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Pin-STRUCTURES ON SURFACES AND QUADRATIC FORMS

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Abstract

A correspondence between various Pin-type structures on a compact surface and quadratic (linear) forms on its homology is constructed. Sum of structures is defined and expressed in terms of these quadratic forms and in terms of Whitney sum of Spin structures.

§1 Introduction

In this paper we made an attempt to clarify the relation between Pin-type structures on a compact surface and quadratic forms on its homology. This relation is well-known and useful in the case of Spin- and Pin^--structures ([6], [7]). It makes it much easier to understand the nature of some fundamental low-dimensional topological objects, such as $\mathbb{Z}/2$-Seifert form on a surface in an oriented 3-manifold and Rokhlin form on a characteristic surface in an oriented 4-manifold (see, e.g., [5], [4]).

Remark. A slightly more general approach of [4] also explains these forms in the case of a non-oriented ambient manifold, as well as the newly found Benedetti-Marin form [2].

We show that a similar correspondence between quadratic forms and structures can also be defined for other Pin-type structures. In this short paper we confine ourselves to a geometrical description in the simplest case, when the structure group G is a $\mathbb{Z}/2$-extension of the orthogonal group O_n. Up to isomorphism, there exist four such extensions, classified by the four element of $H^2(BO_n;\mathbb{Z}/2)$; each element arises as the obstruction to the reduction of an O_n-bundle $P \to X$ to the corresponding group G. More generally, central extensions of a Lie group H with a discrete kernel M are classified up to isomorphism by the elements of $H^2(BH;M)$, which are the obstructions for reduction of H-bundless, see, e.g., [4].

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By a G-structure on P we mean an isomorphism class of such reductions $\widetilde{P} \to P$ or, in other words, a fiberwise homotopy class of liftings $X \to BG$ of a classifying map $X \to BO_n$ of P. When non-empty, the set of all the G- structures forms an affine space over $H^1(X;\mathbb{Z}/2)$, as is seen from the following homotopy exact sequence

$$\mathbb{Z}/2 \to G \to O_n \to K(\mathbb{Z}/2,1) \to BG \to BO_n.$$

If the obstruction is the trivial element of $H^2(BO_n;\mathbb{Z}/2)$, then $G \cong O_n \times \mathbb{Z}/2$ and a G-structure is just a homology class from $H^1(X;\mathbb{Z}/2)$. The obstructions w_2 and $w_2+w_1^2$ characterize Pin^+ and Pin^- structures respectively. Finally, the structures corresponding to the remaining obstruction class w_1^2 are not (as far as we know) mentioned in literature and do not have any special name. We will call them $\widetilde{O}_n$-structures, $\widetilde{O}_n$ standing for the nontrivial semi-direct product $\mathbb{Z}/4 \ltimes SO_n$ (topologically, $\widetilde{O}_n \to O_n$ is a trivial double covering).

Remark. Actually, $\widetilde{O}_n$-structures do appear in literature implicitly, e.g., as framings in the complexification of a vector bundle [1], or as linear forms on a real algebraic variety [8]. Besides, they complete the descending table in [7, Corol. 2.15]: a Pin^--structure on a manifold M descends to an $\widetilde{O}_n$-structure on a codimension on submanifold whose normal bundle is isomorphic to the determinant of the tangent budle of M.

Thus, we show that for each of these four classes there is a one-to-one correspondence between the set of structures on a compact surface and the set of specific quadratic (or, in special cases, linear) forms on the 1-homology of the surface. (This correspondence is known in the Pin^--case (see, e.g., [7]) and is obvious in the case of the trivial extension. In the two other cases it is defined in §2.). Another subject of the paper, which has never (as far as we know) been mentioned explicitly (see, though, a slightly different approach in [3]), is the sum operation for structures. This operation, defined in §3, naturally extends the canonical affine action of $H^1(X;\mathbb{Z}/2)$ on the set of structures. Further, when we add two structures the obstruction class of the result is the sum of the obstruction classes of summands. This gives one more reason to consider $\widetilde{O}_n$-structures: they arise as sums of Pin^- and Pin^+-structures. We give a description of this operation in terms of quadratic forms and interpret it also on the language of Spin structures.

A part of the results of this paper can be extracted from [4], where some further generalization of the notion of Pin-structure is studied. However, we found it reasonable to give a short account of these results for the classical case of $\mathrm{Pin}^{\pm}$-structures (and related $\widetilde{O}$-structures), as it is this case that is mainly used in applications.

§2 Quadratic Forms

1. Pin$^-$**-structures as quadratic forms** $H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/4$. To start with we recall the standard construction of the quadratic form q corresponding to a Pin$^-$-structure on a compact surface F. Pick an integral class $\alpha \in H_1(F;\mathbb{Z})$ and realize it by a collection $S \to F$ of immersed oriented circle whose all self-intersection points are transversal. Let $n(S)$ and $i(S)$ be the number of components and the (geometric) number of self-intersection points of S, respectively. A tangent vector field to S defines a Pin_1^--reduction of the restriction to S of the given Pin_2^--bundle on F. Since $\mathrm{Pin}_1^- \cong \mathbb{Z}/4$ is a discrete abelian group, one can consider the *total holonomy* $h(S) \in \mathrm{Pin}_1^-$ of this bundle along S (which, by definition, is the sum in Pin_1^- of the holonomies along the components of S). Define a map $q : H_1(F;\mathbb{Z}) \to \mathbb{Z}/4$ by $q(\alpha) = h(S) + 2(n(S) + i(S)) \pmod 4$. Now standard arguments apply to show that $q(\alpha)$ does not depend on S and satisfies the identity $q(\alpha+\beta) = q(\alpha) + q(\beta) + 2\langle\alpha,\beta\rangle$, where $\langle\cdot,\cdot\rangle$ is the intersection form on F and $2 : \mathbb{Z}/2 \to \mathbb{Z}/4$ is the unique inclusion. ($q(\alpha)$ obviously does not change during a regular homotopy of S, and elementary transformations like Reidemeister move I and smoothing a self-intersection point can easily be controlled; see, e.g., [7].). Since the mod 2 reduction of q coincides with $w_1 : H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$, the above formula implies, in particular that q factors through the $\mathbb{Z}/2$-homology of F.

2. $\widetilde{O}_2$**-structures as linear forms** $H_1(F;\mathbb{Z}/4) \to \mathbb{Z}/4$. Since the corresponding 1-dimensional group $\widetilde{O}_1$ is also $\mathbb{Z}/4$, this case is similar to the previous one. The only difference is that one should not adjust holonomy by the numbers of components and the self-intersection points, i.e., just put $q(\alpha) = h(S)$. The result is a linear form q which factors through $\mathbb{Z}/4$-homology, $q : H_1(F;\mathbb{Z}/4) \to \mathbb{Z}/4$, and whose restriction mod 2 coincides with $w_1 : H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$.

3. Pin$^+$**-structures as quadratic forms** $H_1(F;\mathbb{Z}/4) \to \mathbb{Z}/2$. The construction goes similar to the case of Pin$^-$-structures. Now $\mathrm{Pin}_1^+ \cong O_1 \times \mathbb{Z}/2$, the projection of $h(S)$ to the first factor O_1 being just the value of w_1 on α. Since this latter term is standard, we drop it and consider the projection $p_2h(S)$ to the second factor $\mathbb{Z}/2$; then we let $q(\alpha) = p_2h(S) = n(S) + i(S) \pmod 2$. This form satisfies the identity $q(\alpha+\beta) = q(\alpha) + q(\beta) + \langle\alpha,\beta\rangle$, which, in particular, implies that it factors through $H_1(F;\mathbb{Z}/4)$.

4. Trivial structures as linear forms $H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$. This case, when structures just *are* cohomology classes, admits a description similar to the previous three: the total holonomy $h(S)$ is an element of $O_1 \times \mathbb{Z}/2$, and we let $q(\alpha) = p_2h(S)$.

We can now uniformize all the four cases and consider quadratic (linear) forms $H_1(F;\mathbb{Z}/4) \to \mathbb{Z}/4$ (In the case of Pin$^+$-and trivial structures $\mathbb{Z}/2$ is embedded in $\mathbb{Z}/4$ via multiplication by 2.). This given the following result:

Theorem A. *Given a compact surface F, there is a canonical affine one-to-one correspondence between structures of F with the characteristic class $aw_2 + bw_1^2$ (for some fixed $a,b \in \mathbb{Z}/2$) and functions $q : H_1(F;\mathbb{Z}/4) \to \mathbb{Z}/4$ satisfying the following conditions:*

(1) $q(\alpha+\beta) = q(\alpha) + q(\beta) + 2a\langle\alpha,\beta\rangle \pmod 2$;
(2) $q(\alpha) = bw_1(\alpha) \pmod 2$.

Proof. The only thing that needs proof is the fact that the constructed map {structures} $\to$ {forms} is one-to-one. Since both the sets are affine over $H^1(F;\mathbb{Z}/2)$, it suffices to show that existence of forms implies existence of structures. This is obvious for Pin^-- and trivial structures, or if the surface is not closed (since structures always exist in such cases). For Pin^+-and $\widetilde{O}_2$-structures on closed surfaces one can easily see that desired forms exist if and only if elements of order 2 in $H_1(F;\mathbb{Z}/4)$ annihilate w_1 (or, equivalently, have trivial self-intersection). This is the case when the surface is the connected sum and an even number $\mathbb{R}P^2$'s, i.e., exactly when $w_2 = w_1^2 = 0$. □

Corollary (classification of Pin^+-structures up to isomorphism). *Two* Pin^+*-structures of a closed surface F are isomorphic (i.e., can be transformed into each other by a diffeomorphism of the surface) if and only if the values of the corresponding quadratic forms on the (unique) 2-torsion element of $H_1(F;\mathbb{Z})$ coincide. In particular, two structures are isomorphic if and only if they are cobordant.*

Proof. The mentioned value is the only algebraic invariant of forms (an easy exercise), and, as usual in 2-dimensional topology, one can find an automorphism of the lattice $H_1(F;\mathbb{Z})$ which is accompanied by a diffeomorphism of F. □

Remark Note that we have to consider integral homology here, since otherwise one cannot distinguish between different 2-torsion elements, and the algebraic invariant disappears.

§3 Sum Of Structures

Given two $\mathbb{Z}/2$-extensions $G_1 \to O_n, G_2 \to O_n$, define their *sum* $G_1 \vee G_2$ to be the quotient $G_1 \times_{O_n} G_2/$ Diag $(\mathbb{Z}/2)$, where Diag is the canonical diagonal map

$$\mathrm{Diag} : \mathbb{Z}/2 = \mathrm{Ker}[G_i \to O_n] \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \mathrm{Ker}[G_1 \times_{O_n} G_2 \to O_n].$$

(In fact, this is one of the standard algebraic approaches to definition of the groups structure on the set of isomorphism classes of $\mathbb{Z}/2$-extensions of O_n, which is isomorphic to $H^2(BO_n;\mathbb{Z}/2)$.). To apply this procedure to structures, one should first fix some representatives $G(\omega)$ of the isomorphism classes of extensions, one for each characteristic class

$\omega \in H^2(BO_n;\mathbb{Z}/2)$, and some maps $G(\omega_1)\vee G(\omega_2) \to G(\omega_1+\omega_2)$. To do that uniformly in all dimensions, it suffices to pick some isomorphisms $\mathrm{Pin}_1^- = \widetilde{O}_1 = \mathbb{Z}/4, \mathrm{Pin}_1^+ = O_1 \times \mathbb{Z}/2$, and $\mathbb{Z}/4 \vee \mathbb{Z}/4 = O_1 \times \mathbb{Z}/2$ (see [4]). For example, let us fix the latter isomorphism via $(x,y) \mapsto (\bar{x}, \frac{1}{2}(x-y))$, where $x,y \in \mathbb{Z}/4, \bar{x} \in O_1 \cong \mathbb{Z}/2$ is the mod 2 reduction of x and $\frac{1}{2}(x-y) \in \mathbb{Z}/2$, because x and y should have the same parity if $(x,y) \in \mathbb{Z}/4 \times \mathbb{Z}/4$ represents and element of $\mathbb{Z}/4 \vee \mathbb{Z}/4$. Once the above isomorphisms are fixed, we can define the sum of structures as follows.

Definition. *Let $P \to X$ be an O_n-bundle. Then, given two structures $\Phi_1 \to P, \Phi_2 \to P$ with the structure groups having obstruction classes $\omega_1, \omega_2 \in H^2(BO_n;\mathbb{Z}/2)$, we define their $\sum \Phi_1 \vee \Phi_2 \to P$ to be the $(\omega_1+\omega_2)$-structure associated with the fibered product $\Phi_1 \times_P \Phi_2 \to P$ via the composed map*

$$G(\omega_1) \times_{O_n} G(\omega_2) \to G(\omega_1) \vee G(\omega_2) \xrightarrow{\cong} G(\omega_1+\omega_2).$$

The proof of the following Theorem B is contained (in a more general setting) in [4].

Theorem B. *$\vee$ is a group operation on the set of all structures on a given O_n-bundle $P \to X$, which extends the canonical affine action of $H^1(X;\mathbb{Z}/2)$ on this set (i.e., $\vee$-sum with an $(O_n \times \mathbb{Z}/2)$-structure coincides with the affine shift by the corresponding cohomology class).*

In terms of quadratic forms $\vee$-sum of structures can be interpreted as follows.

Theorem C. *$\vee$-sum of structures on a compact surface corresponds to the following pointwise operation on quadratic forms: $(q_1,q_2) \mapsto q_1+q_2+2q_1q_2$.*

Remark. Note that if q_1 or q_2 takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$ (i.e., the corresponding structure is Pin^+ or the trivial one), then $2q_1q_2 = 0 \mod 4$, thus $\vee$-sum of structures corresponds in this case to the usual sum of forms.

Proof of Theorem C. Since the construction of §2 is obviously natural with respect to embeddings of surfaces it suffices to prove the formula for a tubular neighborhood of an embedded circle. In this case the statement follows from the commutative diagram

$$\begin{array}{ccc} G_1 \vee G_2 & \xrightarrow{\cong} & G_3 \\ \downarrow & & \downarrow \\ \mathbb{Z}/4 \vee \mathbb{Z}/2 & \xrightarrow{f} & \mathbb{Z}/4, \end{array}$$

where $G_i, i = 1,2,3$ is one of the extensions of O_1, *i.e.*, either $\mathrm{Pin}_1^- = \widetilde{O}_1 = \mathbb{Z}/4$ or $\mathrm{Pin}_1^+ = O_1 \times \mathbb{Z}/2$; the upper horizontal map is either the isomorphism $\mathbb{Z}/4 \vee \mathbb{Z}/4 = O_1 \times \mathbb{Z}/2$ fixed above or one of the canonical isomorphisms $(O_1 \times \mathbb{Z}/2) \vee G_2 = G_2, G_1 \vee (O_1 \times \mathbb{Z}/2) = G_1$; the rightmost map is either the identity $\mathbb{Z}/4 \to \mathbb{Z}/4$ or the product of the projection $O_1 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ to the second factor and the multiplication $2 : \mathbb{Z}/2 \to \mathbb{Z}/4$; the leftmost map is the (obviously defined) $\vee$-sum of the latter maps; and the lower map is induced by $\mathbb{Z}/4 \times \mathbb{Z}/4 \to \mathbb{Z}/4, (x,y) \mapsto x + y + 2xy$. Since the holonomy of $\vee$-sum of structures is obviously the $\vee$-sum of holonomies of summands, this gives the addition low for the holonomy components of q_1, q_2, q_3; clearly the correction term $2(n(s) + i(s))$ (for $\mathrm{Pin}^\pm$ structures) does not change this addition low. □

The introduced $\vee$-sum operation admits an interpretation in terms of Spin-structures. Given an O_n-bundle $\xi : P \to X$, let us denote by $\mathrm{Spin}(\xi), \mathrm{Pin}^\pm(\xi)$, etc. the set of all the Spin-, $\mathrm{Pin}^\pm$-, etc. structures of ξ respectively. Then, according to [7], there are natural isomorphisms $\mathrm{Pin}^-(\xi) = \mathrm{Spin}(\xi \oplus \det \xi)$ and $\mathrm{Pin}^+(\xi) = \mathrm{Spin}(\xi \oplus 3 \det \xi)$. Similar arguments show that, besides, there are isomorphisms $\widetilde{O}_n(\xi) = \mathrm{Spin}(2\xi) = \mathrm{Spin}(2 \det \xi)$. Consider the Whitney sum of the above three bundles:

$$(\xi \oplus \det \xi) \oplus (\xi \oplus 3 \det \xi) \oplus (2\xi) = 4(\xi \oplus \det \xi).$$

This bundle has a canonical Spin-structure (the quaternion Spin-structure, which is defined on 4η for any bundle η, see [4]). Hence, Spin-structures on any two of the three summands define a Spin-structure on the third one, and one can easily see that the obtained maps $\mathrm{Pin}^-(\xi) \times \mathrm{Pin}^+(\xi) \to \widetilde{O}_n(\xi)$, etc. coincide with the $\vee$-sum. This gives an alternative description of this operation in the most interesting cases which are not reduced to the affine action of $H^1(X; \mathbb{Z}/2)$.

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Yüzeyler Üzerinde Pin Yapıları ve Kuadratik Formlar

Özet

Kompakt bir yüzey üzerinde "Pin" tipi yapılarla yüzeyin homolojisi üzerindeki kuadratik (lineer) formlar arasında bir eşleme kuruluyor. Yapıların toplamı tanımlanıp bu, kuadratik formlar cinsinden ifade ediliyor.

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