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## Seiberg-Witten Equations on $\mathbb{R}^8$

*Ayşe Hümeysra Bilge, Tekin Dereli, and Şahin Koçak*

### 1. Introduction

The Seiberg-Witten equations are meaningful on any even-dimensional manifold. To state them, let us recall the general set-up, adopting the terminology of the forthcoming book by D.Salamon ([1]).

A  $spin^c$ -structure on a  $2n$ -dimensional real inner-product space  $V$  is a pair  $(W, \Gamma)$ , where  $W$  is a  $2^n$ -dimensional complex Hermitian space and  $\Gamma : V \rightarrow End(W)$  is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \quad \Gamma(v)^2 = -\|v\|^2$$

for  $v \in V$ . Globalizing this defines the notion of a  $spin^c$ -structure  $\Gamma : TX \rightarrow End(W)$  on a  $2n$ -dimensional (oriented) manifold  $X$ ,  $W$  being a  $2^n$ -dimensional complex Hermitian vector bundle on  $X$ . Such a structure exists iff  $w_2(X)$  has an integral lift.  $\Gamma$  extends to an isomorphism between the complex Clifford algebra bundle  $C^c(TX)$  and  $End(W)$ . There is a natural splitting  $W = W^+ \oplus W^-$  into the  $\pm i^n$  eigenspaces of  $\Gamma(e_{2n}e_{2n-1} \cdots e_1)$  where  $e_1, e_2, \dots, e_{2n}$  is any positively oriented local orthonormal frame of  $TX$ .

The extension of  $\Gamma$  to  $C_2(X)$  gives via the identification of  $\Lambda^2(T^*X)$  with  $C_2(X)$  a map

$$\rho : \Lambda^2(T^*X) \rightarrow End(W)$$

given by

$$\rho\left(\sum_{i < j} \eta_{ij} e_i^* \wedge e_j^*\right) = \sum_{i < j} \eta_{ij} \Gamma(e_i) \Gamma(e_j).$$

The bundles  $W^\pm$  are invariant under  $\rho(\eta)$  for  $\eta \in \Lambda^2(T^*X)$ . Denote  $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$ . The map  $\rho$  (and  $\rho^\pm$ ) extends to

$$\rho : \Lambda^2(T^*X) \otimes \mathbf{C} \rightarrow End(W).$$

(If  $\eta \in \Lambda^2(T^*X) \otimes \mathbf{C}$  is real-valued then  $\rho(\eta)$  is skew-Hermitian and if  $\eta$  is imaginary-valued then  $\rho(\eta)$  is Hermitian.)

A Hermitian connection  $\nabla$  on  $W$  is called a  $spin^c$  connection (compatible with the Levi-Civita connection) if

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi$$

where  $\Phi$  is a spinor (section of  $W$ ),  $v$  and  $w$  are vector fields on  $X$  and  $\nabla_v w$  is the Levi-Civita connection on  $X$ .  $\nabla$  preserves the subbundles  $W^\pm$ .

There is a principal  $Spin^c(2n) = \{e^{i\theta}x | \theta \in \mathbf{R}, x \in Spin(2n)\} \subset C^c(\mathbf{R}^{2n})$  bundle  $P$  on  $X$  such that  $W$  and  $TX$  can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbf{C}^{2^n}, \quad TX = P \times_{Ad} \mathbf{R}^{2n},$$

$Ad$  being the adjoint action of  $Spin^c(2n)$  on  $\mathbf{R}^{2n}$ . We get then a complex line bundle  $L_\Gamma = P \times_\delta \mathbf{C}$  using the map  $\delta : Spin^c(2n) \rightarrow S^1$  given by  $\delta(e^{i\theta}x) = e^{2i\theta}$ .

There is a one-to-one correspondence between  $spin^c$  connections on  $W$  and  $spin^c(2n) = Lie(Spin^c(2n) = spin(2n) \oplus i\mathbf{R}$ -valued connection-1-forms  $\hat{A} \in \mathbf{A}(P) \subset \Omega^1(P, spin^c(2n))$  on  $P$ .

Now consider the trace-part  $A$  of  $\hat{A}$ :  $A = \frac{1}{2^n} trace(\hat{A})$ . This is an imaginary valued 1-form  $A \in \Omega^1(P, i\mathbf{R})$  which is equivariant and satisfies

$$A_p(p \cdot \xi) = \frac{1}{2^n} trace(\xi)$$

for  $v \in T_p P, g \in Spin^c(2n), \xi \in spin^c(2n)$  (where  $p \cdot \xi$  is the infinitesimal action). Denote the set of imaginary valued 1-forms on  $P$  satisfying these two properties by  $\mathbf{A}(\Gamma)$ . There is a one-to-one correspondence between these 1-forms and  $spin^c$  connections on  $W$ . Denote the connection corresponding to  $A$  by  $\nabla_A$ .  $\mathbf{A}(\Gamma)$  is an affine space with parallel vector space  $\Omega^1(X, i\mathbf{R})$ . For  $A \in \mathbf{A}(\Gamma)$  the 1-form  $2A \in \Omega^1(P, i\mathbf{R})$  represents a connection on the line bundle  $L_\Gamma$ . Because of this reason  $A$  is called a *virtual connection* on the *virtual line bundle*  $L_\Gamma^{1/2}$ . Let  $F_A \in \Omega^2(X, i\mathbf{R})$  denote the curvature of the 1-form  $A$ . Finally, let  $D_A$  denote the Dirac operator corresponding to  $A \in \mathbf{A}(\Gamma)$ ,

$$C^\infty(X, W^+) \rightarrow C^\infty(X, W^-)$$

defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A, e_i}(\Phi)$$

where  $\Phi \in C^\infty(X, W^+)$  and  $e_1, e_2, \dots, e_{2n}$  is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows. Fix a  $spin^c$  structure  $\Gamma : TX \rightarrow End(W)$  on  $X$  and consider the pairs  $(A, \Phi) \in \mathbf{A}(\Gamma) \times C^\infty(X, W^+)$ . The SW-equations read

$$D_A(\Phi) = 0, \quad \rho^+(F_A) = (\Phi\Phi^*)_0$$

where  $(\Phi\Phi^*)_0 \in C^\infty(X, End(W^+))$  is defined by  $(\Phi\Phi^*)(\tau) = \langle \Phi, \tau \rangle \Phi$  for  $\tau \in C^\infty(X, W^+)$  and  $(\Phi\Phi^*)_0$  is the traceless part of  $(\Phi\Phi^*)$ .

In dimension  $2n = 4$ ,  $\rho^+(F_A) = \rho^+(F_A^+) = \rho(F_A^+)$  (where  $F^+$  is the self-dual part of  $F$  and the second equality understood in the obvious sense), and therefore self-duality comes intimately into play. The first problem in dimensions  $2n > 4$  is that there is not a generally accepted notion of self-duality. Although there are some meaningful definitions ([2],[3],[4],[5],[6]) (Equivalence of self-duality notions in [2],[3],[5],[6] has been shown in [7], making them more relevant as they separately are), they do not assign a well-defined self-dual part to a given 2-form. Even though  $\rho^+(F_A)$  is still meaningful, it is apparently less important due to the lack of an intrinsic self-duality of 2-forms in higher dimensions.

The other serious problem in dimensions  $2n > 4$  is that the SW-equations as they are given above are overdetermined. So it is improbable from the outset to hope for any solutions. We verify below for  $2n = 8$  that there aren't indeed any solutions.

In dimension  $2n = 4$  it is well-known that there are no finite-energy solutions ([1]), but otherwise whole classes of solutions are found which are related to vortex equations ([8]). It seems to us that it would be desirable to have higher dimensional modifications of Seiberg-Witten equations (at least in the physically important dimension  $2n = 8$ ) having nontrivial solutions (possibly including 4-dimensional Seiberg-Witten solutions as special cases) and related to generalized self-duality referred to above.

## 2. Seiberg-Witten Equations on $\mathbf{R}^8$

We fix the constant *spin*<sup>c</sup> structure  $\Gamma : \mathbf{R}^8 \rightarrow \mathbf{C}^{16 \times 16}$  given by

$$\Gamma(e_i) = \begin{bmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{bmatrix}$$

( $e_i, i = 1, 2, \dots, 8$  being the standard basis for  $\mathbf{R}^8$ ), where

$$\gamma(e_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \gamma(e_2) = \begin{bmatrix} i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \end{bmatrix}$$

$$\gamma(e_3) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad \gamma(e_4) = \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{bmatrix}$$

$$\begin{aligned}
 \gamma(e_5) &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, & \gamma(e_6) &= \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix} \\
 \gamma(e_7) &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}, & \gamma(e_8) &= \begin{bmatrix} 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

(We obtain this  $spin^c$  structure from the well-known isomorphism of the complex Clifford algebra  $C^c(\mathbf{R}^{2n})$  with  $End(\Lambda^*\mathbf{C}^n)$ .)

In our case  $X = \mathbf{R}^8$ ,  $W = \mathbf{R}^8 \times \mathbf{C}^{16}$ ,  $W^\pm = \mathbf{R}^8 \times \mathbf{C}^8$  and  $L_\Gamma = L_\Gamma^{1/2} = \mathbf{R}^8 \times \mathbf{C}$ . Consider the connection 1-form

$$A = \sum_{i=1}^8 A_i dx_i \in \Omega^1(\mathbf{R}^8, i\mathbf{R})$$

on the line bundle  $\mathbf{R}^8 \times \mathbf{C}$ . Its curvature is given by

$$F_A = \sum_{i < j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^8, i\mathbf{R})$$

where  $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$ . The  $spin^c$  connection  $\nabla = \nabla_A$  on  $W^+$  is given by

$$\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi$$

( $i = 1, \dots, 8$ ) where  $\Phi : \mathbf{R}^8 \rightarrow \mathbf{C}^8$ .

$$\rho^+ : \Lambda^2(T^*X) \otimes \mathbf{C} \rightarrow End(W^+)$$

is given by

$$\rho^+(F_A) = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & G_{16} & G_{17} & 0 \\ \bar{G}_{12} & \bar{G}_{22} & \bar{G}_{23} & \bar{G}_{24} & \bar{G}_{25} & \bar{G}_{26} & 0 & G_{17} \\ \bar{G}_{13} & \bar{G}_{23} & \bar{G}_{33} & \bar{G}_{34} & \bar{G}_{35} & 0 & \bar{G}_{26} & -G_{16} \\ \bar{G}_{14} & \bar{G}_{24} & \bar{G}_{34} & G_{44} & 0 & G_{35} & -G_{25} & G_{15} \\ \bar{G}_{15} & \bar{G}_{25} & \bar{G}_{35} & 0 & -G_{44} & G_{34} & -G_{24} & G_{14} \\ \bar{G}_{16} & \bar{G}_{26} & 0 & \bar{G}_{35} & \bar{G}_{34} & -G_{33} & G_{23} & -G_{13} \\ \bar{G}_{17} & 0 & \bar{G}_{26} & -\bar{G}_{25} & -\bar{G}_{24} & \bar{G}_{23} & -G_{22} & G_{12} \\ 0 & \bar{G}_{17} & -\bar{G}_{16} & \bar{G}_{15} & \bar{G}_{14} & -\bar{G}_{13} & \bar{G}_{12} & -G_{11} \end{bmatrix},$$

where

$$\begin{aligned} G_{11} &= iF_{12} + iF_{34} + iF_{56} + iF_{78}, & G_{12} &= F_{13} + iF_{14} + iF_{23} - F_{24}, \\ G_{13} &= F_{15} + iF_{16} + iF_{25} - F_{26}, & G_{14} &= F_{17} + iF_{18} + iF_{27} - F_{28}, \\ G_{15} &= F_{35} + iF_{36} + iF_{45} - F_{46}, & G_{16} &= F_{37} + iF_{38} + iF_{47} - F_{48}, \\ G_{17} &= F_{57} + iF_{58} + iF_{67} - F_{68}, & G_{22} &= -iF_{12} - iF_{34} + iF_{56} + iF_{78}, \\ G_{23} &= -F_{35} - iF_{36} + iF_{45} - F_{46}, & G_{24} &= -F_{37} - iF_{38} + iF_{47} - F_{48}, \\ G_{25} &= F_{15} + iF_{16} - iF_{25} + F_{26}, & G_{26} &= F_{17} + iF_{18} - iF_{27} + F_{28}, \\ G_{33} &= -iF_{12} + iF_{34} - iF_{56} + iF_{78}, & G_{34} &= -F_{57} - iF_{58} + iF_{67} - F_{68}, \\ G_{35} &= -F_{13} - iF_{14} + iF_{23} - F_{24}, & G_{44} &= -iF_{12} + iF_{34} + iF_{56} - iF_{78}. \end{aligned}$$

For  $\Phi = (\phi_1, \phi_2, \dots, \phi_8) \in C^\infty(X, W^+) = C^\infty(\mathbf{R}^8, \mathbf{R}^8 \times \mathbf{C}^8)$ ,

$$(\Phi\Phi^*)_0 = \begin{bmatrix} \phi_1\bar{\phi}_1 - 1/8 \sum \phi_i\bar{\phi}_i & \phi_1\bar{\phi}_2 & \dots & \phi_1\bar{\phi}_8 \\ \phi_2\bar{\phi}_1 & \phi_2\bar{\phi}_2 - 1/8 \sum \phi_i\bar{\phi}_i & \dots & \phi_2\bar{\phi}_8 \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \phi_8\bar{\phi}_1 & \phi_8\bar{\phi}_2 & \dots & \phi_8\bar{\phi}_8 - 1/8 \sum \phi_i\bar{\phi}_i \end{bmatrix}.$$

It was remarked by Salamon([1],p.187) that  $\rho^+(F_A) = 0$  implies  $F_A = 0$ . (i.e.reducible solutions of 8-dim. SW-equations are flat.)

It can be explicitly verified that all solutions are reducible and flat:

**Proposition 2.1.** *There are no nontrivial solutions of the Seiberg-Witten equations on  $\mathbf{R}^8$  with constant standard  $spin^c$  structure, i.e.*

$\rho^+(F_A) = (\Phi\Phi^*)_0$  (alone) implies  $F_A = 0$  and  $\Phi = 0$ .

*Proof.* Trivial but tedious manipulation with the linear system. □

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