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Abstract

The monodromy groupoid was first introduced by Pradines in [7] and developed by Mucuk in [6]. In this paper we give an application of the monodromy groupoid.

Introduction

A groupoid (Definition 1.1) is a category in which every morphism (*arrow*) has an inverse.

A topological groupoid (Definition 1.4) is a groupoid in which all maps are continuous.

For the concepts of free groupoid, normal subgroupoid and quotient groupoid we refer to Section 1.

Let G be a topological groupoid and W an open neighbourhood of the identities in G. So W has a directed graph structure inherited from the groupoid multiplication of G. Hence we have a free groupoid $F(W)$ on W. Let N the normal subgroupoid of $F(W)$ generated by the elements in the from $[v][u][vu]^{-1}$ for $u, v \in W$ such that vu is defined in G and belongs to W. Let $M(G,W)$ be the quotient groupoid of $F(W)$ by N. So we have an inclusion $\bar{i} : W \longrightarrow M(G,W)$ and by the freness of $F(W)$ we have a projection map $p : M(G,W) \longrightarrow G$ induced by the inclusion map $i : W \longrightarrow G$. Further if $f : W \longrightarrow H$ is a local morphism of groupoids as defined in Definition 1.2 then we have a morphism of groupoids (Definition 1.2) $\phi : M(G,W) \longrightarrow H$ such that $\phi\bar{i} = f$. This groupoid $M(G,W)$ is called *monodromy groupoid* of G.

In [6] the monodromy groupoid $M(G,W)$ was given a topological structure making it a topological groupoid such that the projection morphism $p : M(G,W) \longrightarrow G$ is a universal covering on each fibre $M(G,W)_x$. This study was written in Lie groupoid case in [2].

As an example to monodromy groupoid if X is a topological space admitting a universal covering then we can choose a neighbourhood W of the identities in the topological groupoid $G = X \times X$ such that the monodromy groupoid $M(G,W)$ is the fundamental groupoid $\pi_1 X$ described in below. Further if G is a topological group which has a universal covering then the monodromy groupoid of G is the universal covering of G. Hence the notion of the monodromy groupoid generalizes the ideas of fundamental groupoid and universal covering. In this paper we consider the monodromy groupoid $M(G,W)$ just as a groupoid.

In this paper we prove that if $p : E \longrightarrow X$ is a principal bundle in the sense of Definition 2.1, then the nonodromy groupoid $M(G,W)$, with $G = X \times X$, acts on E. (See Definition 2.3 for the action of a groupoid on a set).

1 Groupoids

Definition 1.1. *([5]) A groupoid consists of two sets G and O_G called respectively the set of elements or morphisms and the set of objects of the groupoid, together with two maps $\alpha, \beta : G \longrightarrow O_G$, called respectively the source and target maps, a map $1_{()} : O_G \longrightarrow G, x \mapsto 1_x$ called the object map and a partial multiplication*

$$G * G \longrightarrow G, (h,g) \mapsto hg$$

defined on the fibre product set

$$G * G = \{(h,g) \in G \times G : \alpha(h) = \beta(g)\}.$$

These maps are subject to the following conditions
i) $\alpha(hg) = \alpha(g)$ and $\beta(hg) = \beta(h)$ for all $(h,g) \in G * G$;
ii) $k(hg) = (kh)g$ for all $g,h,k \in G$ such that $\alpha(h) = \beta(g)$ and $\alpha(k) = \beta(h)$;
iii) $\alpha(1_x) = \beta(1_x) = x$ for all $x \in O_G$, where 1_x is the identity at x;
iv) $g1_{\alpha(g)} = g$ and $1_{\beta(g)}g = g$ for all $g \in G$; and
v) each $g \in G$ has an inverse g^{-1} such that $\alpha(g^{-1}) = \beta(g), \beta(g^{-1}) = \alpha(g)$ and $g^{-1}g = 1_{\alpha(g)}, gg^{-1} = 1_{\beta(g)}$.

If (G, O_G) is a groupoid we say G is a groupoid on O_G. For a groupoid G we write G_x for $\alpha^{-1}(x)$ and $G(x,y)$ for $\alpha^{-1}(x) \cap \beta^{-1}(y)$ where $x,y \in O_G$.

For example a group can be considered as a groupoid with only one object. Let X be a topological sapce. Then $G = X \times X$ becomes a groupoid on X, whose morphisms are the pairs (x,y) for $x,y \in X$. In G, α and β are defined by

$$\alpha(x,y) = x, \quad \beta(x,y) = y$$

and the multiplication is defined by

$$(y,z)(x,y) = (x,z).$$

Let X be a topological space. The homotopy classes of the paths in X form a groupoid on X. The composition of the paths in X induces a composition of the homotopy classes. This groupoid is called *fundamental groupoid* of X and denoted by $\pi_1 X$ (see [1] for more details)

Definition 1.2. *Let G and H be groupoids. A local morphism of groupoids is a map $f : W \longrightarrow H$ from a subset of G containing all the identities in G such that for $u \in W, \alpha_H(fu) = f(\alpha_G u), \beta_H(fu) = f(\beta_G u)$ and $f(vu) = f(v)f(u)$ whenever $v, u \in W, vu$ is defined and belongs to W.*

A morphism from G to H is a pair of maps

$$f : G \longrightarrow H \text{ and } O_f : O_G \longrightarrow O_H$$

such that

$\alpha_H o f = O_f \, o \alpha_G, \beta_H o f = O_f o \beta_G$ and $f(vu) = f(v)f(u)$ for all $(v, u) \in G * G$, where

$$G * G = \{(v, u) \in G \times G : \alpha(v) = \beta(u)\}.$$

For such a morphism we simply write $f : G \longrightarrow H$.

The following notions of subgroupoid, normal subgroupoid and quotient groupoid are from [1] and [5].

Definition 1.3 *Let G be a groupoid. A subgroupoid of G is a pair of subsets $H \subseteq G$ and $O_H \subseteq O_G$ such that $\alpha(H) \subseteq O_H$, $\beta(H) \subseteq O_H$, $1_x \in H$ for $x \in O_H$ and H is closed under the partial multiplication and inversion in G.*

A normal subgroupoid of G is subgroupoid N of G such that $O_N = O_G$ and for each $x, y \in O_G, a \in G(x, y)$ we have $aN(x) = N(y)a$.
Let G be a groupoid and N be a normal subgroupoid of G such that $N(x, y) = \emptyset$ if $x \neq y$. Define a groupoid G/N on O_G by

$$G/N(x, y) = \{aN(x) : a \in G(x, y)\}$$

for any $x, y \in O_G$ with the multiplication that if $a \in G(x, y)$ and $b \in G(y, z)$ then $bN(y)aN(x) = baN(x)H(x) = baN(x)$. This groupoid is called quotient groupoid of G by N.

Let W be a directed graph. Let $p = (a_n, \cdots, a_1)$ be sequence of the edges such that the target of a_i is equal to the source of a_{i+1}. Such a p is called directed path. Write $()_x$ for the empty path associated to x. The composition of two directed paths $p = (a_n, \cdots, a_1)$ and $q = (b_m, \cdots, b_1)$ is defined by $qp = (b_m, \cdots, b_1, a_n, \cdots, a_1)$ if the target of a_n is the source of b_1. Then we have a category $P(W)$. Let $\hat{a}$ denote the converse path of a in W. Define an equivalence relation on $P(W)$ as follows: Two ridected paths p, q are equivalent if we can obtain one from the another by adding or deleting a number of $a_i\hat{a}_i$ or $\hat{a}_i ai$. This is an equivalence relation. The set of equivalence classes $[p]$ is denoted by $F(W)$. A groupoid multiplication on $F(W)$ is defined by $[q][p] = [qp]$. So $F(W)$ becomes a groupoid which is called *free groupoid* on the graph $W[3]$.

Let G be a groupoid and R a subset of G. By the normal subgroupoid $N(R)$ generated by R we mean the smallest normal subgroupoid including R. A direct construction of $N(R)$ is given in [1] as follows: Let G be a groupoid and R a subset of G such that $O_R = O_G$ and $R(x,y) = \emptyset$ if $x \neq y$. Let $N(x)$ be the set of all elements

$$\rho = a_n^{-1}\rho_n a_n \cdots a_1^{-1}\rho_1 a_1$$

for $a_i \in G(x, x_i)$ and ρ_i or ρ_i^{-1} an element of $R(x_i)$. Let $N(R)$ be the family of $N(x)$ for all $x \in O_G$. Then $N(R)$ is a normal subgroupoid of G such that $O_{N(R)} = O_G$ and $NR(x,y) = \emptyset$ if $x \neq y$. $N(R)$ is called the *normal subgroupoid generated* by R.

Definition 1.4 *A topological groupoid is a groupoid G in which the sets G and O_G are topological spaces and the following maps are continuous.*

i) partial multiplication $G * G \longrightarrow G, (h,g) \mapsto hg$, where $G * G$ has the relative topology;

ii) inverse map $G \longrightarrow G, g \mapsto g^{-1}$;

iii) source and target maps $\alpha, \beta : G \longrightarrow O_G$;

iv) object map $1_{()} : O_G \longrightarrow G, x \mapsto 1_x$.

A topological group is a topological groupoid with only one object. If X is a topological space then the groupoid $G = X \times X$ described above is a topological groupoid.

2 Action and principal bundles

The following definition is from [5].

Definition 2.1 *Let $p : E \longrightarrow X$ be a continuous surjective map and let G be a topological group acting effectively on E by $G \times E \longrightarrow E, (g,e) \mapsto ge$. By effectively we mean that if $ge = he$ then $g = h$. Then the triple (E, X, p) is called a principal G-bundle if the following conditions are satisfied.*

i) The fibres of p are equal to the orbits of G, that is, for $e, é \in E$ the statement $p(e) = p(é)$ is equivalent to that there is an element $g \in G$ such that $é = eg$.

ii) The map $\delta : E \underset{p}{\times} E \longrightarrow G, (e, eg) \to g$ is continuous. Here

$$E \underset{p}{\times} E = \{(e, é) \in E \times E : p(e) = p(é)\}.$$

iii) There is an open cover $\{U_i : i \in I\}$ of X and there are continuous maps

$s_i : U_i \longrightarrow E$ such that ps_i is identity at U_i. Such maps are called *local sections*

We call the set of these local sections $\{s_i : U_i \longrightarrow E, i \in I\}$

atlas of sections and the maps $s_{ij} : U_i \cap U_j \longrightarrow G$ defined by

$$s_{ij}(x)s_i(x) = s_j(x)$$

are called *transition functions.*

Example 2.2. *Let G be a topological group and $p : E \longrightarrow X$ a principal G-bundle, and for $x \in X$ let $E_x = p^{-1}(x)$. Let S_p denote the set of all bijections $g : E_x \longrightarrow E_y$ for $x, y \in X$. Then S_p becomes a groupoid on X with respect to the following structure: For $g : E_x \longrightarrow E_y$ the source and target maps are defined by $\alpha(g) = x, \beta(g) = y$, and the groupoid multiplication is the composition of the maps.*

Definition 2.3. ([1]p.347) *Let G be a groupoid with $O_G = X, E$ a set and $p : E \longrightarrow X$ a function. Let $G * E$ denote the subset*

$$\{(g, e) \in G \times E : \alpha(g) = p(e)\}$$

of $G \times E$. An action of G on E via p is a function

$$G * E \longrightarrow E,\ (g, e) \mapsto ge$$

such that

i) $p(ge) = \beta(g)$ for $(g, e) \in G * E$;

ii) $h(ge) = (hg)e$ for $(h, g) \in G * G$, where

$$G * G = \{(v, u) \in G \times G : \alpha(v) = \beta(u)\};$$

iii) $(1_{pe})e = e$ for $e \in E$.

From this action in [1] a groupoid called *action groupoid* is obtained and in [6] some useful applications of action groupoid on coverings are given.

In the following theorem, which is the main theorem of this paper, we need a condition that in a principal G-bundle $p : E \longrightarrow X$ the transition functions $s_{ij} : U_i \cap U_j \longrightarrow G$ are constant. But for example in [4] by assuming that G is discrete this condition is guarantied. So we do not loose much by assuming that these transition functions are constant.

We now give the main theorem of this paper.

Let G be a topological group

Theorem 2.4. *Let $p : E \longrightarrow X$ be a principal G-bundle such that the transition functions $s_{ij} : U_i \cap U_j \longrightarrow G$ defined by $s_{ij}(x)s_i(x) = s_j(x)$ are constant. Then for an appropriate subset W of $G = X \times X$ the monodromy groupoid $M(G, W)$ acts on E via p.*

Proof. Let $\{s_i : U_i \longrightarrow E$, where $i \in I\}$ be an atlas of sections of $p : E \longrightarrow X$. Define $\vartheta_i : p^{-1}(U_i) \longrightarrow G$ by $\vartheta_i(e)s_i(pe) = e$. Note that since $ps_i(pe) = p(e)$, such $\vartheta_i(e) \in G$

exists and so ϑ_i is defined. By assumption the transition maps $s_{ij} : U_i \cap Uj \longrightarrow G$ defined by □

$$s_{ij}(x) = \vartheta_j(e)^{-1}\vartheta_i(e)$$

are constant, where $e \in p^{-1}(U_i \cap U_j)$ such that $pe = x$.

On the other hand by the definition of the maps

$$\vartheta_i : p^{-1}(U_i) \longrightarrow \text{G} \quad \text{and } \vartheta_j : p^{-1}(U_j) \longrightarrow \text{G}$$

for each $x \in (U_i \cap U_j)$ we have

$$\vartheta_i(e)s_i(pe) = \vartheta_j(e)s_j(pe) \text{ for } pe = x$$

$$\vartheta_i(e)s_i(x) = \vartheta_j(e)s_j(x)$$

$$\vartheta_j(e)^{-1}\vartheta_i(e)s_i(x) = s_j(x).$$

So we have

$$s_{ij}(x)s_i(x) = s_j(x),\ x \in (U_1 \cap U_j). \qquad (*)$$

Now define a map $g_i : U_i \times U_i \longrightarrow S_p$ as follows: If $(x, y) \in \cup_i \times \cup_i$, then let $g_i(x, y)$ be defined by

$$g_i(x, y) : Gs_i(x) \longrightarrow Gs_i(y), gs_i(x) \mapsto gs_i(y),$$

where $Gs_i(x)$ is the orbit of $s_i(x)$. Note that in a principal G-bundle $p : E \longrightarrow X$ fibres of p ere equal to the orbits we have

$$Gs_i(x) = p^{-1}(x) = E_x.$$

Then by gluing these maps $g_i (i \in I)$ we have a map $g : W \longrightarrow S_p$, where $W = \bigcup_{i \in I}(U_i \times U_i)$ and for $(x, y) \in W$ we have

$$g(x, y) = \begin{cases} g_i(x, y); & \text{if } x, y \in U_i \\ g_i(x, y) = g_j(x, y); & \text{if } x, y \in U_i \cap U_j \end{cases}$$

This map $g : W \longrightarrow S_p$ is well defined. For this we have to prove that if $x, y \in (U_i \cap U_j)$, then $g_i(x, y) = g_j(x, y)$. Since $s_{ij} : U_i \cap U_j \longrightarrow G$ is constant we have

$$Gs_i(x) = Gs_j(x)$$

and by $(*)$ we have

$$\begin{aligned} g_i(x,y)[gs_i(x)] &= gs_i(y) = gs_{ij}(y)^{-1}s_j(y) & (1)\\ g_j(x,y)[gs_i(x)] &= g_i(x,y)[gs_{ij}(x)^{-1}s_j(x)] = gs_{ij}(x)^{-1}s_j(y). & (2)\end{aligned}$$

Since $s_{ij}(x) = s_{ij}(y)$, the right sides of (1) and (2) are equal and so $g_i(x,y) = g_j(x,y)$ for $x, y \in (U_i \cap U_j)$. Morover it can be easily seen that $g : W \longrightarrow S_p$ is a local morphism. By the property of monodromy groupoid $M(G,W)$, with $G = X \times X$, this local morphism extends to a morphism of gropoids $\phi : M(G,W) \longrightarrow S_p$ such that $\phi_{\bar{i}} = g$, where $\bar{i}$ is the inclusion $W \longrightarrow M(G,W)$. Hence each morphism $a \in M(G,W)$ with $\alpha(a) = x, \beta(a) = y$ defines a bijection $\phi(a) : E_x \longrightarrow E_y$. That means we have an action $M(G,W) * E \longrightarrow E$. This completes the proof that the monodromy groupoid $M(G,W)$ acts on E via $p : E \longrightarrow X$.

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Monodromy Groupoidinin Bir Uygulaması

Özet

Monodromy groupoidi ilk olarak referanslardan [7] de Pradines tarafından tanıtıldı ve daha sonra [6] da Mucuk tarafından geliştirildi. Bu matalede monodromy groupoidinin bir uygulamasını veriyoruz.

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