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THE HARDY-LITTLEWOOD-SOBOLEV INEQUALITY FOR NON-ISOTROPIC RIESZ POTENTIALS

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Abstract

In this study the inequality of Hardy-Littlewood-Sobolev type are established for non-isotropic generalized Riesz potential depending on λ -distance.

In this paper we establish analogues well known Hardy-Littlewood-Sobolev inequality (see[3]) for Riesz potentials with non-isotropic kernel depended on λ -distance. Note that different problems for convolution type integrals with kernels, depending on λ -distance were considered in [1] and [2].

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$ and $\|x\|_\lambda = \left(|x_1|^{\frac{1}{\lambda_1}} + \dots + |x_n|^{\frac{1}{\lambda_n}} \right)^{|\lambda|}$, $x \in R^n$. The expression $\|x - y\|_\lambda, x, y \in R^n$ is called the λ -distance between the points x and y . It can be seen that for $\lambda_j = \frac{1}{2}$ $j=1,2,\dots,n$ the λ -distance become ordinary Euclidean distance $|x - y|$.

Now we define the generalized non-isotropic Riesz potentials. We set

$$\Lambda_\alpha f(x) = \int_{R^n} \|x - y\|_\lambda^{\alpha-n} f(y) dy \quad 0 < \alpha < n \quad (1)$$

This integral is called non-isotropic or λ -Riesz potential. We show that this potential has a weak (p,q)- type for some p and q in the following sense [3]: There exist a positive constant $C_{p,q}$ independent on function f such that for any $\beta > 0$ the inequality

$$m\{x : |\Lambda_\alpha f(x)| > \beta\} \leq \left(C_{p,q} \frac{\|f\|_p}{\beta} \right)^q \quad (2)$$

is hold. The following theorem is our main result.

Theorem (Hardy-Littlewood-Sobolev type). Let

$$1 \leq p < q < \infty \text{ and } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \quad (3)$$

- a) If $f \in L_p(R^n)$, then the integral $\Lambda_\alpha f$ is absolutely convergent almost everywhere.

b) If $p > 1$, then

$$\|\Lambda_\alpha f\|_{L_q} \leq A_{p,q} \|f\|_{L_p} \quad (4)$$

c) If $f \in L_1(\mathbb{R}^n)$ then potential $\Lambda_\alpha f$ has weak $(1, q)$ type, where $q = 1 - \frac{\alpha}{n}$.

Proof. a) it is obvious that

$$\begin{aligned} \Lambda_\alpha f(x) &= \int_{\mathbb{R}^n} f(x-y) R_1(y) dy + \int_{\mathbb{R}^n} f(x-y) R_\infty(y) dy \\ &= I_1(x) + I_2(x) \end{aligned} \quad (5)$$

where for some positive μ

$$R_1(y) = \begin{cases} \|y\|_\lambda^{\alpha-n} & \text{if } \|y\|_\lambda \leq \mu \\ 0 & \text{if } \|y\|_\lambda > \mu \end{cases}$$

$$R_\infty(y) = \begin{cases} 0 & \text{if } \|y\|_\lambda \leq \mu \\ \|y\|_\lambda^{\alpha-n} & \text{if } \|y\|_\lambda > \mu \end{cases}$$

Applying the generalized Minkowsky inequality we can see that

$$\|I_1\|_{L_p} \leq \|R_1\|_{L_1} \|f\|_{L_p}$$

Note that the norm $\|R_1\|_{L_1}$ can be easily calculated. Namely passing to spherical coordinates by transformation $x_j = (\gamma\theta_j)^{\lambda_j}$, $j = 1, 2, \dots, n$, where θ_j are the coordinates of the point θ on unit sphere, we can see that the jacobian of this transformation is $\gamma^{2|\lambda|-1} \Omega(\theta)$ and $\Omega(\theta)$ depend only on angles. Consequently, denoting $C_1 = \int_{S^{n-1}} \Omega(\theta) d\theta$, where S^{n-1} is the unite sphere in \mathbb{R}^n , we obtain

$$\|R_1\|_{L_1} = \int_{\|y\|_\lambda \leq \mu} \|y\|_\lambda^{\alpha-n} dy \leq C_1 \mu^{\frac{2|\lambda|}{n}\alpha}. \quad (6)$$

Therefore this it follows that

$$\|I_1\|_{L_p} \leq C_1 \mu^{\frac{2|\lambda|}{n}\alpha} \|f\|_{L_p} < \infty \quad (7)$$

By Hödler inequality, we have

$$|I_2(x)| \leq \|R_\infty\|_{L_{p'}} \|f\|_{L_p} \quad (8)$$

where $p' = \frac{p}{p-1}$

Now we will show that $\|R_\infty\|_{L_{p'}}$ is finite. We have

$$\|R_\infty\|_{L_{p'}} = \left[\int_{\|y\|_\lambda > \mu} \|y\|_\lambda^{(\alpha-n)p'} dy \right]^{\frac{1}{p'}} = C_2 \left(\rho^{\frac{2|\lambda|}{n}(\alpha-n)p'+2|\lambda|} \right)^{\frac{1}{p'}} \Bigg|_\mu^\infty.$$

By (3) it follows that

$$\frac{2|\lambda|}{n}(\alpha-n) + \frac{2|\lambda|}{p'} = 2|\lambda| \left(\frac{\alpha}{n} - \frac{1}{p} \right) < 0$$

and that is, $\|R_\infty\|_{L_{p'}} < C_3$ and $I_2(x)$ is also finite. Due to (5) and (7), λ - Riesz potentials is finite almost everywhere.

We now prove the part c)- Obviously, it is sufficient to prove this fact in case $\|f\|_{L_p} = 1$ and with 2β replaced β in (2).

Since $\Lambda_\alpha(x) = I_1(x) + I_2(x)$ in view of (5) we have the inequality

$$m \{x : |\Lambda_\alpha f(x)| > 2\beta\} \leq m \{x : |I_1(x)| > \beta\} + m \{x : |I_2(x)| > \beta\}. \quad (9)$$

Consider the right side of this inequality. We have

$$\beta^p m \{x : |I_1(x)| > \beta\} \leq \int_{\{x:|I_1(x)|>\beta\}} I_1(x)^p dx \leq \int_{R^n} |I_1(x)|^p dx = \|I_1\|_{L_p}^p$$

and therefore

$$m \{x : |I_1(x)| > \beta\} \leq \frac{\|I_1\|_{L_p}^p}{\beta^p}.$$

Since

$$\|I_1\|_{L_p} \leq \|R_1\|_{L_1} \|f\|_{L_p} = \|R_1\|_{L_1}$$

and by (6) the inequality

$$m \{x : |I_1(x)| > \beta\} \leq \frac{\|R_1\|_{L_1}}{\beta^p} \leq C_1 \frac{\mu^{\frac{2|\lambda|\alpha}{n}}}{\beta^p} \quad (10)$$

holds.

Now by (8) we have the inequality

$$|I_2(x)| \leq \|R_\infty\|_{L_{p'}} \|f\|_{L_p} = C_3 \mu^{\frac{2|\lambda|}{n}(\alpha-n) + \frac{2|\lambda|}{p'}} = C_3 \mu^{-\frac{2|\lambda|}{q}}$$

so it follows that if we choose

$$\mu = (C_3^{-1}\beta)^{-\frac{q}{2|\lambda|}}$$

then for all x $|I_2(x)| \leq \beta$ and $m\{x : |I_2(x)| > \beta\} = 0$.

By (10), we have

$$\begin{aligned} m\{x : |\Lambda_\alpha f(x)| > 2\beta\} &< C_1 \left(\frac{\mu^{\frac{2|\lambda|}{n}\alpha}}{\beta} \right)^p = C_1 C_4 \left(\mu^{\frac{2|\lambda|}{n}\alpha + \frac{2|\lambda|}{q}} \right)^p \\ &= C_5 \left(\mu^{2|\lambda|(\frac{\alpha}{n} - \frac{1}{q})} \right)^p = C_5 \left(\mu^{2|\lambda|\frac{1}{p}} \right)^p = C_5 \mu^{2|\lambda|} = C_5 C_6 \beta^{-q} = C_7 \left(\frac{\|f\|_{L_p}}{\beta} \right)^q. \end{aligned}$$

Consequently under condition (3) $\Lambda_\alpha f(x)$ has a weak (p, q) type.

b)- To prove this part we use the Marcinkiewicz Interpolation theorem [3].

By part (c) the operator $\Lambda_\alpha f(x)$ is the weak type (p, q) where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In special case $p = 1$ this operator is the weak type $(1, q)$ where $\frac{1}{q} = 1 - \frac{\alpha}{n}$. Using the Marcinkiewicz Interpolation theorem between (p_0, q_0) and (p_1, q_1) where $p_0 = 1, q_0 = \left(1 - \frac{\alpha}{n}\right)^{-1}, p_1 = p_1, q_1 = \left(\frac{1}{p_1} - \frac{\alpha}{n}\right)^{-1}$. We have that for potential $\Lambda_\alpha f(x)$ hods (4) and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.
The proof is completed. □

Remark. The condition (3) is also the necessary for (4). To prove this we assume that (4) holds for every function $f \in L_p(\mathbb{R}^n)$ and consider the function $g(x) = f(t^\lambda x)$, where $t^\lambda x = (t^{\lambda_1} x_1, \dots, t^{\lambda_n} x_n)$. Then the simple calculations show that

$$\|g\|_{L_p} = \left[\int_{\mathbb{R}^n} |f(t^\lambda x)|^p dx \right]^{1/p} = t^{-\frac{|\lambda|}{p}} \|f\|_{L_p}$$

and

$$\|\Lambda_\alpha g\|_{L_q} = t^{-\left(\frac{|\lambda|}{q} + \alpha \frac{|\alpha|}{n}\right)} \|\Lambda_\alpha f\|_{L_q}.$$

Hence, by (4) we have

$$\|\Lambda_\alpha f\|_{L_q} \leq A_{p,q} t^{|\lambda|[\frac{1}{q} - (\frac{1}{p} - \frac{\alpha}{n})]} \|f\|_{L_p} \quad (11)$$

The contradiction, which can be obtained from this inequality when $t \rightarrow 0$ (if $\frac{1}{q} > \frac{1}{p} - \frac{\alpha}{n}$) and when $t \rightarrow \infty$ (if $\frac{1}{q} < \frac{1}{p} - \frac{\alpha}{n}$) show that (4) holds only for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Note that (4) does not hold for $p = q$. Really from the inequality (11) it may be seen that in the case $p = q$

$$\|\Lambda_\alpha f\|_{L_p} \leq A_{p,q} t^{\frac{|\lambda|}{n}\alpha} \|f\|_{L_p}.$$

But this is possible only when $\alpha = 0$. That is the potential $\Lambda_\alpha f$ can not acting from $L_p(R^n)$ to $L_q(R^n)$.

This remark was added after the referee comments.

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İsotropik Olmayan Riesz Potansiyelleri İçin Hardy-Littlewood Sobolev Tipi Eşitsizlik

Özet

Bu çalışmada λ - uzaklığa bağlı Riesz potansiyeli tanımlanarak bu potansiyeller için Hardy-Littlewood-Sobolev tipinde bir teorem ispatlanmaktadır.

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