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A. Bülent EKİN

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THE RANK AND THE CRANK MODULO 5

A.Bülent Ekin

Abstract

Let $p(n)$ denote the number of partitions of n. Ramanujan's partition congruences are $p(5n+4)$, $p(7n+5)$ and $p(11n+6) \equiv \bmod 5,\ 7,$ and 11, respectively. These have been proved in number of ways. Atkin and Swinnerton-Dyer proved the congruences and some more relations about partition in the case of mod5 and 7 in terms of *rank*, Garvan proved them in three cases in terms of *crank*.

In this study, we give an another proof of their results in the case of mod5 by using the theory of modular forms. Although our method is more tedious and complicated, it shows us how Modular forms of integral weight on a certain subgroups of $SL_2(\mathbb{Z})$ play role in partition theory. Our method could be applied to the case mod7, but not mod11 since the components of $\prod_{n=1}^{\infty}(1-q^n)^{-1}$ are not known explicitly.

1 Introduction

A *partition* of a positive integer n is a representation of n as the sum of any number of positive integers. The number of partitions of n is denoted by $p(n)$. Let $p(0)=1$. The *generating function* of $p(n)$ is the power series

$$\mathbf{P} = \sum_{n\geq 0} p(n)q^n = \prod_{r=1}^{\infty}(1-q^r)^{-1} \tag{1.1}$$

where $|q|<1$, (see [9]). The well-known Ramanujan's congruences are

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 & (1.2)\\ p(7n+5) &\equiv 0 \pmod 7 & (1.3)\\ p(11n+6) &\equiv 0 \pmod{11} & (1.4)\end{aligned}$$

Dyson[4] conjectured and Atkin Swinnerton-Dyer proved combinatorial results which imply the congruences (1.2) and (1.3). Let

$$\pi = \pi_0 + \pi_1 + \cdots + \pi_{s-1} \tag{1.5}$$

be a partition with $\pi_0 \geq \pi_1 \geq \cdots \geq \pi_{s-1}$. Dyson[4] defined the *rank* of π to be the number $\pi_0 - s$. He denoted by $N(r,m,n)$ the number of partitions of n with rank congruent to r modulo m and remarked that several relations appeared to hold between the numbers $N(r,m,mn+c)$ when $m=5$ and $m=7$. Of particular interest are the relations

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4)$$
$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5)$$

since they provide a combinatorial interpretation to the congruences (1.2) and (1.3). Dyson also conjectured the existence of a "*crank*" which would likewise imply the congruence (1.4). In 1987, Andrews and Garvan[2] discovered the crank as follows

$$crank(\pi) := \begin{cases} \pi_0, & \text{if } \ w(\pi) = 0 \\ \mu(\pi) - w(\pi), & \text{if } \ w(\pi) > 0 \end{cases}$$

where $w(\pi)$ is the number of ones in π and $\mu(\pi)$ is the number of parts of π bigger than $w(\pi)$. Let $M(r,m,n)$ denote the number of partitions of n with crank r modulo m. Several relations are known to hold between the numbers $M(r,m,an+c)$ when $(m,a) = (5,5)$, $(7,7)$ and $(11,11)$ [7] and when $(m,a) = (8,4)$, $(9,3)$ and $(10,15)$ [8].

Dyson[5] (following Andrews and Garvan) defined the crank of π in the following way, set $t = \pi_0 - \pi_1$,

$$crank(\pi) := \begin{cases} -s, & \text{if } \ t = 0 \\ t - \pi_t, & \text{if } \ t > 0 \end{cases}.$$

(supposing $\pi_r = 0$ when $r \geq s$). Since the Andrews-Garvan crank of a partition, π, is the negative of the Dyson crank of the conjugate of π, both definitions give the same value for $M(r,m,n)$ unless $n = 1$ Let $N(m,n)$ and $M(m,n)$ denote the number of partitions of n with rank and crank respectively congruent to r modulo m. We change this definition of $M(m,n)$ just a little, setting $M(0,1) = -1$ and $M(-1,1) = 1 = M(1,1)$, and modify $M(r,m,n)$ accordingly. We shall also suppose that the empty partition of zero has rank zero.

For convenience, we write N_1 for M and N_3 for N. By (12) in [11] and (1.11) in [2] when $k = 1$, and (2.12) in [3] when $k = 3$, we have

$$\sum_{m\in\mathbb{Z}} \sum_{n\geq 0} N_k(m,n) z^m q^n = \mathbf{P}(1-z) \sum_{m\in\mathbb{Z}} (-1)^n \frac{q^{n(kn+1)/2}}{1 - zq^n} \tag{1.6}$$

where $\mathbf{P} = \prod_{n=1}^{\infty}(1-q^n)^{-1}$, $|q| < 1$ and $|q| < |z| < |q^{-1}|$

For a power series $X = X(q) = \sum_{n\geq 0} a_n q^n$, we define

$$X^{(r)} := q^r \sum_{n\geq 0} a_{5n+r} y^n \quad (\ r = 0, \ 1, ..., \ m-1 \ )$$

and say that $X^{(r)}$ is rth component of X. Here, and below, $y = q^5$.
We introduce some notations

$$(z;q)_\infty := \prod_{r=1}^{\infty}(1 - zq^{r-1}),$$

$$J(z;q) := (z;q)_\infty(z^{-1}q;q)_\infty = \prod_{r=1}^{\infty}(1 - zq^{r-1})(1 - z^{-1}q^r).$$

Note that

$$-zJ(zq;q) = -zJ(z^{-1};q) = J(z;q) = J(z^{-1}q;q). \tag{1.7}$$

We also write

$$J(a) := J(y^a;y^5) = \prod_{r=1}^{\infty}(1 - y^{a+5(r-1)})(1 - y^{5r-a})$$

$$J(0) := \prod_{r=1}^{\infty}(1 - y^{5r})$$

It should be noted that $J(0)$ is not the expression that would be obtained by writing 0 instead of a in the definition of $J(a)$. From (1.7), we have

$$J(5-a) = J(a), \qquad J(-a) = J(5+a) = -y^{-a}J(a). \tag{1.8}$$

we also define

$$\mathbf{R}_{ij}(k) := N(i)^{(k)} - N(j)^{(k)} \quad \text{and} \quad \mathbf{C}_{ij}(k) := M(i)^{(k)} - M(j)^{(k)} \tag{1.9}$$

so that

$$\mathbf{R}_{ij}(k) = q^k \sum_{n \geq 0} (N(i,5,5n+k) - N(j,5,5n+k))\, y^n$$

and

$$\mathbf{C}_{ij}(k) = q^k \sum_{n \geq 0} (M(i,5,5n+k) - M(j,5,5n+k))\, y^n.$$

Theorem 1 (Thm.4 of [3])

$$\mathbf{R}_{12}(0) = \frac{y}{J(0)}T(y,1,y^5) \tag{1.10}$$

$$\mathbf{R}_{02}(3) = -q^3\frac{y}{J(0)}T(y^2,1,y^5) \tag{1.11}$$

$$\mathbf{R}_{12}(1) = \mathbf{R}_{02}(2) = \mathbf{R}_{02}(4) = \mathbf{R}_{12}(4) = 0 \tag{1.12}$$

$$\mathbf{R}_{02}(0) + 2\mathbf{R}_{12}(0) = \frac{J(2)J(0)}{J(1)^2} \tag{1.13}$$

$$\mathbf{R}_{02}(1) = q\frac{J(0)}{J(1)} \tag{1.14}$$

$$\mathbf{R}_{12}(2) = q^2\frac{J(0)}{J(2)} \tag{1.15}$$

$$\mathbf{R}_{01}(3) + \mathbf{R}_{02}(3) = q^3\frac{J(1)J(0)}{J(2)^2} \tag{1.16}$$

Theorem 2 (Thm.4.7 of [7])

$$\mathbf{C}_{01}(0) = \frac{J(2)J(0)}{J(1)^2} \tag{1.17}$$

$$\mathbf{C}_{01}(1) = -2q\frac{J(0)}{J(1)} \tag{1.18}$$

$$\mathbf{C}_{12}(1) = q\frac{J(0)}{J(1)} \tag{1.19}$$

$$\mathbf{C}_{12}(2) = -q^2\frac{J(0)}{J(2)} \tag{1.20}$$

$$\mathbf{C}_{01}(3) = q^3\frac{J(1)J(0)}{J(2)^2} \tag{1.21}$$

$$\mathbf{C}_{12}(3) = -q^3\frac{J(1)J(0)}{J(2)^2}. \tag{1.22}$$

$$\textit{and all other functions } \mathbf{C}_{i,i+1}(d) = 0, \quad \textit{where} \quad i = 0 \textit{ or } 1, \quad \textit{are zero.} \tag{1.23}$$

2 Preparation

We define

$$N_k(r) := \sum_{n\geq 0} N_k(r,5,n)q^n,$$

and

$$S_k(b) := \sum_n{}' (-1)^n \frac{q^{n(kn+1)/2+bn}}{1-q^{5n}}$$

where the sum is taken over all non-zero integers n, and

$$T_k(z,\zeta,q) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{\zeta^n\, q^{3n(n+1)/2}}{1-z\,q^n}$$

which is an analytic function of z in every region $0 < r_1 \leq z \leq r_2$, except for simple poles at $z = q^{-n}$. From (3.6) in [6], we have

$$\begin{aligned} N(0) &= \mathbf{P}\left(1-2S_3(4)\right), & M(0) &= \mathbf{P}\left(2S_1(0)+1\right) \\ N(1) &= \mathbf{P}\left(S_3(1)+S_3(4)\right), & M(1) &= \mathbf{P}\left(S_1(1)-S_1(0)\right) \\ N(2) &= -\mathbf{P}S_3(1), & M(2) &= -\mathbf{P}S_1(1) \end{aligned} \tag{2.1}$$

From [10],

$$\mathbf{P}^{(0)} = \frac{J(0)^5}{(y;y)_\infty^6}\left\{\frac{J(2)^4}{J(1)^4} - 3y\frac{J(1)}{J(2)}\right\} \tag{2.2}$$

$$\mathbf{P}^{(1)} = q\frac{J(0)^5}{(y;y)_\infty^6}\left\{\frac{J(2)^3}{J(1)^3} + 2y\frac{J(1)^2}{J(2)^2}\right\} \tag{2.3}$$

$$\mathbf{P}^{(2)} = q^2\frac{J(0)^5}{(y;y)_\infty^6}\left\{2\frac{J(2)^2}{J(1)^2} - y\frac{J(1)^3}{J(2)^3}\right\} \tag{2.4}$$

$$\mathbf{P}^{(3)} = q^3\frac{J(0)^5}{(y;y)_\infty^6}\left\{3\frac{J(2)}{J(1)} + y\frac{J(1)^4}{J(2)^4}\right\} \tag{2.5}$$

$$\mathbf{P}^{(4)} = 5q^4\frac{J(0)^5}{(y;y)_\infty^6} \tag{2.6}$$

We have the components of $S_3(1)$ and $S_3(4)$ from (6.21) and (6.22) of [3] and the components of $S_1(0)$ and $S_1(1)$ from (3.7) of [6].

Finally, we can state the components of $N(r)$ and $M(r)$:
for $k = 0,1,2,3,4$

$$\begin{aligned} N(0)^{(k)} &= -2\delta_{0k}\frac{y}{J(0)}T_3(y,1,y^5) + \frac{\mathbf{P}^{(k)}}{5} \\ &+ \left\{\frac{\mathbf{P}^{(k)}}{5}\left(4\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3}\right) - \mathbf{P}^{(k+3)}\frac{2q^2}{J(1)}\right\}J(0)^2\ , \end{aligned}$$

$$\begin{aligned} N(1)^{(k)} &= \delta_{0k}\frac{y}{J(0)}T_3(y,1,y^5) - \delta_{3k}q^3\frac{y}{J(0)}T_3(y^2,1,y^5) + \frac{\mathbf{P}^{(k)}}{5} \\ &+ \left\{\frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3} + 3y\frac{J(1)^2}{J(2)^3}\right) + \mathbf{P}^{(k+3)}\frac{q^2}{J(1)} - \mathbf{P}^{(k+2)}\frac{q^3}{J(2)}\right\}J(0)^2\ , \end{aligned}$$

$$\begin{aligned} N(2)^{(k)} &= \delta_{3k}q^3\frac{y}{J(0)}T_3(y^2,1,y^5) + \frac{\mathbf{P}^{(k)}}{5} \\ &+ \left\{\frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3} - 2y\frac{J(1)^2}{J(2)^3}\right) + \mathbf{P}^{(k+2)}\frac{q^3}{J(2)}\right\}J(0)^2 \end{aligned} \tag{2.7}$$

where $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise , and

$$
\begin{aligned}
M(0)^{(k)} &= \frac{\mathbf{P}^{(k)}}{5}+\left\{\frac{\mathbf{P}^{(k)}}{5}\left(4\frac{J(2)^2}{J(1)^3}-2y\frac{J(1)^2}{J(2)^3}\right)-\mathbf{P}^{(k+4)}2q\frac{J(2)}{J(1)^2}+\mathbf{P}^{(k+2)}2\frac{q^3}{J(2)}\right\}J(0)^2\,,\\
M(1)^{(k)} &= \frac{\mathbf{P}^{(k)}}{5}+\left\{\frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3}+3y\frac{J(1)^2}{J(2)^3}\right)+\mathbf{P}^{(k+4)}q\frac{J(2)}{J(1)^2}\right.\\
&\qquad\left.-\mathbf{P}^{(k+3)}\frac{q^2}{J(1)}-\mathbf{P}^{(k+2)}\frac{q^3}{J(2)}+\mathbf{P}^{(k+1)}q^4\frac{J(1)}{J(2)^2}\right\}J(0)^2\\
M(2)^{(k)} &= \frac{\mathbf{P}^{(k)}}{5}+\left\{\frac{\mathbf{P}^{(k)}}{5}\left(-\frac{J(2)^2}{J(1)^3}-2y\frac{J(1)^2}{J(2)^3}\right)+\mathbf{P}^{(k+3)}\frac{q^2}{J(1)}\right.\\
&\qquad\left.-\mathbf{P}^{(k+1)}q^4\frac{J(1)}{J(2)^2}\right\}J(0)^2
\end{aligned}
\tag{2.8}
$$

(we take $\mathbf{P}^{(i)}=\mathbf{P}^{(j)}$ when $i\equiv j(\mathrm{mod}5)$)

3 Modular Forms

By (2.7) and (2.8), (1.10)-(1.12) and (1.23) are straightforward. For the rest, let, for $i=1,\ 2,\ 3,\ 4$, H_i be the right hand-sides minus the left hand-sides of (1.13)-(1.16), respectively and let, for $j=1,\ 2,\ 3,\ 4,\ 5,\ 6$, G_j be the r.h.s minus the l.h.s of (1.17)-(1.22) respectively. But, (2.7) and (2.8) also give that

$$
\begin{array}{lll}
G_1=H_1 & G_2=-2G_3 & G_3=H_2\\
G_4=-H_3 & G_5=H_4 & G_6=-H_4
\end{array}
$$

Therefore, we must only show that $H_i\equiv 0$ for $i=1,\ 2,\ 3,\ 4$. In what follows, τ denotes a variable ranging over $\mathbb{H}:=\{z\in\mathbb{C}\ :\ \mathrm{im}z>0\}$ and $q:=\exp(2\pi i\tau)$.

We now define, for $r=1$ and 5, the functions on $\mathbb{H}$

$$\eta(r)=\eta(r,\tau):=\exp(\pi i\tau/12)(q^r;q^r)_\infty$$

and, for k an integer,

$$
\begin{aligned}
s(k)=s(k,\tau) &:= q^{(5/2-k)^2/10}J(q^r;q^5)(q^5;q^5)_\infty\\
&= -\exp(k^2\pi\tau/5)\Theta_{1,1}(k\tau|5\tau).
\end{aligned}
$$

where $\Theta_{1,1}$ is the Mock theta function (see §76 of [12]).
(1.8) gives that

$$s(-k)=-s(k)=s(k+5) \tag{3.1}$$

We set, $i=1,\ 2,\ 3,\ 4$

$$\mathcal{H}_i:=\exp(-\pi i\tau/60)H_i(q^{1/5})$$

so that

$$\mathcal{H}_1 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^6}{s(1)^7} - 11\frac{s(2)}{s(1)^2} - \frac{s(1)^3}{s(2)^4}\right) - \frac{s(2)}{s(1)^2}\eta(5)^2 \tag{3.2}$$

$$\mathcal{H}_2 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^5}{s(1)^6} - 11\frac{1}{s(1)} - \frac{s(1)^4}{s(2)^5}\right) - \frac{1}{s(1)}\eta(5)^2 \tag{3.3}$$

$$\mathcal{H}_3 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^4}{s(1)^5} - 11\frac{1}{s(2)} - \frac{s(1)^5}{s(2)^6}\right) - \frac{1}{s(2)}\eta(5)^2 \tag{3.4}$$

$$\mathcal{H}_4 = \frac{\eta(5)^8}{\eta(1)^6}\left(\frac{s(2)^3}{s(1)^4} - 11\frac{s(1)}{s(2)^2} - \frac{s(1)^6}{s(2)^7}\right) - \frac{s(1)}{s(2)^2}\eta(5)^2 \tag{3.5}$$

From these, we see that

$$\mathcal{H}_1 = \frac{s(2)}{s(1)}\mathcal{H}_2 = \frac{s(2)^2}{s(1)^2}\mathcal{H}_3 = \frac{s(2)^3}{s(1)^3}\mathcal{H}_4 \tag{3.6}$$

Thus, we only show that

$$\mathcal{H}_1 \equiv 0. \tag{3.7}$$

We define the subgroups

$$\Gamma_0(10) := \{A \in SL_2(\mathbb{Z}) \ : \ c \equiv 0 \pmod{10}\}$$
$$\Gamma_1(10) := \{A \in \Gamma_0(10) \ : \ a \equiv d \equiv \mp 1 \ .\!\pmod{10}\}$$

of $SL_2(\mathbb{Z})$. Here, and below, A denotes a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The subgroup $\Gamma_0(10)$ has index 18 in $SL_2(\mathbb{Z})$ (see (1.4.28) in [13]) and $\Gamma_1(10)$ has index 36 in $SL_2(\mathbb{Z})$. We establish (3.7) by showing that $\mathcal{H}_1$ is a modular form on $\Gamma_1(10)$ of weight $1/2$ and that

$$\sum_{\zeta \in \mathcal{F}^*} \operatorname{ord}(\mathcal{H}_1, \zeta, \Gamma_1(10)) > 3/2 \tag{3.8}$$

where $\mathcal{F}^*$ is a fundamental region of $\Gamma_1(10) \cup \{\text{cups}\}$. (See [13] for the definitions of these terms.) We then appeal to the following well-known result.

Lemma 1 (Thm.4.1.4 in [13]) *Suppose that Γ is a subgroup of $SL_2(\mathbb{Z})$ (containing $-I$) of finite index μ. If f is a modular form of weight k on Γ, then either f is identically zero, or*

$$\sum_{\zeta \in \mathcal{F}^*} ord(f, \zeta, \Gamma) = \mu k/12.$$

The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ in the usual way. Suppose $A \in \Gamma_0(10)$. The transformation rules for the η and Θ functions ((74.11) and (81.2) in [12]) give

$$\eta(A\tau) = \varepsilon(A)\sqrt{c\tau + d}\,\eta(\tau) \tag{3.9}$$

where $\varepsilon(A)$ is a root of unity.

$$s(k, A\tau) = (-1)^{kb}\varepsilon(A_{10})^3 \exp(k^2\pi iab/5)\sqrt{c\tau + d}\, s(ak, \tau) \tag{3.10}$$

where $A_{10} = \begin{pmatrix} a & 10b \\ c/10 & d \end{pmatrix}$. Now, with the help of (3.1), (3.9) and (3.10), we see that $\mathcal{H}_i$'s are modular forms of weight 1/2 on $\Gamma_1(10)$

The set of cusps of $\Gamma_1(10)$ may be identified with the set

$$C = C_1 \cup C_2 \cup C_5 \cup C_0$$

where
$C_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$, $C_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$, $C_3 = \left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$,

$C_0 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$. Here $\begin{bmatrix} x \\ y \end{bmatrix}$ denotes the equivalence class of $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}_1^2$ under the equivalence relation

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \pmod{18}$$

for some $A \in \Gamma_1(10)$.

For $r \neq 0$, the cusps in C_r have width $10/r$ and those in C_0 have width 1. We now give the orders of $\eta(1)$, $\eta(5)$, $s(1)$ and $s(2)$ at the cusps in C_r, $(r \neq 0)$,

	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
$\eta(1)$	5/24	5/24	1/12	1/2
$\eta(5)$	1/24	1/24	5/12	5/12
$s(1)$	1/8	1/8	9/20	1/20
$s(2)$	1/8	1/8	1/20	9/20

cusp	0/1	0/3	1/2	1/4	1/5	2/5
ord $\mathcal{H}_1$	$\geq -\frac{25}{12}$	$\geq -\frac{25}{12}$	$\geq -\frac{25}{24}$	$\geq -\frac{25}{24}$	$\geq -\frac{1}{60}$	$\geq \frac{71}{60}$

Therefore

$$\sum_{\zeta \in C \setminus C_0} \operatorname{ord}(\mathcal{H}_1, \zeta, \Gamma_1(10)) \geq -\frac{61}{12} \tag{3.11}$$

where $\mathcal{H}_i = q^{-1/120} H_i(q^{1/5})$ for $i = 1, 2, 3, 4$. By (3.8) and (3.11), we need to show that $\mathcal{H}_1$ has order bigger than

$$\frac{1}{2}\left(\frac{3}{2} + \frac{61}{12}\right) = \frac{89}{24} < 4 - \frac{1}{120}$$

at each of $1/0$ and $3/0$,

$$H_1(q^{1/5}) = \sum_{n\geq 0} (N(0,5,5n) + 2N(1,5,5n) - 3N(2,5,5n))\, q^n - \frac{J(q^2;q^5)(q^5;q^5)_\infty}{j(q;q^5)^2} \quad (3.12)$$

$$\begin{aligned} H_4(q^{1/5}) &= q^{3/5} \left(\sum_{n\geq 0} (2N(0,5,5n+3) - N(1,5,5n+3) - N(2,5,5n+3))\, q^n \right. \\ &\qquad \left. - \frac{J(q;q^5)(q^5;q^5)_\infty}{J(q^2;q^5)^2} \right) \end{aligned} \quad (3.13)$$

If we calculate the first four coefficients of $H_1(q^{1/5})$ and $H_4(q^{1/5})$, we see that they are zero.

From (3.10) we have

$$\text{ord}\left(\mathcal{H}_1, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \Gamma_1(10)\right) = \text{ord}\left(\mathcal{H}_4, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \Gamma_1(10)\right).$$

Consequently,

$$\sum_{\zeta\in C_0} \text{ord}(\mathcal{H}_1, \zeta, \Gamma_1(10)) \geq (4 - \frac{1}{120}) + (4 + \frac{3}{5} - \frac{1}{120}) = \frac{89}{24} \quad (3.14)$$

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Rank ve Crank Modülo 5

Özet

$p(n)$, n'nin kısıtlanmamış ayrışımları sayısını göstersin. Ramanujan'nın ayrışım kongrüansları $p(5n+4)$, $p(7n+5)$ ve $p(11n+6) \equiv \bmod 5,\ 7,$ ve 11, sırasıyla. Bunlar değişik yollarla ispat edildi. Atkin ve Swinnerton-Dyer bu kongrüanslarla birlikte bazı ayrışım ilişkilerini mod 5 ve 7 durumlarında *rank* yöntemiyle ispatladılar, Garvan da benzeri ilişkileri her üç durum için *crank* yöntemini kullanarak ispatladı.

Bu çalışmada, modüler formlar teorisini kullanarak, mod 5 durumda bunların sonuclarına başka bir ispat veriyoruz. Bizim yöntemimiz her nekadar daha karmaşık olsa da, bu yöntem $SL_2(\mathbb{Z})$'in bir kongrüans altgrubu üzerinde yarım tamsayı ağırlıklı Modüler formların Ayrışım Teoride nasıl bir rol oynadığını gösterir. Yöntemimiz mod 7 durumuna da uygulanabilir fakat $\prod_{n=1}^{\infty}(1-q^n)^{-1}$'nın komponentleri açıkca bilinmediğinden mod 11 durumuna uygulanamaz.

A.Bülent EKİN
Ondokuz Mayıs Üniversitesi
Fen-Edebiyat Fakültesi,
Matematik Bölümü,
55139, Samsun-TURKEY

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