Turkish Journal of Mathematics

Volume 21 | Number 1 Article 3

1-1-1997

On irreducible simply-connected 4-manifolds

Zoltan SZABO

Tr. J. of Mathematics
21 (1997) , 41 – 46.

Proceedings of 5^{th} Gökova
Geometry-Topology Conference

On irreducible simply-connected 4-manifolds

Zoltán Szabó

1. Introduction

In the last few years, based on results of Donaldson, Gompf, Mrowka and Taubes, see [DK], [G], [GM], [T1], [T2], an ambitious classification scheme emerged regarding smooth simply-connected closed 4-manifolds:

Conjecture 1.1. Minimal Conjecture. ([T5], [Ko]). *Every simply-connected smooth closed 4-manifold X can be decomposed as $X = X_1 \# \cdots \# X_n$, where X_i are symplectic 4-manifolds with both the symplectic and the opposite orientations allowed, and where S^4 corresponds to the empty sum.*

The first counter-examples to the Minimal Conjecture were constructed in [Sz1], and were later generalized in [Sz2] and by Fintushel and Stern in [FS].

This is an expository paper, that aims to give a short account of the counter-examples of [Sz1] and [Sz2]. For a detailed description see [Sz1], [Sz2]. The paper also contains additional examples, and gives an interesting family of irreducible simply-connected smooth closed 4-manifolds with small signature, presented in Section 3. (A simply-connected smooth closed 4-manifold is irreducible if any connected sum decomposition satisfies that one of the summands is a homotopy S^4).

Acknowledgements. I wish to thank the organizors of the Gokova Conference, Selman Akbulut and Turgut Onder for their kind hospitality.

2. Counter-examples

Let us start by recalling the original examples from [Sz1]. It is easy to see that the problem is equivalent to constructing irreducible simply-connected 4-manifold X with non-trivial intersection form, such that X doesn't have symplectic structures with either orientation. In order to show irreducibility one is more or less forced to consider simply-connected 4-manifolds with non-trivial gauge invariants. Note that this implies b_2^+ odd, and the existence of almost complex structures. The obstruction to have symplectic structures is provided by results of Taubes on symplectic 4-manifolds:

Theorem 2.1. (See [T1], [T2]). *Let X be a simply-connected symplectic 4-manifold with $b_2^+(X) > 1$ and symplectic structure ω. Then we have*

$$SW_X(c_1(\omega)) = \pm 1.$$

where $c_1(\omega) \in H^2(X, \mathbf{Z})$ denotes the canonical class of the symplectic structure and SW_X denotes the Seiberg-Witten invariant of X. Furthermore if $K \in H^2(X, \mathbf{Z})$ satisfies $SW_X(K) \neq 0$, then $|K \cdot \omega| \leq |c_1(\omega) \cdot \omega|$, and equality implies that $K = \pm c_1(\omega)$.

We start by constructing a 4-manifold Y with $\pi_1(Y) = \mathbf{Z}$ and then performing logarithmic transformations of multiplicity n to get the non-symplectic manifolds X_n. Y is built from the four-torus T^4 and two copies of the rational elliptic surface $\mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$ in the following way.

Let $T_{12} \hookrightarrow T^4$, $T_{13} \hookrightarrow T^4$, $T_{14} \hookrightarrow T^4$ be smoothly embedded 2-tori in the 4-torus: $T_{12} = S^1 \times S^1 \times p_1 \times p_2$, $T_{13} = S^1 \times p_3 \times S^1 \times p_4$, $T_{14} = S^1 \times p_5 \times p_6 \times S^1$, where $p_i \in S^1$, $i = 1, ..., 6$ are disjoint points. Then T_{12}, T_{13} and T_{14} are disjoint and have self-intersection 0.

Take also two copies of the rational elliptic surface $E_1 = \mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$, $E_2 = \mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$ and fix one-one generic fiber in both copies $F_1 \hookrightarrow E_1$, $F_2 \hookrightarrow E_2$. Note that F_1, F_2 are also smoothly embedded tori with square 0. Now we make the fiber sum of T^4, E_1 and E_2 along T_{12}, T_{13}, F_1 and F_2. This is done by fixing diffeomorphisms $f_1 : F_1 \to T_{12}$, $f_2 : F_2 \to T_{13}$ and lifting them to orientation reversing diffeomorphisms g_1, g_2 between the closed tubular neighborhoods. Then the fiber sum Y is:

$$Y = (E_1 \setminus ndF_1) \cup_{g_1} (T^4 \setminus ndT_{12} \setminus ndT_{13}) \cup_{g_3} (E_2 \setminus ndF_2),$$

where nd denotes the open tubular neighborhood.

Now we have $T_{14} \hookrightarrow Y$, and it is easy to see that $\pi_1(Y \setminus ndT_{14}) = \mathbf{Z}$, where the fundamental group is generated by a loop $\gamma = q_1 \times q_2 \times q_3 \times S^1 \hookrightarrow \partial(Y \setminus ndT_{14})$. The next step is to perform a certain logarithmic transformation along T_{14} in order to kill the fundamental group. So for all $n \geq 0$ let us fix an orientation reversing diffeomorphism $\phi_n : \partial(D^2 \times T^2) \to \partial(Y \setminus ndT_{14})$ satisfying $(\phi_n)_*(e) = [\gamma] + n\beta$, where $\beta \in H_1(\partial(Y \setminus ndT_{14}), \mathbf{Z})$ is the homology class of the meridian around T_{14}, $[\gamma]$ is the homology class of γ, and $e \in H_1(\partial(D^2 \times T^2), \mathbf{Z})$ is defined by $e = [\partial(D^2 \times p)]$, where $p \in T^2$.

We define X_n by:

$$X_n = (Y \setminus ndT_{14}) \cup_{\phi_n} (D^2 \times T^2).$$

The computation of Seiberg-Witten invariants of X_n uses a joint work with John Morgan and Tom Mrowka, see [MMSz], on how logarithmic transformation changes Seiberg-Witten invariants. In this case we get a surgery formula, see [Sz1], [MMSz], that gives a relation between Seiberg-Witten invariants of X_0, Y and X_n. After computing SW_{X_0} and SW_Y this relation gives the following result.

Theorem 2.2. (See [Sz1]). *For all $n \geq 0$ let X_n denote the smooth closed 4-manifolds defined above. Then $\pi_1(X_n) = 1$, $b_2^+(X_n) = 3$, $b_2^-(X_n) = 19$, X_n has odd intersection form, and*

- *$SW_{X_n}(\pm K_1 \pm K_2)) = \pm n$, where $K_1 = PD[T_{12}]$, $K_2 = PD[T_{13}]$,*
- *$SW_{X_n}(L) = 0$ for $L \neq \pm K_1 \pm K_2$,*
- *The Seiberg-Witten invariant of $\overline{X}_n$ vanishes.*

Now it follows from standard arguments that X_n is irreducible for all $n \geq 1$. On the other hand it follows from the result of Taubes, see Theorem 2.1, that X_n is non-symplectic for all $n \geq 2$. So we have:

Theorem 2.3. (See [Sz1]). *For all $n \geq 2$, the simply connected smooth closed 4-manifold X_n is irreducible and doesn't have symplectic structure. In particular X_n is a counter-example to the Minimal Conjecture.*

The above construction admits several generalizations. One could for example replace the rational elliptic surfaces with other interesting 4-manifolds (e.g other elliptic surfaces), and get counter-examples realizing many homotopy types, see [Sz1, Section 4]. A more subtle generalization is given in [Sz2], where we construct counter-examples with $b_2^+ = 1$. Let us recall the construction, since it will also be used in the next section.

Let $\phi : T^2 \to T^2$ be an orientation preserving self-diffeomorphism satisfying $\phi_*(a_1) = a_1 + a_2$, $\phi_*(a_2) = a_2$, where $a_1, a_2 \in H_1(T^2, \mathbf{Z})$ form a basis. Let Z_ϕ denote the mapping torus of ϕ, and $W = Z_\phi \times S^1$ denote the Kodaira-Thurston manifold, see [Th]. The fibration $T^2 \to Z_\phi \to S^1$, gives fibration $T^2 \to W \to T^2$. It easy to see that W has a symplectic structure, and admits a symplectic section $S \hookrightarrow W$. Let us fix a simple loop $\delta \hookrightarrow Z_\phi$ that lies in a fiber and represents a_1 and let $T = \delta \times S^1 \hookrightarrow W$. It is easy to see that S, and T are smoothly embedded tori of square 0.

Let us fix a generic fiber $F \hookrightarrow \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ in the rational elliptic surface. After forming the fiber sum of W and $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ along S and F we get a 4-manifold Z, with $b_2^+(Z) = 2$, $b_2^-(Z) = 10$, $\pi_1(Z) = \mathbf{Z}$. By performing certain logarithmic transformations of multiplicity $n \geq 0$ along $T \hookrightarrow Z$, we get simply-connected 4-manifolds Y_n with $b_2^+(Y_n) = 1$, $b_2^-(Y_n) = 9$.

Note that for 4-manifolds X with $b_2^+(X) = 1$ the Seiberg-Witten invariants depend on the riemannian metric g, and the perturbing self-dual two form h, that are used in the definition of the Seiberg-Witten equations. In our case however, since $b_2^-(Y_n) = 9$, it follows from standard arguments that $SW_{Y_n}(L, g_1, h_1) = SW_{Y_n}(L, g_2, h_2)$, provided that h_1, h_2 are small enough. The corresponding chamber is called the metric chamber. Adopting the surgery formula and the arguments of [Sz1] to the $b_2^+ = 1$ case we get the following result.

Theorem 2.4. (See [Sz2]). *Let Y_n be as above, and let SW_{Y_n} denote the Seiberg-Witten invariant in the metric chamber. Then*

- $SW_{Y_n}(\pm L) = \pm n$, *where* $L = PD[S]$
- $SW_{Y_n}(L') = 0$ *for all* $L' \neq \pm L$.

Furthermore the Seiberg-Witten invariant of $\overline{Y}_n$ vanishes.

While this theorem already implies that if $n \neq m$ then Y_n is not diffeomorphic to Y_m, one has to be more careful when studying the blown-ups of these manifolds because of the chamber structure. Similar problem arises when one compares Theorem 2.4 with computations of Taubes on the Seiberg-Witten invariants of symplectic 4-manifolds with $b_2^+ = 1$, see [T1], [T2], since in that case the invariant is computed by perturbing the equations with a large enough multiple of the symplectic 2-form and the invariant no longer corresponds to the metric chamber. After studying of the chamber structures in both cases we get the following results.

Theorem 2.5. (See [Sz2]). *For all $n \geq 2$ we have the following*

- *(i) Y_n is irreducible.*
- *(ii) If $k \geq 0$ and $n \neq m$ then $Y_n \# k\overline{\mathbf{CP}}^2$ is not diffeomorphic to $Y_m \# k\overline{\mathbf{CP}}^2$.*
- *(iii) If $k \geq 0$, then $Y_n \# k\overline{\mathbf{CP}}^2$ is not diffeomorphic to any Kähler surface.*

Theorem 2.6. (See [Sz2]). *For all $n \geq 2$ neither Y_n nor $\overline{Y}_n$ have symplectic structure..*

3. Irreducible 4-manifolds with small signature

The building blocks for our construction are the simply-connected non-symplectic 4-manifolds Y_n, defined in the previous section, and a symplectic 4-manifold M. M was introduced by Gompf in [G] as the twisted double of the Kodaira-Thurston manifold. The construction is as follows. Take two copies W_1, W_2 of the Kodaira-Thurston manifold, fix fibers $F_1 \hookrightarrow W_1$, $F_2 \hookrightarrow W_2$, and a symplectomorphism $f : F_1 \to F_2$ that induces $f_*(a_1) = a_2$, $f_*(a_2) = -a_1$. Then the corresponding fiber sum $M = W_1 \#_f W_2$ has a symplectic structure by the symplectic sum theorem [G]. Note that the sections $S_1 \hookrightarrow W_1$, $S_2 \hookrightarrow W_2$ can be glued together to give a symplectic section $\Sigma_0 \hookrightarrow M$. Then Σ_0 is a smoothly embedded genus 2 surface with trivial self-intersection. It is shown in [G], that Σ_0 has a canonical normal framing. Using this framing let us fix parallel copies Σ_i, $1 \leq i \leq k$, where the parameter k will be fixed later.

We claim that Y_n contains a genus 2 surface $\Gamma \hookrightarrow Y_n$, with self-intersection 1, and $S \cdot \Gamma = 1$. To see that let us recall from the previous section that Z is the fiber sum of W and $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ along $S \hookrightarrow W$, and $F \hookrightarrow \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$. The name fiber sum is somewhat misleading, since in fact S is a section of W and F is a fiber of $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$. Now by gluing together a fiber of W and a section of $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ we get an embedded torus with self-intersection -1 that is disjoint from T and intersects S at a transverse point. Then forming the union with S and resolving the self-intersection point gives a genus 2 surface $\Gamma \hookrightarrow Z$ with self-intersection 1. Since Γ is disjoint from T, we also get a corresponding $\Gamma \hookrightarrow Y_n$ with the desired properties. After blowing up Y_n in such a way that the base point is in Γ, we get a genus 2 surface $\Gamma' \hookrightarrow Y_n \# \overline{\mathbf{CP}}^2$ with self intersection 0.

Let $X(0, n)$ denote the fiber sum of M and $Y_n \# \overline{\mathbf{CP}}^2$ along Σ_0 and Γ'. Note that the disjoint smoothly embedded genus 2 surfaces $\Sigma_i \hookrightarrow M$ give rise to $\Sigma_i \hookrightarrow X(0, n)$ for $i = 1, ..., k$. By induction we get $X(i+1, n)$ as the fiber sum of $X(i, n)$ and a disjoint copy of M along $\Sigma_{i+1} \hookrightarrow X(i, n)$ and $\Sigma_0 \hookrightarrow M$.

We claim the following:

Theorem 3.1. *For all $k \geq 0$, $n \geq 2$ let $X(k, n)$ denote the smooth closed 4-manifold defined above. Then $\pi_1(X(k, n)) = 1$, $b_2^+(X(k, n)) = 2k + 3$, $b_2^-(X(k, n)) = 2k + 12$. In particular $|sign(X(k, n))| = 9$. Furthermore we have the followings*

- *(i) There is a* $K \in H^2(X(k,n), \mathbf{Z})$*, such that* $SW_{X(k,n)}(\pm K) = \pm n$*, and* $SW_{X(k,n)}(L) = 0$ *for* $L \neq K$.
- *(ii)* $X(k,n)$ *is irreducible*
- *(iii)* $X(k,n)$ *doesn't have symplectic structures with either orientations.*

Proof. Let us start with the homotopy type of $X(k,n)$. It is shown in [G], that $\pi_1(M \setminus nd(\Sigma_0))$ is normally generated by the image of $i_* : \pi_1(\Sigma) \to \pi_1(M \setminus nd(\Sigma_0))$ where Σ is a parallel copy of Σ_0. Since $Y_n \# \overline{\mathbf{CP}}^2 \setminus nd(\Gamma')$ is simply-connected, it follows that $\pi_1(X(0,n)) = 1$. Similarly if $\Sigma \hookrightarrow \partial(X(i,n) \setminus nd(\Sigma_{i+1}))$ is a parallel copy of Σ_{i+1}, then $j_* : \pi_1(\Sigma) \to \pi_1(X(i,n) \setminus nd(\Sigma_{i+1}))$ is trivial, and it follows from induction and the Van Kampen theorem that $X(i+1,n)$ is simply-connected. The computation for the b_2^+, b_2^- follows from the additivity of signature under fiber sum.

Note that the symplectic structure of M satisfies $c_1(\omega) = \pm 2G$, where G is the Poincare dual of the fiber of M. It follows then from [T1], that $SW_M(\pm 2G) = \pm 1$. The invariants of $Y_n \# \overline{\mathbf{CP}}^2$ can be computed from Theorem 2.4 and the blow-up formulas. Note that $PD(\Gamma') = PD(\Gamma) - E$, where E is the exceptional class. Let $SW_{Y_n \# \overline{\mathbf{CP}}^2}$ correspond to the chamber that contains $PD(\Gamma')$ on its boundary. Then $SW_{Y_n \# \overline{\mathbf{CP}}^2}(S + E) = \pm n$ follows from Theorem 2.4. Now using $k+1$ times the product formula of [MSzT], we get a $K \in H^2(X(k,n), \mathbf{Z})$, such that $SW_{X(k,n)}(\pm K) = \pm n$. On the other hand using the adjunction inequality of [KM], cf. [Sz1], it is an easy exercise to show that $SW_{X(k,n)}(L) = 0$ for $L \neq \pm K$. This proves (i). The irreducibility of $X(k,n)$ follows from (i) and standard arguments. Since $X(k,n)$ contains smoothly embedded spheres with self-intersection -2, it follows that the Seiberg-Witten invariant of $\overline{X}(k,n)$ vanishes. Then (iii) follows from (i) and Theorem 2.1.

References

[DK] S.K. Donaldson and P.B. Kronheimer, *'The Geometry of Four-Manifolds'*, Clarendon, Oxford, 1990.

[FS] R. Fintushel and R. Stern, *Knots, links and 4-manifolds*, preprint 1996 Oct.

[G] R.E. Gompf, *New construction of symplectic manifolds*, Annals of Math. **142** (1995), 527-595.

[GM] R.E. Gompf and T.S. Mrowka, *Irreducible 4-manifolds need not be complex*, Annals of Math. **138** (1993), 61-111.

[Ko] D. Kotschick, *The Seiberg-Witten invariants of symplectic four-manifolds, [after C.H. Taubes]*, to appear in Asterisque.

[KM] P.B. Kronheimer and T.S Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters **1** (1994), 797-808.

[M] J.W. Morgan, *'The Seiberg-Witten Equations and applications to the topology of smooth four-manifolds'*, Mathematical Notes 44, Princeton University Press, 1996.

[MMSz] J.W Morgan, T.S Mrowka and Z. Szabó, *Product formulas along* T^3 *for Seiberg-Witten invariants*, in preparation.

[MSzT] J.W. Morgan, Z. Szabó and C.H. Taubes, *A product formula for the Seiberg-Witten invariants and the generalized Thom-Conjecture*, to appear in Journal of Differential Geometry.

[Sz1] Z. Szabó, *Simply-connected irreducible 4-manifolds with no symplectic structures*, preprint 1996 July.

[Sz2] Z. Szabó, *Exotic smooth 4-manifolds with* $b_2^+ = 1$, to appear in Math. Research Letters.

[T1] C.H. Taubes, *The Seiberg-Witten invariants and symplectic form*, Math. Research Letters **1**, (1994), 809-822.
[T2] C.H. Taubes, *More constraints on symplectic manifolds from Seiberg-Witten invariants*, Math. Research Letters **2**, (1995), 9-14.
[T3] C.H. Taubes, *The Seiberg-Witten and the Gromov invariants*, Math. Research Letters **2** (1995), 221-238.
[T4] C.H. Taubes, $SW \to Gr$, *From the Seiberg-Witten equations to pseudoholomorphic curves*, to appear in Jour. Amer. Math. Soc.
[T5] C.H.. Taubes, I. International Press Lectures, UC Irvine, March 1996.
[Wi] E. Witten, *Monopoles and four-manifolds*, Math. Research Letters **1** (1994), 769-796.

Mathematics Department, Princeton University, Princeton, NJ 08544, USA
E-mail address: szabo@math.princeton.edu