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Yıldıray OZAN

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Abstract

In this paper, we will study finite algebraic group actions on real algebraic sets and compare the topological quotient X/G with the algebraic quotient $X//G$. We will give a different and shorter proof of a result of Procesi and Schwarz, stating that if the order of the group G, acting algebraically on a real algebraic set X, is odd then X/G is equal to $X//G$. In the case of even order groups, we will a give sufficient condition (and a necessary condition in the case $G = \mathbb{Z}_2$) for the X/G to be equal to $X//G$.

1. Introduction and Preliminaries

The problem of real algebraic realization of topological or smooth objects has been studied by many authors. In [11], Seifert showed that any closed smooth submanifold $M \subseteq \mathbb{R}^n$ with trivial normal bundle is isotopic to a nonsingular component of a real algebraic set $X \subseteq \mathbb{R}^n$. Nash showed that every closed smooth manifold is diffeomorphic to a component of a nonsingular real algebraic set in some $\mathbb{R}^N$ ([7]). Later, Tognoli proved that any closed smooth manifold is diffeomorphic to a nonsingular real algebraic set in some $\mathbb{R}^N$ ([13]). In [2, 3] Akbulut and King improved Tognoli's result by showing that any closed smooth submanifold of $\mathbb{R}^n$ can be isotoped to the nonsingular points of an algebraic subset of $\mathbb{R}^n$. Dovermann and Masuda showed that, in some cases, smooth manifolds with group actions can be realized as equivariant nonsingular algebraic sets ([6]).

In this paper, we will study finite algebraic group actions on real algebraic sets and compare the topological quotient X/G with the algebraic quotient $X//G$. If the order of the group G, acting algebraically on a real algebraic set X is odd, then X/G is canonically equal to $X//G$ (Theorem 2.1). When the order $|G|$ is even, in general, this is not true (see the counterexample after Theorem 2.1). In the case of even order groups, we will a give sufficient condition (and a necessary condition in the case $G = \mathbb{Z}_2$) for the X/G to be equal to $X//G$ (Theorem 2.2 and Theorem 2.3).

In [9] Procesi and Schwarz had proved Theorem 2.1.a in the case of linear G actions. However, the proof we give is shorter and does not require linear G actions.

Definition 1.1. *1) Let $X \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^m$ be semialgebraic sets. A map $F : X \longrightarrow Z$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i = 1, \ldots, m$, such that each g_i vanishes nowhere on X and*

$$F = (f_1/g_1, \ldots, f_m/g_m).$$

We say X and Z are isomorphic if there are entire rational maps $F : X \longrightarrow Z$ and $G : Z \longrightarrow X$ such that $FoG = id_Z$ and $GoF = id_X$.

2) Let $X \subseteq \mathbb{R}^n$ be a semialgebraic set and G be a finite group acting on X. Then G is said to be acting algebraically on X, if for each $g \in G$ the map $g : X \longrightarrow X$, $x \longmapsto g \cdot x$ is the restriction of some polynomial map $P_g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Remark. By the last theorem in Section 9 in [6], in the case that X is an algebraic set, any algebraic G action on X is equivalent to a linear action if we are willing to replace X with an isomorphic copy of it possibly in a larger Euclidean space. Nevertheless, our proofs work in the polynomial case and hence we will assume that the action is given by polynomial maps.

Let X be an algebraic set in $\mathbb{R}^n$. Suppose that G is a finite group acting algebraically on X. Let $S = \mathbb{R}[x_1, \ldots, x_n]/J(X)$ be the ring of polynomial functions on X where $J(X) \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is the ideal of vanishing polynomials on X. We define a G action on S as follows: for $g \in G$ and $f \in S$, let $g \cdot f = fog^{-1}$. Consider the subring T of S defined by

$$T = S^G = \{f \in S \mid g \cdot f = f, \ \forall g \in G\}.$$

Both T and S are $\mathbb{R}$ algebras. Moreover, it is well known that S is an algebraic extension of T and therefore T is also a finitely generated $\mathbb{R}$ algebra (Exercise 5.12 and Proposition 7.8 in [4]). Say T is generated by $y_1, \ldots, y_m$ over the reals. Consider the complexification $T_\mathbb{C}$ and $S_\mathbb{C}$ of T and S defined by

$$T_\mathbb{C} = T \otimes_\mathbb{R} \mathbb{C} \quad \text{and} \quad S_\mathbb{C} = S \otimes_\mathbb{R} \mathbb{C}.$$

Clearly, these are finitely generated $\mathbb{C}$ algebras and $S_\mathbb{C}$ is the ring of polynomial functions on the complexification $X_\mathbb{C}$ of X. The above action of G on S extends to $S_\mathbb{C}$, linearly over $\mathbb{C}$. With this definition of G action, immediately, we have that $S_\mathbb{C}^G = T_\mathbb{C}$.

Consider the maps $F_\mathbb{C} : X_\mathbb{C} \longrightarrow \mathbb{C}^m$ and $F = F_{\mathbb{C}|X} : X \longrightarrow \mathbb{R}^m$, where both are given by

$$x \longrightarrow (y_1(x), \ldots, y_m(x)).$$

If Z is the complex algebraic set corresponding to the $\mathbb{C}$ algebra $T_\mathbb{C}$ (i.e. the embedding of the maximal spectrum of $T_\mathbb{C}$ into $\mathbb{C}^m$ via $F_\mathbb{C}$), then $F_\mathbb{C} : X_\mathbb{C} \longrightarrow Z$ is the map corresponding to the inclusion of $\mathbb{C}$ algebras $i : T_\mathbb{C} \longrightarrow S_\mathbb{C}$ and $F_\mathbb{C}(X_\mathbb{C}) = Z$.

Let $Y_0 = F(X) \subseteq \mathbb{R}^n$ and Y denote the Zariski closure of the semialgebraic set Y_0 in $\mathbb{R}^m$. We endow Y_0 with its subspace topology and X/G with the quotient topology. Now we state a well known result which we will need later. We refer the reader to Section 1.3 in [5] or Chapter 1 in [8] or [10] for a proof this lemma.

Lemma 1.2. $F : X \longrightarrow Y_0$ induces a homeomorphism $\overline{F} : X/G \longrightarrow Y_0$. Moreover, if X is nonsingular and the G action on X is free, then Y_0 is a subset of nonsingular points of its Zariski closure Y and the induced map $\overline{F} : X/G \longrightarrow Y_0$ becomes a diffeomorphism.

Lemma 1.3. $Y = Z \cap \mathbb{R}^m$ or $Z = Y_{\mathbb{C}}$.

Proof. Since $Y_0 = F(X) \subseteq F_{\mathbb{C}}(X_{\mathbb{C}}) \cap \mathbb{R}^m = Z \cap \mathbb{R}^m$ we see that $Y \subseteq Z \cap \mathbb{R}^m \subseteq Z$. Hence, $Y_{\mathbb{C}} \subseteq Z$, where $Y_{\mathbb{C}}$ is the complexification of the real algebraic set Y. Also, since $X \subseteq {F_{\mathbb{C}}}^{-1}(F_{\mathbb{C}}(X)) \subseteq {F_{\mathbb{C}}}^{-1}(Y) \subseteq {F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})$, we have $X_{\mathbb{C}} \subseteq {F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})$. Therefore, $Z = F_{\mathbb{C}}(X_{\mathbb{C}}) \subseteq F_{\mathbb{C}}({F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})) \subseteq Y_{\mathbb{C}} \subseteq Z$ and thus $Y_{\mathbb{C}} = Z$ and $Y = Z \cap \mathbb{R}^m$. □

This is nothing but the first paragraph of the proof of Proposition 2.10.3 in [1]. In the literature, Y is usually denoted by $X//G$, the algebraic quotient of X by G.

2. Results

Theorem 2.1. *Let X be an irreducible real algebraic set and G is a finite group acting algebraically on X. Let Y be the Zariski closure of Y_0. Then:*

a) if $|G|$ is odd, then $Y_0 = Y$; i.e. Y_0 is algebraic;

b) if G acts freely on X and X is nonsingular, then Y_0 is a union of topological components of $Nonsing(Y)$ (G might have even order);

c) if $|G|$ is odd, X is nonsingular and G acts freely on X, then Y_0 is a nonsingular algebraic set.

Proof. First let us prove (a): Assume that the conclusion of the theorem is not true. So there exists a point $p \in Y - Y_0$. Let $\{q_1, \ldots, q_l\}$ be the preimage of p under $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$. Since G acts transitively on the fibers $l = |G|/Stab_G(q_1)$. But G is of odd order and hence l is an odd number. However, since $p \in Y - Y_0$ none of the q_i's is contained in the real part of $X_{\mathbb{C}}$, because all the real points of $X_{\mathbb{C}}$ are sent to Y_0. Moreover, since $F_{\mathbb{C}}$ is a real polynomial map, and $X_{\mathbb{C}}$ is defined over the reals, the complex conjugation of $\mathbb{C}^n$ preserves the fibers of $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ over real points. In particular, it preserves the fiber over p. So l should be an even number, which is a contradiction. Hence, $Y = Y_0$.

For part (b) consider the map $F : X \longrightarrow Y$. Since this map is a local diffeomorphism at each point of X (Lemma 1.2) and dim(X)=dim(Y) we have that Y_0 is an open subset of $Nonsing(Y) \subseteq Y$. The map $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ is a closed mapping ([8] or Corollary A on page 49 in [12]). Now since $X \subseteq X_{\mathbb{C}}$ is closed, $Y_0 = F_{\mathbb{C}}(X)$ is closed in $Y_{\mathbb{C}}$ and hence in Y. So Y_0 is a union of topological components of $Nonsing(Y)$. Finally, part (c) follows from parts (a) and (b). □

Remark. The following example shows that Theorem 2.1.a. does not hold for groups of even order. Let X be the zero set of the irreducible polynomial $x^4 + y^4 - 1$ in $\mathbb{R}^2$. Then, X is a nonsingular irreducible algebraic set diffeomorphic to the unit circle S^1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the involution of $\mathbb{R}^2$ given by

$$f((x, y)) = (-x, -y).$$

Note that $G = \{id, f\}$ acts freely on X. Let S,T be as in Section 1. Then,

$$S = \mathbb{R}[x, y]/(x^4 + y^4 - 1) \quad \text{and} \quad T = \mathbb{R}[x^2, y^2, xy]/(x^4 + y^4 - 1).$$

The map $F : X \longrightarrow \mathbb{R}^3$ is given by $(x, y) \longrightarrow (x^2, y^2, xy)$, and, as before, let us denote its image by Y_0, which is a smooth manifold diffeomorphic to S^1. The Zariski closure Y of Y_0 in $\mathbb{R}^3$ is given by the polynomial equations $t_1^2 + t_2^2 - 1 = 0$ and $t_1 t_2 - t_3^2 = 0$. In Y_0 the first two coordinates are always non negative, whereas in Y these two coordinates can be negative. Actually, $Y_0 \cup Y_0' = Y$ (Example 2 after Theorem 2.2), where $Y_0' = \{(t_1, t_2, t_3) \mid (-t_1, -t_2, -t_3) \in Y_0\}$ and $Y_0 \cap Y_0' = \emptyset$. Therefore, Y_0 can not be a Zariski open set and thus $Y_0 = F(X)$ is not algebraic.

Even order group case: To be able to get a result similar to Theorem 2.1 in the case of even order groups we will assume that X is a nonsingular real algebraic set and the G action on X is free. First let us consider the case where $G = \mathbb{Z}_2 =< g >$. By Theorem 2.1.b Y_0 is a union of the components of $Nonsing(Y)$. Let us look at Y carefully and see when Y_0 is an algebraic set or a Zariski open subset of Y.

The points in $Y - Y_0$ are coming from the non real points of $X_\mathbb{C}$. So, the preimage of any point in $Y - Y_0$ under $F_\mathbb{C} : X_\mathbb{C} \longrightarrow Y_\mathbb{C}$ consists of two complex conjugate points in $X_\mathbb{C} - X$. Let W be the set of the points in all such fibers. Note that W is nothing but the subset of $X_\mathbb{C}$ on which complex conjugation and g agree. So it is a real algebraic subset of $X_\mathbb{C}$.

G acts freely on X and complex conjugation acts trivially on X and thus $X \cap W = \emptyset$. Since $F_\mathbb{C} : X_\mathbb{C} \longrightarrow Y_\mathbb{C}$ is a finite to one map, the semialgebraic set $F_\mathbb{C}(W) = Y - Y_0$ has the same dimension as W. So, $\dim(W) \leq \dim(Y) = k$. If $\dim(W) \leq k - 1$, then $Y - Y_0$ is contained in $Sing(Y)$ and therefore Y_0 is a Zariski open subset of Y. If $W = \emptyset$ then $Y_0 = Y$ and hence is algebraic in $\mathbb{R}^m$. So we have proved the following theorem.

Theorem 2.2. *Let $G = \mathbb{Z}_2 =< g >$ act freely on a k dimensional nonsingular real algebraic set $X \subseteq \boldsymbol{R}^n$. Let W be as above, then $Y = Y_0 \cup F^\mathbb{C}(W)$. In particular,*

a) *if $W = \emptyset$, then $Y_0 = Y$ and hence Y_0 is algebraic in $\boldsymbol{R}^m$;*

b) *if $\dim(W) \leq k - 1$, then Y_0 is a Zariski open subset of Y;*

c) *if $\dim(W) = k$, then Y_0 is not a Zariski open subset of Y.*

Remark. Assume that the above G action is linear. So, by a linear change of coordinates we have

$$g(x_1,\dots,x_n)=(x_1,\dots,x_{j-1},-x_j,\dots,-x_n)$$

for some $1\le j\le n$. In this case $W=X_{\mathbb{C}}\cap V_g$, where V_g is the linear subspace of $\mathbb{R}^{2n}=\mathbb{C}^n$ given by

$$x_l=0,\quad l=j,\dots,n \quad\text{and}\quad y_l=0,\quad l=1,\dots,j-1,$$

where $(x_1,y_1,\dots,x_n,y_n)\in\mathbb{R}^{2n}$ ($\mathbb{R}^{2n}=\mathbb{C}^n$ by $z_l=x_l+iy_l$). By identifying

$$(x_1,0,x_2,0,\dots,x_{j-1},0,0,y_j,0,y_{j+1},\dots,0,y_n) \text{ with } (x_1,\dots,x_{j-1},y_j,\dots,y_n),$$

we can identify W with the algebraic subset $\tilde{W}\subseteq\mathbb{R}^n$, given by

$$\tilde{W}=\{(X,Y)=(x_1,\dots,x_{j-1},y_j,\dots,y_n)\mid f(X,iY)=0\ ,\forall f\in J(X)\}.$$

Examples. 1) Let $X=S^k\subseteq\mathbb{R}^{k+1}$ be the standard k-sphere and g be the antipodal map. Then,

$$\tilde{W}=\{(x_1,\dots,x_{k+1})\in\mathbb{R}^{k+1}\mid (ix_1)^2+\cdots+(ix_{k+1})^2=1\}=\emptyset.$$

So $W=\emptyset$. The coordinate functions of F are x_ix_j for $i\le j=1,\dots,k+1$. So $\mathbb{R}P^k$ sits in $\mathbb{R}^{(k+1)(k+2)/2}$ as an algebraic set.

2) Let us look at the counterexample following Theorem 2.1 once more. $W\subseteq\mathbb{R}^4=\mathbb{C}^2$ is equal to

$$W=\{(0,ix,0,iy)\mid x,y\in\mathbb{R},\ x^4+y^4=1\},$$

so that, W has dimension one. Moreover, W is sent onto Y_0' by the map $F_{\mathbb{C}}(x,y)=(x^2,y^2,xy)$. Therefore $Y=Y_0\cup Y_0'$.

3) Let $X\subseteq\mathbb{R}^2$ be given by $x^4+y^2=1$. X is diffeomorphic to S^1. Let g be as above. Then, Y_0 is diffeomorphic to S^1 but, the Zariski closure Y of Y_0 contains an unbounded curve. Hence, Y may not be compact even if X is compact.

A weaker version of Theorem 2.2 generalizes to arbitrary finite groups as follows.

Theorem 2.3. *Let G be any finite group acting freely on a k dimensional nonsingular real algebraic set X. Let W be the set*

$$\{p\in X_{\mathbb{C}}\mid g(p)=\bar{p},\ \textit{for some order two element } g\in G\},$$

where $\bar{p}$ denotes the complex conjugate of p. Now, if $\dim(W)\le k-1$ *then $Y_0=Nonsing(Y)$ and therefore Y_0 is a Zariski open subset of Y.*

Proof. By Theorem 2.1.b Y_0 is union of topological components of $Nonsing(Y)$. Assume that the conclusion of the theorem is not true. So $Nonsing(Y)$ has components other than the ones contained in Y_0. In particular, $Y - Y_0$ has real dimension $k = \dim(Y)$. Let $p \in Nonsing(Y) - Y_0$ and $\Delta = (F_{\mathbb{C}})^{-1}(p)$. Since p is a real point and everything is defined over the reals, Δ is invariant under complex conjugation. Moreover, $\Delta \cap X = \emptyset$ and G acts transitively on Δ. Note that the set Z of points in $X_{\mathbb{C}}$ on which G does not act freely is a proper algebraic subset of $X_{\mathbb{C}}$. So $F_{\mathbb{C}}(Z)$ has complex dimension at most $k-1$ and thus its real part $Y \cap F_{\mathbb{C}}(Z)$ has real dimension not more than $k-1$.

Let $q \in \Delta$, then $\bar{q} \in \Delta$. Assume that $q \notin Z$; then $\Delta \cap Z = \emptyset$. Thus G acts freely and transitively on Δ and therefore, there exists an element $g \in G$ so that $g(q) = \bar{q}$. Since $g : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ is defined over the reals we get $g(\bar{q}) = q$ and therefore $g^2 = id_G$. Hence, $\Delta \subseteq W$ and we conclude that any fiber $(F_{\mathbb{C}})^{-1}(p)$, for $p \in Nonsing(Y) - Y_0$, is contained in $W \cup Z$. Hence, $Nonsing(Y) - Y_0 \subseteq F_{\mathbb{C}}(W \cup Z)$ which is a contradiction since the real dimension of $Y \cap F_{\mathbb{C}}(W \cup Z)$ is less than $k = \dim(Y)$. □

Let us now consider the entire rational functions on Y_0 and Y. For any semi-algebraic subset X of $\mathbb{R}^n$ let $\Gamma(X)$ denote the ring of entire rational functions on X. Assuming the previous notation we have a G action on $\Gamma(X)$ and the map $F : X \longrightarrow Y_0$ induces a homomorphism $F^* : \Gamma(Y_0) \longrightarrow \Gamma(X)$ via composition by F. Moreover, we have the following.

Proposition 2.4. *$(\Gamma(X))^G = F^*(\Gamma(Y_0))$ and $\Gamma(Y) \subseteq \Gamma(Y_0)$, where the equality holds if and only if $Y_0 = Y$. In particular, if X and Z are irreducible algebraic sets with algebraic G actions and if they are G equivariantly isomorphic to each other, then their quotients $X//G$ and $Z//G$ are isomorphic.*

Proof. First let us show that $(\Gamma(X))^G = F^*(\Gamma(Y_0))$. Clearly $F^*(\Gamma(Y_0)) \subseteq (\Gamma(X))^G$. Let $f_1/g_1 \in (\Gamma(X))^G$. Then, $l \cdot f_1/g_1 = f_1/g_1 + \cdots + f_l/g_l \in (\Gamma(X))^G$ where the sets $\{f_1, \ldots, f_l\}$ and $\{g_1, \ldots, g_l\}$ are the G orbits of f_1 and g_1 respectively. Now

$$f_1/g_1 = \frac{h_1 + \cdots + h_l}{l \cdot g_1 \cdots g_l}$$

where $h_i = f_i \cdot g_1 \cdots g_{i-1} \cdot g_{i+1} \cdots g_l$. Note that $h_1 + \cdots + h_l$ and $g_1 \cdots g_l$ are in the invariant subring T and hence $f_1/g_1 \in F^*(\Gamma(Y_0))$.

For the second statement, evidently we have that $\Gamma(Y) \subseteq \Gamma(Y_0)$. If $Y_0 \neq Y$ let $P = (p_1, \ldots, p_m) \in Y - Y_0$ and consider the function

$$\frac{1}{(x_1 - p_1)^2 + \cdots + (x_m - p_m)^2}$$

which is entire rational on Y_0 but not on Y. So $\Gamma(Y_0) \neq \Gamma(Y)$ and therefore $\Gamma(Y_0) = \Gamma(Y)$ if and only if $Y_0 = Y$. The third statement follows easily. □

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Yıldıray OZAN
Middle East Technical University,
Department of Mathematics,
e-mail: ozan@rorqual.cc.metu.edu.tr
06531, Ankara-TURKEY

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