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ON ONE SIDED (σ,τ)-LIE IDEALS IN PRIME RINGS

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Abstract

In this paper, we proved some results for one-sided (σ,τ)-Lie ideals in prime rings.

1. Introduction

Let R be a ring and U an additive subgroup of R, σ and $\tau : R \to R$ two mappings. In [5] the following definitions were given: (i) $[U,R]_{\sigma,\tau} \subset U$ then U is called (σ,τ)-right Lie ideal of R (ii) U is (σ,τ)-left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$ (iii) U is said to be (σ,τ)-Lie ideal of R if U is both a (σ,τ)-left Lie ideal of R and a (σ,τ)-right Lie ideal of R, where the commutator $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ for $x,y \in R$.

In this paper the following results are proved. Let R be a prime ring and $\sigma,\tau \in AutR$, the set of automorphisms of R. (1) Let U be a (σ,τ)-left Lie ideal of R. If $[R,U]_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $\sigma(u) = \tau(u)$, for all $u \in U$ or R is commutative. (2) Let U be a (σ,τ)-left Lie ideal such that $[U,U]_{\sigma,\tau} = 0$ and $[U,U] = 0$.Then $U \subset Z$. (3) Let U be a (σ,τ)-left Lie ideal of R such that $\tau(u) \neq \sigma(u)$ and $\tau(v) + \sigma(v) \notin Z$, for some $u,v \in U$. (a) There exist a nonzero left ideal A of R and a nonzero right ideal B of R such that $[R,A]_{\sigma,\tau} \subset U$ and $[R,B]_{\sigma,\tau} \subset U$; but $[R,A]_{\sigma,\tau} \not\subset Z$ and $[R,B]_{\sigma,\tau} \not\subset Z$. (b) Suppose a, $b \in R$ such that $aUb = 0$. Then $a = 0$ or $b = 0$. (4). Let R be of characteristic not 2. Suppose U is a nonzero (σ,τ)-right Lie ideal of R such that $U \subset Z$. Then $\sigma = \tau$ or R is commutative. (5) Let R be of characteristic not (2). Suppose U is a nonzero (σ,τ)-Lie ideal of R such that $U \subset C_{\sigma,\tau}$. Then $\sigma = \tau$ or R is commutative.

Throughout this paper R will be a prime ring, $\sigma,\tau \in AutR$, and Z, the center of R, $C_{\sigma,\tau} = \{c \in R : c\sigma(x) = \tau(x)c, \forall x \in R\}$ and C the extended centroid of R (See [7] and [4,p20-31] for the notion of the extended centroid). We will often use the identities: (i) $[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = [y,\sigma(z)] + [x,z]_{\sigma,\tau}y$ and (ii) $[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)$.

2. Results

Lemma 1. *[6, Lemma 3] Let R be a prime ring. If $ab, b \in C_{\sigma,\tau}$ for $a, b \in R$ then $a \in Z$ or $b = 0$.*

Lemma 2. *[1, Lemma 4] Let R be a prime ring and $(0) \neq U$ a (σ,τ)-left Lie ideal of R such that $U \subset C_{\sigma,\tau}$ then $U \subset Z$.*

Lemma 3. *[2, Lemma 3] Let R be a prime ring and $a \in R$ such that $aU = (0)$ (or $Ua = (0)$). (i) if U is a (σ,τ)-left Lie ideal of R then $a = 0$ or $U \subset Z$. (ii) if U is a (σ,τ)-right Lie ideal of R then $a = 0$ or $U \subset C_{\sigma,\tau}$.*

Lemma 4. *[3, Lemma 2.3] Let R be a prime ring and d, f, g and h be derivations of R. Suppose that*

$$d(x)g(y) = h(x)f(y) \text{ for all } x, y \in R.$$

If $d \neq 0$ and $f \neq 0$ then there exists $\lambda \in C$ such that $g(x) = \lambda f(x)$ and $h(x) = \lambda d(x)$ for all $x \in R$.

Lemma 5. *Let U be a (σ,τ)-left Lie ideal of R. If $U \subset Z$ then $\sigma(u) = \tau(u)$, for all $u \in U$ or R is commutative.*

Proof. For all $x \in R, u \in U, [x,u]_{\sigma,\tau} \in U$. Therefore $\sigma(u) - \tau(u), [x,u]_{\sigma,\tau} \in Z$. Then $[x,u]_{\sigma,\tau} = x\sigma(u) - \tau(u)x = x(\sigma(u) - \tau(u)) \in Z$. By the primeness of R, we conclude $\sigma(u) = \tau(u)$, for all $u \in U$ or R is commutative. □

Theorem 1. *Let R be a prime ring and U be (σ,τ)-left Lie ideal of R. If $[R,U]_{\sigma,\tau} \subset C_{\sigma,\tau}$, then $\sigma(u) = \tau(u)$, for all $u \in U$ or R is commutative*

Proof. For all $x \in R, u \in U, [\tau(u)x, u]_{\sigma,\tau} = \tau(u)[x,u]_{\sigma,\tau} + [\tau(u), \tau(u)]x = \tau(u)[x,u]_{\sigma,\tau} \in C_{\sigma,\tau}$. By Lemma 1 we have for any $u \in U, u \in Z$ or $[x,u]_{\sigma,\tau} = 0$. That is, U is the union of its additive subgroups $L = \{u \in U : u \in Z\}$ and $K = \{u \in U : [R,u]_{\sigma,\tau} = 0\}$. Since a group cannot be the union of two proper subgroups we arrive at $U = L$ or $U = K$. If $U = K$ then $0 = [xy, u]_{\sigma,\tau} = x[y, \sigma(u)] + [x,u]_{\sigma,\tau}y = x[y,\sigma(u)]$, for all $x, y \in R$ and all $u \in U$. Since R is prime we have $U \subset Z$. By Lemma 5 we prove the theorem. □

Example. *In [2], N. Aydın, and H. Kandamar, proved that if U is (σ,τ)-Lie ideal and $a \in R, [a,U] = 0$ then $a \in Z$ or $U \subset Z$. The following easy example shows that this is not the case when U is a (σ,τ)-left Lie ideal of R.*

Let $R=\left\{\begin{pmatrix} x & y \\ & z^t \end{pmatrix} : x,y,z,t\in I,\ \textit{the set of integers}\right\}$ *and* $U=\left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x,y\in I\right\}$. *Let* $\tau : R\to R, \tau(x)=b\times b$, *where* $b=\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\in R$, *then* τ *is an automorphism of* $R.U$ *is a* $(1,\tau)$*-left Lie ideal of* R *such that* $U\not\subset Z$. *If* $a=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\in R$, *then* $a\notin Z$ *and* $[a,U]=0$.

From now on, we will assume that $\sigma\neq\tau$ *on* (σ,τ)*-left Lie ideal* U *of* R.

Lemma 6. *Let* R *be a prime ring and* U *be* (σ,τ)*-left Lie ideal of* R. *Suppose there exists* $a\in R$ *such that* $[a,U]=0$, *then* $\tau(u)+\sigma(u)\in Z$, *for all* $u\in U$ *or* $a\in Z$.

Proof. Assume that $a\notin Z$. From the definition of U for all $x\in R$ and all $u\in U$. $[\tau(u)x,u]_{\sigma,\tau}=\tau(u)[x,u]_{\sigma,\tau}+[\tau(u),\tau(u)]x=\tau(u)[x,u]_{\sigma,\tau}\in U$. Therefore $0=[\tau(u)[x,u]_{\sigma,\tau},a]=\tau(u)[[x,u]_{\sigma,\tau},a]+[\tau(u),a][x,u]_{\sigma,\tau}=[\tau(u),a][x,u]_{\sigma,\tau}$. Consequently,

$$[\tau(u),a][x,u]_{\sigma,\tau}=0,\ \text{for all } x\in R \text{ and all } u\in U. \tag{1}$$

Taking xy for x in (1) we get $[\tau(u),a]x[y,\sigma(u)]=0$ for all $x,y\in R$ and all $u\in U$. The primeness of R implies that for any $u\in U$, either $[\tau(u),a]=0$ or $u\in Z$. It implies that $[\tau(u),a]=0$. That is,

$$[\tau(U),a]=0. \tag{2}$$

On the other hand, for $u\in U, x,y\in R$, expanding $0=[[x,u]_{\sigma,\tau},a]$ and using (2) we have

$$\tau(u)[x,a]=x\sigma(u)a-ax\sigma(u)\ \text{for all } x\in R,\ \text{and all } u\in U. \tag{3}$$

Replacing xby v, $v\in U$, in (3) we arrive at $U[\sigma(v),a]=0$, for all $v\in U$. By Lemma 3(i) we obtain

$$[\sigma(U),a]=0. \tag{4}$$

Considering (3) together with (4), one obtains

$$0=[x,a]\sigma(u)+\tau(u)[a,x]\ \text{for all } x\in R,\ \text{and all } u\in U. \tag{5}$$

Replacing x by xy in (5) and using (5) we have

$$[x,\tau(u)][y,a]=[a,x][y,\sigma(u)]\ \text{for all } x\in R, \textit{all } u\in U. \tag{6}$$

Now let $d(x)=[x,\tau(u)], g(y)=[y,a], h(x)=[a,x]$ and $f(y)=[y,\sigma(u)]$ be derivations of R. Moreover, $d(x)g(y)=h(x)f(y)$ by (6). If $d=0$ and $f=0$ then it is clear that $\sigma(u)+\tau(u)\in Z$ for all $u\in U$. Therefore we may assume that $d\neq 0$ and $f\neq 0$. Then by Lemma 4, (6) implies that there exists $\lambda\in C$ such that

$$\lambda[x,\sigma(u)] = [x,a] = \lambda[\tau(u),x] \text{ for all } x \in R \tag{7}$$

Therefore, from (7) we have $\tau(u)+\sigma(u) \in Z$, for all $u \in U$. □

Lemma 7. *Let R be a prime ring and U be (σ,τ)-left Lie ideal of R. Suppose there exists $a \in R$ such that $[a,U]_{\sigma,\tau}=0$ and $[a,U]=0$, then $\tau(u)+\sigma(u) \in Z$ for all $u \in U$ or $a=0$.*

Proof. By assumption, there exists a $(0 \neq)u_0 \in U$ such that $\sigma(u_0) \neq \tau(u_0)$. That is, $\sigma(u_0)-\tau(u_0) \neq 0$. By Lemma 6, we have $a \in Z$ or $\tau(u)+\sigma(u) \in Z$, for all $u \in U$. If $a \in Z$ then $0=[a,u_0]_{\sigma,\tau}=a\sigma(u_0)-\tau(u_0)a=a(\sigma(u_0)-\tau(u_0))$. Since R is prime we have $a=0$. □

Theorem 2. *Let R be a prime ring of characteristic not 2 and U be a (σ,τ)-left Lie ideal of R such that $[U,U]_{\sigma,\tau}=0$ and $[U,U]=0$. Then $U \subset Z$.*

Proof. Suppose $U \not\subset Z$. Then by Lemma 7 we get $\tau(u)+\sigma(u) \in Z$ for all $u \in U$. $[xv,u]_{\sigma,\tau} = x[v,u]_{\sigma,\tau}+[x,\tau(u)]v = [x,\tau(u)]v \in U$. By hypothesis we also have $0 = [w,[x,\tau(u)]v] = [w,[x,\tau(u)]]v$, for all $x \in R$ and all $u,v,w, \in U$. Therefore we have $[w,[x,\tau(u)]]U=0$. By Lemma 3 (i), we obtain $[w,[\tau(u),x]]=0$, for all $u,v \in U$ and all $x \in R$. Now let I_w and $I_{\tau(u)}$ be two inner derivations determined by w and $\tau(u)$ respectively. Then it implies that $I_w I_{\tau(u)}(R)=0$. By [8, Theorem 1] we arrive at $U \subset Z$. A contradiction. □

Lemma 8. *Let R be a prime ring and U be both a (σ,τ)-left Lie ideal of R and a subring of R. Then either $\tau(u)+\sigma(u) \in Z$, for all $u \in U$ or U contains a nonzero left ideal of R and a nonzero right ideal of R.*

Proof. Suppose that for $u_0 \in U, \sigma(u_0)+\tau(u_0) \notin Z$. Then for $x \in R, v \in U, [xu_0,v]_{\sigma,\tau} = x[u_0,\sigma(v)]+[x,v]_{\sigma,\tau}u_0 \in U$. Then second member of this is in U (since U is both (σ,τ)-left Lie ideal and subring). And os we have

$$x[u_0,\sigma(v)] \in U \text{ for all } v \in U \text{ and all } x \in R$$

We have shown that the left ideal $R[u_0,\sigma(U)]$ is in U. If $R[u_0,\sigma(U)]=0$, by the primeness of R we obtain $[\sigma^{-1}(u_0),U]=0$. By Lemma 6, we have $u_0 \in Z$. And so, $\sigma(u_0)+\tau(u_0) \in Z$ gives a contradiction. Similarly, using the identity $[ux,v]_{\sigma,\tau}=u[x,v]_{\sigma,\tau}+[u,\tau(u)]x \in U$ one can obtains that U contains a nonzero right ideal of R. □

Theorem 3. *Let R be a prime ring. Let U be a (σ,τ)-left Lie ideal of R such that $\tau(v)+\sigma(v)\notin Z$, for some $v\in U$. Then there exist a nonzero left ideal A of R and a nonzero right ideal B of R such that $[R,A]_{\sigma,\tau}\subset U$ $[R,B]_{\sigma,\tau}\subset U$ but $[R,A]_{\sigma,\tau}\not\subset Z$ and $[R,B]_{\sigma,\tau}\not\subset Z$.*

Proof. Let $T=\{x\in R:[R,x]_{\sigma,\tau}\subset U\}$. By the previous note, Theorem 1 in [2], we know T is both a (σ,τ)-left Lie ideal of R and a subring of R such that $U\subset T$. Since $U\not\subset Z$ we have $T\not\subset Z$. By Lemma 8, T contains a nonzero left ideal A of R and a nonzero right ideal B of R. From the definition of T, we obtain $[R,A]_{\sigma,\tau}\subset U$ and $[R,B]_{\sigma,\tau}\subset U$. If $[R,A]_{\sigma,\tau}\subset Z$ then for $x\in R$ and $a\in A,[\tau(a)x,a]_{\sigma,\tau}=\tau(a)[x,a]_{\sigma,\tau}\in Z$. And so, we arrive at $a\in Z$ or $[x,a]_{\sigma,\tau}=0$. If $[x,a]_{\sigma,\tau}=0$, for all $x\in R$ then replacing x by xy, we obtain $[x,\tau(a)]y=0$ for all $x,y\in R$. The primeness of R implies that $a\in Z$. Therefore we have $A\subset Z$. Then for all $x,y\in R$ and all $a\in A, 0=[x,ya]=[x,y]a$, this implies that $A=(0)$ or R is commutative. So this is a contradiction to $\tau(u)+\sigma(u)\notin Z$ for some $u\in U$. Similarly, using the identity $[x\sigma(b),b]_{\sigma,\tau}=[x,b]_{\sigma,\tau}\sigma(b)$ one can prove easily that $[R,B]_{\sigma,\tau}\notin Z$. □

Theorem 4. *Let R be a prime ring. Let U be a (σ,τ)-left Lie ideal of R such that $\tau(v)+\sigma(v)\notin Z$, for some $v\in U$ and $a,b\in R$. If $AUb=0$. Then $a=0$ or $b=0$.*

Proof. Assume $b\neq 0$. By Theorem 3, there exists a nonzero right ideal B of R such that $[R,B]_{\sigma,\tau}\subset U$, but $[R,B]_{\sigma,\tau}\not\subset Z$. Therefore, for all $x\in R$ and all $s\in B$, $a[x,s]_{\sigma,\tau}b=0$. Replacing x by xy, we obtain $0=ax[y,s]_{\sigma,\tau}b+a[x,\tau(s)]yb$. In this equation, taking $ub, u\in U$, for x we get

$$a[ub,\tau(s)]yb=0 \text{ for } \forall y\in R,\forall u\in U,\forall s\in B.$$

Since R is prime and $b\neq 0$, we have $0=a[ub,\tau(s)]=aub\tau(s)-a\tau(s)ub=-a\tau(s)ub$. It implies that $a\tau(B)RUb=0$ because B is a right ideal of R. By Lemma 3(i) and since $b\neq 0, Ub\neq 0$. Thus we have $a\tau(B)=0$ since R is prime. Then $0=a[x,s]_{\sigma,\tau}b=ax\sigma(s)b-a\tau(s)xb=ax\sigma(s)b$ for all $x\in R$ and all $s\in B$. That is

$$aR\sigma(B)b=0$$

Since $R\sigma(B)$ is a nonzero ideal of R and $b\neq 0$, the primeness of R implies that $a=0$. □

Lemma 9. *Let R be a prime ring of characteristic not 2. Suppose U is a nonzero (σ,τ)-right Lie ideal of R such that $U\subset Z$. Then $\sigma=\tau$ or R is commutative*

Proof. Assume that R is not commutative. For all $x\in R$ and all $u\in U$, $[u,x]_{\sigma,\tau}=u\sigma(x)-\tau(x)u=u(\sigma(x)-\tau(x))\in Z$. Since R is prime we have $u=0$ or $\sigma(x)-\tau(x)\in Z$, for all $x\in R$ and all $u\in U$. Since $U\neq(0)$ we get $\sigma(x)-\tau(x)\in Z$, for all $x\in R$. Hence for all $x,y\in R$, $0=[\sigma(x)-\tau(x),y]=[\sigma(x),y]-[\tau(x),]$, from which we get

$$[\sigma(x), y] - [\tau(x), y] = 0 \text{ for all } x, y \in R. \tag{8}$$

Replacing x by x^2 in (8) and using (8) and char $R \neq 2$, we obtain $(\sigma(x)-\tau(x))[\sigma(x), y] = 0$. Since R is prime we get $\sigma(x) = \tau(x)$ or $x \in Z$, for any $x \in R$. Therefore R is the union of its additive subgroups $\{x \in R : \sigma(x) = \tau(x)\}$ and $\{x \in R : x \in Z\}$. Since a group cannot be the union of two proper subgroups and we have assumed that R is not commutative, it follows that $\sigma(x) = \tau(x)$, for all $x \in R$. □

Theorem 5. *Let R be a prime ring of characteristic not 2. Suppose U is a nonzero (σ,τ)-Lie ideal of R such that $U \subset C_{\sigma,\tau}$. Then $\sigma = \tau$ or R is commutative*

Proof. By Lemma 2 we have $U \subset Z$. And so, by Lemma 9 the proof of theorem is completed. □

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Asal Halkalarda Tek Yanlı (σ, τ)- Lie İdealler

Özet

Bu makalede, asal halkalarda tek yanlı (σ, τ)-Lie idealler için bazı sonuçlar ispatlanmıştır.

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