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## CHARACTERIZATION OF SOME RINGS BY FUNCTOR $Z^*(.)$

Ayşe Çiğdem Özcan, Abdullah Harmancı

### Abstract

Let  $\underline{X} = \{M : Z^*(M) = 0\}$  and  $\underline{X}^* = \{M : Q \leq P \leq M, P/Q \in \underline{X} \text{ implies } P/Q = 0\}$  be classes of  $R$ -modules. In this note we study the structure of rings  $R$  over which every module  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ .

Let  $R$  be a ring with identity. Throughout all modules will be unital right  $R$ -modules and  $RadM, E(M), Z(M)$  will denote the radical, injective hull and singular submodule of a module  $M$ .  $J(R)$  is the Jacobson radical of  $R$ .

A module  $N$  is called a small *submodule* in a module  $M$  if whenever  $N+L = M$  for some submodule  $L$  of  $M$  we have  $M = L$ . A module  $M$  is said to be *small* if  $M$  is small in  $E(M)$ . Let  $M$  be an  $R$ -module. We set  $Z^*(M) = \{m \in M : mR \text{ is small}\}$  and we define inductively  $Z_n^*(M) : Z_1^*(M) = Z^*(M), Z^*(M/Z_{n-1}^*(M)) = Z_n^*(M)/Z_{n-1}^*(M) (n = 2, 3, \dots)$ . It is well-known that  $Z_2(M) = Z_3(M) = \dots$  for  $Z(M)$ . But it is not known in  $Z_2^*(M) = Z_3^*(M) = \dots$ . In this note we consider the classes  $\underline{X} = \{M : MR\text{- module and } Z^*(M) = 0\}, \underline{X}^* = \{M : MR\text{- module and whenever } Q \leq P \leq M, P/Q \in \underline{X} \text{ implies } P/Q = 0\}$ , following [5]. Since  $RadM$  is the sum of small submodules of  $M$ , then  $RadM \leq Z^*(M)$ .

A class  $\Omega$  of modules is called *s-closed* if  $\Omega$  is closed under submodules and *q-closed* if  $\Omega$  is closed under homomorphic images, and  $\{s, q\}$ -*closed* if  $\Omega$  is s-closed and q-closed. It is known that  $\underline{X}^*$  is  $\{s, q\}$ -closed. Let  $H_{\underline{X}}(M)$  denote the sum of  $\underline{X}^*$ -submodules of  $M$ . Then  $H_{\underline{X}}(M) \in \underline{X}^*, H_{\underline{X}}(M/H_{\underline{X}}(M)) = 0$ , and  $H_{\underline{X}}$  is fully invariant [5], and  $\underline{X} \cap \underline{X}^* = 0$ . It is known that the class  $\underline{X}$  is closed under submodules, direct products, direct sums, essential extensions and module extensions.

In [9] it is proved that if  $R$  is a quasi-Frobenius ring then every module is a direct sum of an injective module and a small module. In this note we show that every module  $M$  over a quasi-Frobenius ring has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ . We also deal with the question: Let  $R$  be a ring such that every module  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ , then  $R$  is quasi-Frobenius?

**Lemma 1.** *Let  $M$  be an  $R$ -module. Then*

- (i) *If  $M$  is small then  $Z^*(M) = M$ ,*
- (ii) *If  $Z^*(M) = M$  then  $M \in \underline{X}^*$ ,*
- (iii) *If  $M$  is semisimple injective then  $M \in \underline{X}$ .*

**Proof.** (i) Clear from definitions.

(ii) Let  $M$  be a module such that  $Z^*(M) = M$ . Assume  $Q \leq P \leq M$  and  $P/Q \in \underline{X}$ . Then  $Z^*(P/Q) = 0$ . Since  $Z^*(M) = M$  and any homomorphic image of a small module is small, then  $P/Q = Z^*(P/Q)$ . Hence  $P/Q = 0$ , and so  $M \in \underline{X}^*$ .

(iii) Assume first  $M$  is simple injective. Let  $0 \neq m \in M$  be such that  $mR$  is small in  $E(mR) = M$ . This is a contradiction for  $mR = M$ . Hence  $Z^*(M) = 0$  and  $M \in \underline{X}$ . Assume  $M$  is semisimple injective. Since  $\underline{X}$  is closed under direct sums, then  $M \in \underline{X}$ .  $\square$

**Lemma 2.** *Let  $R$  be a right perfect ring. Then a module  $M$  is small if and only if  $Z^*(M) = M$ .*

**Proof.** Let  $R$  be a right perfect ring. Assume  $M$  is small module. Let  $0 \neq m \in M$ . Then  $mR$  is small in  $E(M)$  and so in  $E(mR)$ . Hence  $m \in Z^*(M)$  and then  $Z^*(M) = M$ . Conversely suppose that  $Z^*(M) = M$ . Since  $R$  is right perfect and  $Z^*(M) = M$ , then  $Z^*(M) \leq \text{Rad}E(M)$  and  $\text{Rad}E(M)$  is small in  $E(M)$  [1]. Hence  $M$  is small.  $\square$

**Theorem 3.** *Let  $R$  be a right hereditary ring. Then  $\underline{X}^* = \{M : Z^*(M) = M\}$ .*

**Proof.** Let  $R$  be a right hereditary ring and  $M$  a module with  $Z^*(M) = M$ . By Lemma 1(ii),  $M \in \underline{X}^*$ . Assume  $M$  is a module with  $M \in \underline{X}^*$ . Let  $m \in M$  be such that  $m \notin Z^*(M)$ . Then  $mR$  is not small in  $E(mR)$ . Hence there is a submodule  $L$  of  $E(mR)$  such that  $mR + L = E(mR)$ . Since  $R$  is right hereditary, then  $E(mR)/L$  is injective and so the cyclic module  $mR/(mR \cap L)$  is injective. Let  $K/(mR \cap L)$  be a maximal submodule in  $mR/(mR \cap L)$ . Then  $mR/K$  is simple, and injective as a quotient of injective module. By Lemma 1(iii),  $mR/K \in \underline{X}$ . Since  $M \in \underline{X}^*$  and  $\underline{X}^*$  is  $\{s\text{-}q\}$ -closed, then  $mR/K \in \underline{X}^*$ . Hence  $mR/K$  is a zero module. This is a contradiction. Thus  $Z^*(M) = M$ .  $\square$

**Theorem 4.** *Let  $R$  be a ring such that  $R/J(R)$  is a semisimple ring. Then  $\underline{X} = \{M : M \text{ is semisimple injective } R\text{-module}\}$ .*

**Proof.** Let  $M$  be an  $\underline{X}$ -module. Since  $R/J(R)$  is a semisimple ring, then  $\text{Rad}M = MJ(R)$  [1]. Since  $\text{Rad}M$  is contained in  $Z^*(M)$  and  $Z^*(M) = 0$ , then  $\text{Rad}M = 0$ . It follows that  $M$  is semisimple. Since  $M \in \underline{X}$ , implies  $E(M) \in \underline{X}$  then  $E(M)$  is semisimple. Hence  $M = E(M)$  and so  $M$  is injective. Conversely suppose  $M$  is a semisimple injective module. By Lemma 1(iii),  $M \in \underline{X}$ . It completes the proof.  $\square$

**Lemma 5.** *Let  $M$  be a semisimple module with  $M = \bigoplus_{i \in I} M_i$ ,  $M_i$  simple for each  $i \in I$ . Then  $M$  has a decomposition  $M = M_1 \oplus M_2$  where  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ .*

**Proof.** Let  $M_i (i \in I)$  be a simple module. Then  $M_i$  is either small or injective. Let  $M = \bigoplus_{i \in I} M_i$ ,  $M_i$  simple, write  $I = I_1 \cup I_2, I_1 \cap I_2 = \emptyset$  with  $i \in I_1$  implies  $M_i$  is injective and  $i \in I_2$  implies  $M_i$  is small. Then  $M = M_1 \oplus M_2$  with  $M_1 = \bigoplus_{i \in I_1} M_i$  and  $M_2 = \bigoplus_{i \in I_2} M_i$ . It is clear that  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$   $\square$

**Lemma 6.** *Let  $R$  be a quasi-Frobenius ring. Then every  $R$ -module  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ .*

**Proof.** Assume  $R$  quasi-Frobenius ring. Let  $M$  be an  $R$ -module. Then  $M = N_1 \oplus N_2$  with  $N_1$  is injective and  $N_2$  is small by [9].  $N_2 \in \underline{X}^*$  by Lemma 1(ii). Since  $R$  is Noetherian ring and  $N_1$  is injective  $R$ -module, then  $N_1 = \bigoplus_{i \in I} L_i$  with  $L_i$  indecomposable injective [1]. Now if  $H_{\underline{X}}(L_i) = 0$  then  $L_i \in \underline{X}$ . If not,  $L_i/H_{\underline{X}}(L_i) = K_1 \oplus K_2$  where  $K_1$  is injective and  $K_2$  is small. Then  $K_2 \in \underline{X}^*$ , and since  $H_{\underline{X}}(L_i/H_{\underline{X}}(L_i)) = 0$ , then  $K_2 = 0$ . Hence  $L_i/H_{\underline{X}}(L_i)$  is injective, and so  $L_i/H_{\underline{X}}(L_i)$  is projective since  $R$  is a quasi-Frobenius ring. Thus  $L_i/H_{\underline{X}}(L_i) = L_i$  and then  $L_i \in \underline{X}^*$ . Hence for  $i \in I$ , either  $L_i \in \underline{X}$  or  $L_i \in \underline{X}^*$ . Thus  $N_1 = L \oplus K$  with  $L \in \underline{X}$  and  $K \in \underline{X}^*$  as in the proof of Lemma 5. Therefore  $M = L \oplus K \oplus N_2$  with  $L \in \underline{X}, K \oplus N_2 \in \underline{X}^*$ . This completes the proof.  $\square$

**Corollary 7.** *Every module  $M$  over a semisimple ring has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}, M_2 \in \underline{X}^*$ .*

**Proof.** Let  $M$  be a module over a semisimple ring  $R$ . Then  $M$  is semisimple. Corollary is now clear from Lemma 5.  $\square$

In this note we investigate the converse statements of Lemma 5, Lemma 6 and Corollary 7. For this we set<sup>(\*)</sup>.

**Lemma 8.** *We assume  $R$  satisfies<sup>(\*)</sup>. Then  $\underline{X}^*$  is closed under essential extensions.*

**Proof.** Let  $M$  be an  $\underline{X}^*$ -module. It is enough to show  $E(M) \in \underline{X}^*$ . By hypothesis,  $E(M) = M_1 \oplus M_2, M_1 \in \underline{X}, M_2 \in \underline{X}^*$ . Since  $M$  is essential in  $E(M), M_1 \in \underline{X}$  and  $M_1 \cap M \in \underline{X} \cap \underline{X}^* = 0$ , then  $M_1 = 0$ . Hence  $E(M) \in \underline{X}^*$ .  $\square$

**Lemma 9.** *Assume  $\underline{X}^*$  closed under essential extensions. Then every injective module  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ .*

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<sup>(\*)</sup> Every module  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}, M_2 \in \underline{X}^*$ .

**Proof.** Let  $M$  be an injective  $R$ -module. We note that  $H_{\underline{X}}(M) \in \underline{X}^*$  and then by assumption,  $E(H_{\underline{X}}(M)) \in \underline{X}^*$ . Since  $E(M) = E(H_{\underline{X}}(M)) \oplus K$  for some submodule  $K$  of  $E(M)$ , then  $E(M) = H_{\underline{X}}(E(M)) + K$ . Let  $x \in K$  be such that  $xR \in \underline{X}^*$ . Since  $xR \cap M \in \underline{X}^*$ , then  $xR \cap M \leq H_{\underline{X}}(M) \leq E(H_{\underline{X}}(M))$ . Since  $K \cap E(H_{\underline{X}}(M)) = 0$  then  $xR \cap M = 0$  and so  $xR = 0$  for all  $x \in K$  with  $xR \in \underline{X}^*$ . Hence  $H_{\underline{X}}(K) = 0$ . Then  $0 = H_{\underline{X}}(K) = K \cap H_{\underline{X}}(E(M))$  implies  $E(M) = H_{\underline{X}}(E(M)) \oplus K$ . Since  $H_{\underline{X}}(E(M))$  is the largest submodule of  $E(M)$  belonging to  $\underline{X}^*$ , then  $K \in \underline{X}$ . This completes the proof.  $\square$

Let  $M$  be an  $R$ -module and  $A, L$  submodules of  $M$ .  $L$  is called a *supplement* of  $A$  in  $M$  if it is minimal with the property  $A + L = M$ . A submodule  $K$  of  $M$  is called a *supplement* (in  $M$ ) if  $K$  is a supplement of some submodule of  $M$ . It is easy to check that  $L$  is a supplement of  $A$  in  $M$  if and only if  $M = A + L$  and  $A \cap L$  is small in  $L$ .  $M$  is called a *supplemented module* if every submodule has a supplement in  $M$ . The following lemma is in [6]. We prove for the sake of completeness.

**Lemma 10.** *Let  $M$  be a supplemented module. Then  $M$  has a decomposition  $M = M_1 \oplus M_2$  with  $M_1$  semisimple and  $RadM_2$  is essential in  $M_2$ .*

**Proof.** Let  $M_1$  be a submodule of  $M$  such that  $RadM \oplus M_1$  is essential in  $M$ . Since  $M$  is supplemented, then there exists a submodule  $M_2$  of  $M$  such that  $M = M_1 + M_2$  and  $M_1 \cap M_2$  is small in  $M_2$ . Hence  $M_1 \cap M_2$  is submodule of both  $RadM$  and  $M_1$ . It follows that  $M = M_1 \oplus M_2$ , and then  $RadM = RadM_2$  is essential in  $M_2$ .  $M_1$  is semisimple because  $M_1 \cap RadM = 0$  and  $M$  is supplemented.  $\square$

**Lemma 11.** *We assume  $\underline{X}^*$  is closed under essential extensions. Then every supplemented module  $M$  has a decomposition  $M = M_1 \oplus M_2$ ,  $M_1 \in \underline{X}$ ,  $M_2 \in \underline{X}^*$ .*

**Proof.** Let  $M$  be a supplemented module. Then  $M = M_1 \oplus M_2$  with  $M_1$  semisimple and  $RadM_2$  is essential in  $M_2$  by Lemma 10. By Lemma 5,  $M_1 = N \oplus K$ ,  $N \in \underline{X}$ ,  $K \in \underline{X}^*$ . Also  $RadM_2 \in \underline{X}^*$ . By hypothesis,  $M_2 \in \underline{X}^*$ . Then  $M = N \oplus K \oplus M_2$ ,  $N \in \underline{X}$ ,  $K \oplus M_2 \in \underline{X}^*$ .  $\square$

**Proposition 12.** *Let  $R$  be a ring such that every module has a projective cover (i.e. right perfect ring). Then the following are equivalent.*

- (1)  $\underline{X}^*$  is closed under essential extensions.
- (2)  $R$  satisfies  $(^*)$ .

**Proof.** We combine Lemma 8, Lemma 4.40 of [7] and Lemma 11 to prove the equivalence of (1) and (2).  $\square$

**Theorem 13.** *Let  $R$  be a right hereditary ring. Then  $R$  is a right perfect and  $R$  satisfies  $(^1)$  if and only if  $R$  is a right H-ring and  $\underline{X}^*$  is closed under essential extensions.*

**Proof.** Oshiro [8] class a ring  $R$  a right H-ring if every right  $R$ -module is a direct sum of an injective module and a small module. Now, if  $R$  is a right hereditary, right perfect ring satisfying  $(^1)$  then by Lemma 2 and Theorem 3,  $\underline{X}^* = \{M : Z^*(M) = M\} = \{M : M \text{ is small}\}$ , and by Theorem 4  $\underline{X} = \{M : M \text{ is semisimple injective}\}$ . Let  $M$  be an  $R$ -module. By hypothesis,  $M = M_1 \oplus M_2$  with  $M_1 \in \underline{X}$  is injective and  $M_2 \in \underline{X}^*$  is small. Then  $R$  is a right H-ring. By Lemma 8, we have  $\underline{X}^*$  is closed under essential extensions. Assume  $R$  is a right H-ring and  $\underline{X}^*$  is closed under essential extensions. Then  $R$  is a right perfect. By Proposition 12, every module  $M$  has a decomposition  $M = M_1 \oplus M_2, M_1 \in \underline{X}$  and  $M_2 \in \underline{X}^*$ . This completes the proof.  $\square$

**Example 14.** The ring of integers is a (right) hereditary ring. It is not a quasi-Frobenius ring. Let  $K$  be a field and  $G$  a finite group such that the characteristic of  $K$  divides the order of  $G$ . Then by Mascke's Theorem [10] the group ring  $KG$  is not semisimple but a quasi-Frobenius ring [2, Proposition 9.6]. Then the following lemma shows that the quasi-Frobenius ring  $KG$  is not right hereditary.

**Lemma 15.** *Let  $R$  be a quasi-Frobenius ring. Then  $R$  is a right hereditary if and only if  $R$  is semisimple.*

**Proof.** Let  $R$  be a quasi-Frobenius ring. Assume  $R$  is semisimple. Then every  $R$ -module, in particular, every right ideal of  $R$  is projective [1]. Hence  $R$  is right hereditary. Conversely, suppose that  $R$  is right hereditary. Let  $x \in Z(R)$ . Then  $xR$  is a projective  $R$ -module. Since  $xR \cong R/r(x)$  then the essential right ideal  $r(x)$  is a direct summand of  $R$ . Hence  $xR = 0$ . It follows that  $Z(R) = 0$ . Since  $R$  is a quasi-Frobenius ring, then  $J(R) = Z(R)$  by [8, Theorem 4.3] and  $R$  is artinian. Hence  $R$  is semisimple.  $\square$

**Remark.** Let  $R$  be a ring. Then every direct sum of small modules is small if and only if for every injective module  $M, RadM$  is small in  $M$  [9]. In this case  $Z^*(M) = M$  if and only if  $M$  is small. To prove this only note that  $Z^*(M) = M \cap Rad(E(M))$  for any module  $M$ .

**Theorem 16.** *Let  $R$  be a right hereditary ring. Then the following are equivalent.*

- (1)  $R$  is a right perfect ring and satisfies  $(^1)$ ,
- (2)  $R/J(R)$  is semisimple and direct sum of small modules is small and  $R$  satisfies  $(^1)$ ,
- (3)  $R$  is a quasi-Frobenius ring,
- (4)  $R$  is a semisimple ring,
- (5)  $R$  is a right self-injective ring.

**Proof.** (1)  $\implies$  (2) Then  $R/J(R)$  is semisimple and for every injective  $R$ -module  $M$ ,  $RadM$  is small in  $M$  [1]. Hence direct sum of small modules is small by remark.

(2)  $\implies$  (3) Let  $M$  be an  $R$ -module. Then by (2)  $M = M_1 \oplus M_2$ ,  $M_1 \in \underline{X}$ ,  $M_2 \in \underline{X}^*$ . Since  $R/J(R)$  is semisimple, then  $M_1$  is semisimple injective by Theorem 4. Since  $R$  is a right hereditary and direct sum of small modules is small, then  $M_2$  is small by remark. Thus  $R$  is a right H-ring. We write  $R = I_1 \oplus I_2$  where  $I_1 \in \underline{X}$ ,  $I_2 \in \underline{X}^*$ . Then  $I_1$  is injective by Theorem 4, and  $E(R) = I_1 \in E(I_2)$ . Since  $I_2 \in \underline{X}^*$  then  $E(I_2) \in \underline{X}^*$  by Lemma 8, and so  $E(I_2)$  is small. Hence  $E(I_2) = 0$ . It follows that  $H_{\underline{X}}(R) = 0$  and  $R$  is a right self-injective. Since  $J(R)$  is  $\underline{X}^*$ -submodule of  $M$ , then  $J(R) = 0$ . It is clear that every non-zero right ideal of  $R$  is injective and then direct summand of  $R$ . Hence  $Z(R) = 0$ . By [8, Theorem 4.3]  $R$  is quasi-Frobenius ring.

(3)  $\implies$  (4) By Lemma 15.

(4)  $\implies$  (5) Clear.

(5)  $\implies$  (4) Let  $R$  be a right hereditary ring. Assume  $R$  right self-injective. Let  $M$  be a non-zero  $R$ -module and  $0 \neq m \in M$ . Then  $mR \cong R/r(m)$  is injective. Hence  $M$  contains a non-zero injective  $R$ -module. By [3, Lemma 15.10]  $R$  is a semisimple ring.

(4)  $\implies$  (1) Clear.  $\square$

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**$Z^*(.)$  Yardımıyla Bazı Halkaların Karakterizasyonu**

**Özet**

$\underline{X} = \{M : Z^*(M) = 0\}$  ve  $\underline{X}^* = \{M : Q \geq M, P/Q \in \underline{X} \text{ ise } P/Q = 0 \text{ dir.}\}$  modüllerin, sınıfları olsun. Bu çalışmada bir  $R$  halkası için her  $R$ -modül  $M$ ,  $M = M_1 \oplus M_2$ ,  $M_1 \in \underline{X}$   $M_2 \in \underline{X}^*$  olacak şekilde bir ayrışımına sahipse  $R$ 'nin yapısı belirleniyor.

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