y^{(1)} \# Y^{(2)} \#_g R$ , where  $g = b_1(B)$  and  $R = S^2 \times S^2$  or  $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$  (it is a corollary of [15], cf. [17]). This implies complete decomposability of  $Y_\varepsilon$  if  $Y^{(i)}$  are completely decomposable.  $\square$

Now we proceed by proving Theorem 2.4.

*Proof.* Assume that  $\varepsilon \in \mathbb{R}$  and  $\varepsilon > 0$  are sufficiently small. Then  $X_\varepsilon$  intersects  $X_0^{(i)}$ ,  $i = 1, 2$  transversally along the curve  $C_i = X_0^{(i)} \cap X$ . Consider first the blow-up,  $\tilde{V} \rightarrow V$ , along  $C_1$  and denote by  $\tilde{C}_i$ ,  $\tilde{X}_0^{(i)}$ , and  $\tilde{X}_t$  the proper images of  $C_i$ ,  $X_0^{(i)}$  and  $X_t$ . The pencil  $\tilde{X}_t$  has the base-curve  $\tilde{C}_2$ , therefore, the next blow-up  $\hat{V} \rightarrow \tilde{V}$  along  $\tilde{C}_2$  gives a fibering over  $\mathbb{C}P^1$  with fibers  $\hat{X}_t$  (here and below we mark with a hat the proper image in  $\hat{V}$ ).

The projections  $\hat{X}_\varepsilon \rightarrow X_\varepsilon$ ,  $\hat{X}_0^{(1)} \rightarrow X_0^{(1)}$  are biregular, as well as  $\hat{X}_0^{(2)} \rightarrow \tilde{X}_0^{(2)}$ , whereas  $\tilde{X}_0^{(2)} \rightarrow X_0^{(2)}$  is the blow-up at  $C_1 \cap X_0^{(2)} = A \cap X$ .

The real structure on  $V$  can be obviously lifted to the real structure,  $\text{conj}_{\widehat{V}} : \widehat{V} \rightarrow \widehat{V}$ , and we have  $\widehat{Y}^{(1)} \cong Y^{(1)}$ ,  $\widehat{Y}_\varepsilon \cong Y_\varepsilon$  and  $\widehat{Y}^{(2)} \cong Y^{(2)} \#_k(\overline{\mathbb{C}P^2})$ , where  $\widehat{Y}^{(1)}$ ,  $\widehat{Y}_\varepsilon$ ,  $\widehat{Y}^{(2)}$  denote the quotients by  $\text{conj}_{\widehat{V}}$  of  $\widehat{X}_0^{(1)}$ ,  $\widehat{X}_\varepsilon$  and  $\widehat{X}_0^{(2)}$ . The latter diffeomorphism follows from that blow-ups at real points do not change the diffeomorphism type of the quotient, since  $\mathbb{C}P^2/\text{conj} \cong S^4$ , whereas a pair of blow-ups at conjugated imaginary points descends to a blow-up in the quotient. Restrictions give diffeomorphisms between  $M^{(i)}$  and regular neighborhoods of  $\widehat{A}/\text{conj}_{\widehat{V}}$  in  $\widehat{X}_0^{(i)}/\text{conj}_{\widehat{V}}$  for  $\widehat{A} = \widehat{X}_0^{(1)} \cap \widehat{X}_0^{(2)}$ .

To complete the proof we apply the Equivariant Deformation Theorem 3.1, proven below, and obtain  $\widetilde{\varphi}$  from the quotient of  $\varphi$  by  $\text{conj}_W$ .  $\square$

### 3. Equivariant Deformation Theorem

#### 3.1. Statement of the Theorem

Assume that  $f : W \rightarrow \Delta$  is a nonconstant proper holomorphic mapping of a nonsingular real variety  $W$ ,  $\dim_{\mathbb{C}} W = n \geq 2$ , into a disc,  $\Delta \subset \mathbb{C}$ , around zero, such that  $f \circ \text{conj}_W = \text{conj} \circ f$ , where  $\text{conj} : \Delta \rightarrow \Delta$  is the complex conjugation on  $\mathbb{C}$ . Assume further that  $f$  has a critical value only at zero and the zero divisor  $X_0$  of  $f$  splits into two nonsingular  $\text{conj}_W$ -invariant components  $X_0^{(i)}$ ,  $i = 1, 2$ , of multiplicity 1 crossing transversally along a nonsingular subvariety  $A$  (which inherits a real structure from  $W$ ). Fix a  $\text{conj}_W$ -invariant metric on  $W$ , and consider a sufficiently small tubular  $\varepsilon$ -neighborhood  $N \subset W$  of  $A$ , with the projection  $p : N \rightarrow A$ . Then  $N^{(i)} = N \cap X_0^{(i)}$  is a tubular compact  $\text{conj}_W$ -invariant neighborhood of  $A$  in  $X_0^{(i)}$ ,  $i = 1, 2$ .

**Theorem 3.1.** *Assume that  $\delta \in \Delta \cap \mathbb{R}$ ,  $\delta \neq 0$ . Then there exists a  $\text{conj}_W$ -equivariant bundle isomorphism  $\varphi : \partial N^{(1)} \rightarrow \partial N^{(2)}$  reversing orientations of fibers, such that  $X_\delta = f^{-1}(\delta)$  is equivariantly diffeomorphic to  $\text{Cl}(X_0^{(1)} - N^{(1)}) \cup_\varphi \text{Cl}(X_0^{(2)} - N^{(2)})$ .*

*Remark 3.1.* To avoid bulky considerations, I do not deal with the non-holomorphic setting as is done in [18] (nicely 2-degenerating families of manifolds), although the arguments below can be applied in this case as well.

#### 3.2. Proof of Deformation Theorem by Mandelbaum–Moishezon

Let us recall first the proof of Theorem 3.1 in a non-equivariant setting, which is given in [18]. For sufficiently small  $\varepsilon$  and  $\delta \in \Delta$ ,  $0 < |\delta| \ll \varepsilon$  the restriction,  $p_\delta : N \cap X_\delta \rightarrow A$ , of  $p$  is a smooth fibering with the fibers homeomorphic to an annulus,  $Q = S^1 \times [0, 1]$  (which follows from straightforward calculations). On the other hand, the complement  $\text{Cl}(X_\delta - N)$  is diffeomorphic to  $\text{Cl}(X_0 - N) = \text{Cl}(X_0^{(1)} - N^{(1)}) \amalg \text{Cl}(X_0^{(2)} - N^{(2)})$ . Furthermore, the natural embedding  $\text{Diff}^+(S^1) \rightarrow \text{Diff}_0^+(Q)$ ,  $\psi \mapsto \psi \times \text{id}_{[0,1]}$ , is known to be a homotopy equivalence (here  $\text{Diff}^+$  mean that the diffeomorphisms are orientation-preserving; the subscript 0 means that the boundary components are not permuted).

Thus, we obtain a reduction of the structure group of  $p_\delta$  to  $\text{Diff}^+(S^1)$  and therefore a diffeomorphism  $N \cap X_\delta \cong K_\delta^{(i)} \times [0, 1]$ ,  $i = 1, 2$ , where  $K_\delta^{(i)} = \partial N \cap X_0^{(i)}$ , which determines a diffeomorphism  $K_\delta^{(1)} \rightarrow K_\delta^{(2)}$  and provides the required  $\varphi$ .

### 3.3. Proof of the Equivariant Deformation Theorem

Assume from now on that  $\delta \in \mathbb{R} \cap \Delta - \{0\}$ , and moreover  $0 < \delta$  (without loss of generality in Theorem 3.1). First, we need to choose an equivariant diffeomorphism  $\text{Cl}(X_\delta - N) \cong \text{Cl}(X_0 - N)$ . It is defined by the standard construction: consider an isotopy  $h_t : \text{Cl}(X_0 - N) \rightarrow W$ ,  $t \in [0, 1]$ , defined by a smooth vector field along  $\text{Cl}(f^{-1}([0, \delta]) - N)$  which is mapped into the unit tangent vectors along  $[0, \delta] \subset \mathbb{R}$ ,  $\frac{d}{dt}$ , by the differential  $df$  and tangent to  $\partial N$ ; then  $h_t(\text{Cl}(X_0 - N)) = \text{Cl}(X_{t\delta} - N)$ . This isotopy will be equivariant if the vector field is  $\text{conj}_W$ -invariant, which can be achieved by taking the average of any sample such vector field with its image under the complex conjugation.

Next, we need to prove the equivariant reducibility of the structure group. Denote by  $V \subset A$  a small compact tubular neighborhood of  $A_\mathbb{R} = A \cap W_\mathbb{R}$ , by  $r : V \rightarrow A_\mathbb{R}$  the projection map, and by  $p_V : p_\delta^{-1}(V) \rightarrow V$ ,  $p_\mathbb{R} : p_\delta^{-1}(A_\mathbb{R}) \rightarrow A_\mathbb{R}$  the restrictions of  $p_\delta$ . Note that if we could obtain an equivariant reduction of  $\text{Diff}_0^+(Q)$  to  $\text{Diff}^+(S^1)$  over  $V$ , then we would extend it over the whole  $A$ , since such an equivariant extension would be equivalent to a usual (relative with respect to  $\partial(V/\text{conj}_W)$ ) reduction of the structure group in the quotient fiber bundle  $p_\delta^{-1}(\text{Cl}(A - V))/\text{conj}_W \rightarrow \text{Cl}(A - V)/\text{conj}_W$ , the obstruction for which would vanish. The rest of the paper is devoted to proving the existence of an equivariant reduction for  $p_V$ , which is all that remains to complete the proof of Theorem 3.1.

In what follows, we mean by an equivariant isomorphism of bundles a smooth  $\mathbb{Z}/2$  equivariant isomorphism (for generalities on equivariant bundles we refer to [14]).

### 3.4. Equivariant reduction of the structure group over $V$

Define an involution  $c_Q : Q \rightarrow Q$ ,  $c_Q(x, y, t) = (x, -y, t)$ , where  $(x, y) \in S^1 \subset \mathbb{R}^2$ , which permute the halves  $Q^\pm = \{(x, y, z) \in Q \mid \pm y \geq 0\}$  of  $Q$ . Denote by  $\text{Diff}(Q, c_Q)$  the subgroup of  $\text{Diff}_0^+(Q)$  consisting of  $c_Q$ -equivariant diffeomorphisms (the lemma below implies that this is the structure group of  $p_\mathbb{R}$ ). Let  $\text{Diff}_0(Q, c_Q) \subset \text{Diff}(Q, c_Q)$  be the subgroup whose elements do not permute  $Q^+$  and  $Q^-$ ; it is the kernel of the homomorphism  $\text{Diff}(Q, c_Q) \rightarrow \mathbb{Z}/2$ , which admits a splitting defined by the involution  $T : Q \rightarrow Q$ ,  $T(x, y, t) = (-x, -y, t)$ , thus  $\text{Diff}(Q, c_Q) \cong \text{Diff}_0(Q, c_Q) \oplus \mathbb{Z}/2$ .

**Lemma 3.2.** *For any  $x \in A_\mathbb{R}$ , the fiber  $F = p_\mathbb{R}^{-1}(x)$  with the involution  $\text{conj}_W|_F$  is equivariantly isomorphic to  $(Q, c_Q)$ .*

*Proof.* It can be easily checked that the intersection of a fiber of  $p_\delta$  with the fixed point set of  $\text{conj}_W$  consists of a pair of arcs connecting the boundary circles of this fiber. This determines the topological type of the involution  $\text{conj}_W|_F$ .  $\square$

**Lemma 3.3.**  *$\text{Diff}_0(Q, c_Q)$  is contractible.*



*Proof.* The restriction of diffeomorphisms to  $Q^+ \subset Q$  defines an embedding of  $\text{Diff}_0(Q, c_Q)$  into the diffeomorphism group of  $Q^+$  keeping the corner points fixed. The parametrization  $\sigma : [0, 1]^2 \rightarrow Q^+$ ,  $\sigma(\tau, t) = (\cos(\pi\tau), \sin(\pi\tau), t)$ , identifies the latter group with the group  $\text{Diff}_0([0, 1]^2)$ , which consists of similar diffeomorphisms of  $[0, 1]^2$ . The composition gives an embedding  $\Sigma : \text{Diff}_0(Q, c_Q) \rightarrow \text{Diff}_0([0, 1]^2)$ , whose image,  $\mathcal{E}$ , consists of  $\psi \in \text{Diff}_0([0, 1]^2)$ , whose differentials,  $d_{(\tau, t)}\psi$ , has vectors  $\frac{d}{d\tau}$  (as well as  $\frac{d}{dt}$ ) as eigenvectors along the fixed point set of  $c_Q$ , (the sides  $(i, t)$ ,  $i = 0, 1$ ,  $t \in [0, 1]$ ). It is an exercise to construct a deformation retraction of  $\mathcal{E}$  to the subgroup  $\mathcal{F} \subset \text{Diff}_0([0, 1]^2)$  whose elements  $\psi$  have identity differential  $d\psi$  along the boundary of  $[0, 1]^2$  (see [20] for the technique). Then it is a standard fact of 2-dimensional topology (also originated from [20]) that  $\mathcal{F}$  is contractible.  $\square$

**Corollary 3.4.** *The structure group of  $p_{\mathbb{R}}$  can be equivariantly reduced to  $\{id_Q, T\} \subset \text{Diff}(Q, c_Q)$ .*

**Lemma 3.5.**  *$p_V$  is equivariantly isomorphic to the bundle induced from  $p_{\mathbb{R}}$  by the projection  $r : V \rightarrow A_{\mathbb{R}}$ .*

*Proof.* Consider the equivariant homotopy  $h_\delta : V \rightarrow V$  which contracts the fibers of  $r$  and connects the identity  $h_0 = id_V$  with  $h_1 = r$ . According to [14] (Theorem 2.10), the pull backs of a smooth equivariant bundle via equivariantly homotopic maps are equivariantly isomorphic. Therefore  $p_V = h_0^* p_V$  is isomorphic to  $h_1^* p_V = r^* p_{\mathbb{R}}$ .  $\square$

**Corollary 3.6.** *The structure group of  $p_V$  can be equivariantly reduced to  $\mathbb{Z}/2 = \{id_Q, T\}$  and therefore to  $\text{Diff}^+(S^1) \supset \text{SO}(2) \supset \mathbb{Z}/2$ .*

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