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## IMMERSIONS PRESERVED UNDER ROTATIONS WITH TOTALLY REDUCIBLE FOCAL SET

*Rıdvan Ezentas*

### Abstract

In [1] Carter and the author introduced the idea of an immersion  $f : M^m \rightarrow R^n$  with totally reducible focal set (TRFS). Such an immersion has the property that, for all  $p \in M$ , the focal set with base  $p$  is a union of hyperplanes in the normal plane to  $f(M)$  at  $f(p)$ . Here we show that if we take two immersions with TRFS then we can construct new immersions with TRFS. In particular, rotating an immersion with TRFS about an axis gives a new immersion with TRFS.

**Keywords:** Critical point theory, Focal points, Focal sets, Totally reducible focal sets, Flat normal bundles. M.R. Number. 53C40, 53C42.

### 1. Introduction

Let  $f : M \rightarrow R^n$  be a smooth immersion of a connected smooth ( $C^\infty$ )  $m$ -dimensional manifold without boundary into Euclidean  $n$ -space. For each  $p \in M$ , the focal set of  $f$  with base  $p$  is an algebraic variety. In this paper we consider immersions for which this variety is a union of hyperplanes. Trivially, this always holds if  $n = m + 1$  so we only consider  $n > m + 1$ . In the section 2 we give a known definition and results and in the section 3 we shall give very interesting constructions of immersions with totally reducible focal set (TRFS).

### 2. Definition and Basic Properties of TRFS

For  $p \in M$ , let  $U$  be a neighbourhood of  $p$  in  $M$  such that  $f|U : U \rightarrow R^n$  is an embedding and let  $v_f(p)$  denote the  $(n - m)$ -plane which is normal to  $f(U)$  at  $f(p)$ . Then the total space of the normal bundle is  $N_f = \{(p, x) \in M \times R^n | x \in v_f(p)\}$ . The projection map  $\eta_f : N_f \rightarrow R^n$  is defined by  $\eta_f(p, x) = x$  and the set of focal points with base  $p$  is  $\Gamma_f(p) = \{x \in R^n | (p, x) \text{ is a singularity of } \eta_f\}$ . For each  $p \in M$ ,  $\Gamma_f(p)$  is a real algebraic variety in  $v_f(p)$  which can be defined as the zeros of polynomial on  $v_f(p)$  of degree  $\leq m$ .

**Definition 1.** *The immersion  $f : M \rightarrow R^n$  has totally reducible focal set (TRFS) if, for all  $p \in M$ ,  $\Gamma_f(p)$  can be defined as the zeros of a real polynomial which is a product of real linear factors.*

So each irreducible component of  $\Gamma_f(p)$  is an affine plane in  $v_f(p)$ , and  $\Gamma_f(p)$  is a union of  $(n - m - 1)$ -planes (possible  $\Gamma_f(p) = \phi$ ).

Examples of embeddings  $f : M \rightarrow R^n$  with TRFS are embeddings whose image is an isoparametric submanifolds [2, 6]. For these examples the pattern of the planes of  $\Gamma_f(p)$  in  $v_f(p)$  is the same for all  $p \in M$ , i.e. for all  $p, q \in M$ ,  $\Gamma_f(p)$  is isometric to  $\Gamma_f(q)$ . In general, for an immersion with TRFS the pattern of planes of  $\Gamma_f(p)$  will vary with  $p$ . It is shown in [5, 7] that  $f$  has TRFS if and only if  $f$  has flat normal bundle, where  $M$  is thought of as a Riemannian manifold with metric  $g$  induced from  $R^n$ . We will give explicit ways of constructing immersions and embeddings with TRFS.

In calculating focal sets it is often easiest to work with distance functions. For  $x \in R^n$  the distance function  $L_x : M \rightarrow R$  is defined by  $L_x(p) = \|f(p) - x\|^2$ . Then  $x \in R^n$  is a focal point of  $f$  with base  $p$  if and only if  $p$  is a degenerate critical point of  $L_x$ , where at  $p$ ,  $\frac{\partial L_x}{\partial p_i} = 0$  and  $H = \left[ \frac{\partial^2 L_x}{\partial p_i \partial p_j} \right]$  is singular for  $i, j = 1, 2, \dots, m$  [4].

**Proposition 1.** [1] *Every immersion  $f : S^1 \rightarrow R^n$ ,  $n > 2$ , has TRFS.*

**Proposition 2.** [1] *Let  $f : M_1 \rightarrow R^n$  and  $g : M_2 \rightarrow R^n$  be immersions with TRFS. Then  $f \times g : M_1 \times M_2 \rightarrow R^{n_1+n_2}$  defined by  $(f \times g)(p, q) = (f(p), g(q))$  has TRFS.*

From Proposition 1 and 2 we can construct more examples of immersions with TRFS by taking products of immersions of circles.

### 3. Immersions Preserved Under Rotations

**Theorem 1.** *Let  $f : M^m \rightarrow R^n$  and  $g : N^p \rightarrow S^q \subset R^{q+1}$  be immersions with TRFS and assume that  $f_n(\theta) \neq 0$  for all  $\theta \in M^m$ . Then  $h : M^m \times N^p \rightarrow R^{n+q}$  defined by  $h(\theta, \varphi) = (f_1(\theta), \dots, f_{n-1}(\theta), f_n(\theta)g(\varphi))$  is an immersion with TRFS.*

**Proof.** Let  $x \in R^n$  and  $L_x(\theta) = \sum_{k=1}^n (x_k - f_k(\theta))^2$ . Then  $x \in \Gamma_f(\theta)$  if and only if, at  $\theta$ ,

$$\frac{\partial L_x}{\partial \theta_i} = -2 \sum_{k=1}^n (x_k - f_k) \frac{\partial f_k}{\partial \theta_i} = 0, \quad i = 1, 2, \dots, m \tag{1}$$

and if

$$A_{ij} = \frac{\partial^2 L_x}{\partial \theta_i \partial \theta_j} = -2 \left( \sum_{k=1}^n (x_k - f_k) \frac{\partial^2 f_k}{\partial \theta_i \partial \theta_j} - \left( \frac{\partial f_k}{\partial \theta_i} \frac{\partial f_k}{\partial \theta_j} \right) \right) \tag{2}$$

at  $\theta$ , then  $\det(A_{ij}) = 0$ . □

Thus  $f$  has TRFS if and only if using (1), (2) factorizes into linear factors. Let  $y \in R^{q+1}$  and

$$L_y(\varphi) = \sum_{s=1}^{q+1} (y_s - g_s(\varphi))^2.$$

Then  $y \in \Gamma_g(\varphi)$  if and only if, at  $\varphi$ ,

$$\sum_{s=1}^{q+1} g_s^2 = 1 \Rightarrow \sum_{s=1}^{q+1} g_s \frac{\partial g_s}{\partial \varphi_r} = 0, \quad r = 1, 2, \dots, p \quad (3)$$

$$\frac{\partial L_y}{\partial \varphi_r} = -2 \sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial \varphi_r} = 0, \quad r = 1, 2, \dots, p \quad (4)$$

and if

$$B_{rt} = \frac{\partial^2 L_y}{\partial \varphi_r \partial \varphi_t} = -2 \left( \sum_{s=1}^{q+1} y_s \frac{\partial^2 g_s}{\partial \varphi_r \partial \varphi_t} \right) \quad (5)$$

at  $\varphi$  then  $\det(B_{rt}) = 0$ .

Thus  $g$  has TRFS if and only if using (3), (4), (5) factorizes into linear factors. For

$$z = (x_1, \dots, x_{n-1}, y_1, \dots, y_{q+1}) \in R^{n+q} \quad \text{and} \quad L_z(\theta, \varphi) = \sum_{k=1}^{n-1} (x_k - f_k)^2 + \sum_{s=1}^{q+1} (y_s - f_n g_s)^2.$$

Then  $z = (x_1, \dots, x_{n-1}, y_1, \dots, y_{q+1}) \in \Gamma_h(\theta, \varphi)$  if and only if at  $(\theta, \varphi)$ , and we get

$$\frac{\partial L_z}{\partial \theta_i} = -2 \sum_{k=1}^{n-1} (x_k - f_k) \frac{\partial f_k}{\partial \theta_i} - 2 \left( \sum_{s=1}^{q+1} y_s g_s - f_n \right) \frac{\partial f_n}{\partial \theta_i} = 0 \quad (6)$$

and using (3)

$$\frac{\partial L_z}{\partial \varphi_r} = -2 f_n \sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial \varphi_r} = 0, \quad (7)$$

and if

$$C_{ij} = \frac{\partial^2 L_z}{\partial \theta_i \partial \theta_j} = -2 \left( \sum_{t=1}^{n-1} (x_t - f_t) \frac{\partial^2 f_t}{\partial \theta_i \partial \theta_j} - \left( \frac{\partial f_t}{\partial \theta_i} \frac{\partial f_t}{\partial \theta_j} \right) + \left( \left( \sum_{s=1}^{q+1} y_s g_s \right) - f_n \right) \frac{\partial^2 f_n}{\partial \theta_i \partial \theta_j} - \left( \frac{\partial f_n}{\partial \theta_i} \frac{\partial f_n}{\partial \theta_j} \right) \right),$$

$$\text{then} \quad \det(C_{ij}) = 0; \quad (8)$$

and if

$$D_{rt} = \frac{\partial^2 L_z}{\partial \varphi_r \partial \varphi_t} = -2f_n \left( \sum_{s=1}^{q+1} y_s \frac{\partial^2 g_s}{\partial \varphi_r \partial \varphi_t} \right)$$

then  $\det(D_{rt}) = 0,$  (9)

since if  $\frac{\partial L_z}{\partial \varphi_r} = 0$  then

$$\frac{\partial^2 L_z}{\partial \theta_i \partial \varphi_r} = \frac{\partial^2 L_z}{\partial \varphi_r \partial \theta_i} = -2 \left( \sum_{s=1}^{q+1} y_s \frac{\partial g_s}{\partial \varphi_r} \right) \frac{\partial f_n}{\partial \theta_i} = 0 \text{ as } f_n \neq 0.$$

**Case (i)** If (6), (7) and (8) hold then since (7) is linear in  $y_1, \dots, y_{q+1}$  and  $(x_1, \dots, x_{n-1}, w_n)$  satisfies (1) and (2) where  $w_n = \sum_{s=1}^{q+1} y_s g_s$  and using (6), (8) factorizes into linear factors in  $x_1, \dots, x_{n-1}, w_n$  since  $f$  has TRFS. But  $w_n$  is linear in  $y_1, \dots, y_{q+1}$ , so (8) factorizes into linear factors in  $x_1, \dots, x_{n-1}, y_1, \dots, y_{q+1}$ .

**Case (ii)** If (6), (7) and (9) hold then (7) is linear in  $x_1, \dots, x_{n-1}, y_1, \dots, y_{q+1}$  and  $(y_1, \dots, y_{q+1})$  satisfies (4) and (5) and using (7), (9) factorizes into linear factors in  $y_1, \dots, y_{q+1}$ , since  $g$  has TRFS.

Hence the equations defining  $\Gamma_h(\theta, \varphi)$  factorize into linear factors and therefore  $h$  has TRFS.

□

**Corollary 1.** *If  $f : M^m \rightarrow R^n$  has TRFS and the immersion  $h : M^m \times S^q \rightarrow R^{n+q}$  is defined by  $h(p, \theta) = (f_1(p), \dots, f_{n-1}(p), f_n(p)g(\theta))$ , where  $g : S^q \rightarrow S^q \subset R^{q+1}$  is defined by*

$$g(\theta) = \left( \prod_{i=1}^q \cos \theta_i, \prod_{j=2}^q \cos \theta_j \sin \theta_1, \prod_{k=3}^q \cos \theta_k \sin \theta_2, \dots, \cos \theta_q \sin \theta_{q-1}, \sin \theta_q \right),$$

*Then  $h$  has TRFS.*

**Proof.** Since  $g$  has TRFS as the embedding  $g$  has codimension one, the results follows immediately from Theorem 2.1. □

As a particular case consider an immersion  $f : M^m \rightarrow R^{m+1}$ , with  $f_{m+1}(p) \neq 0$  for all  $p \in M^m$ . Then the immersion  $h : M^m \times S^1 \rightarrow R^{m+2}$  is defined by  $h(p, \theta) = (f_1(p), \dots, f_m(p), f_{m+1}(p) \cos \theta, f_{m+1}(p) \sin \theta)$  has TRFS. This immersion is obtained by rotating the image of  $f$  about the hyperplane  $R^m \times \{(0, 0)\}$  in  $R^{m+2}$ .

**Theorem 2.** Let  $f : M^m \rightarrow R^n$  ( $f(M^m) \subset S^{n-1}$ ) and  $g : N^k \rightarrow R^d$  ( $g(N^k) \subset S^{d-1}$ ) be immersions with TRFS. Then the embedding  $h : M^m \times N^k \times (0, \pi/2) \rightarrow R^{n+d}$  defined by  $h(p, q, \theta) = (\cos \theta \cdot f(p), \sin \theta \cdot g(q))$  has TRFS.

**Proof.** Let  $x = (x_1, \dots, x_n) \in R^n$ ,  $y = (y_1, \dots, y_d) \in R^d$  and  $z = (x, y) \in R^{n+d}$ . We know that

$$\sum_{i=1}^n f_i^2 = 1 \Rightarrow \sum_{i=1}^n f_i \frac{\partial f_i}{\partial p_s} = 0, \quad s = 1, 2, \dots, m \tag{10}$$

$$\sum_{j=1}^d g_j^2 = 1 \Rightarrow \sum_{j=1}^d g_j \frac{\partial g_j}{\partial q_r} = 0, \quad r = 1, 2, \dots, k. \tag{11}$$

So from (10),

$$\frac{\partial L_z}{\partial p_s} = 2 \cos \theta \left( \sum_{i=1}^n x_i \frac{\partial f_i}{\partial p_s} \right) \tag{12}$$

since

$$\cos \theta \neq 0, \quad \frac{\partial L_z}{\partial p_s} = 0 \quad \text{if and only if} \quad \left( \sum_{i=1}^n x_i \frac{\partial f_i}{\partial p_s} \right) = 0. \tag{13}$$

Also, from (11),

$$\frac{\partial L_z}{\partial q_r} = 2 \sin \theta \left( \sum_{j=1}^d y_j \frac{\partial g_j}{\partial q_r} \right) \tag{14}$$

since

$$\sin \theta \neq 0, \quad \frac{\partial L_z}{\partial q_r} = 0 \quad \text{if and only if} \quad \left( \sum_{j=1}^d y_j \sum \frac{\partial g_j}{\partial q_r} \right) = 0. \tag{15}$$

From (10), (11),

$$\frac{\partial L_z}{\partial \theta} = 2 \sin \theta \left( \sum_{i=1}^n x_i f_i \right) - 2 \cos \theta \left( \sum_{j=1}^d y_j g_j \right). \tag{16}$$

So

$$\frac{\partial L_z}{\partial \theta} = 0 \quad \text{if and only if} \quad 2 \sin \theta \left( \sum_{i=1}^n x_i f_i \right) - 2 \cos \theta \left( \sum_{j=1}^d y_j g_j \right) = 0. \tag{17}$$

From (12), (14),  $\frac{\partial^2 L_z}{\partial p_s \partial q_r} = \frac{\partial^2 L_z}{\partial q_r \partial p_s} = 0$ . From (12), (16),

$$\frac{\partial^2 L_z}{\partial p_s \partial \theta} = \frac{\partial^2 L_z}{\partial \theta \partial p_s} = 2 \sin \theta \left( \sum_{i=1}^n x_i \frac{\partial f_i}{\partial p_s} \right).$$

So  $\frac{\partial^2 L_z}{\partial p_s \partial \theta} = 0$  if and only if  $\frac{\partial L_z}{\partial p_s} = 0$  from (13). From (14), (16),

$$\frac{\partial^2 L_z}{\partial q_r \partial \theta} = \frac{\partial^2 L_z}{\partial \theta \partial q_r} = -2 \cos \theta \left( \sum_{j=1}^d y_j \frac{\partial g_j}{\partial q_r} \right).$$

So  $\frac{\partial^2 L_z}{\partial q_r \partial \theta} = 0$  if and only if  $\frac{\partial L_z}{\partial q_r} = 0$  from (15), and

$$\begin{aligned} A_{st} &= \frac{\partial^2 L_z}{\partial p_s \partial p_t} = -2 \cos \theta \left( \sum_{i=1}^n x_i \frac{\partial^2 f_i}{\partial p_s \partial p_t} \right), \\ B_{rv} &= \partial^2 L_z \partial q_r \partial q_v = 2 \sin \theta \left( \sum_{j=1}^d y_j \frac{\partial^2 g_j}{\partial q_r \partial q_v} \right) \\ C &= \frac{\partial^2 L_z}{\partial \theta^2} = -2 \cos \theta \left( \sum_{i=1}^n x_i f_i \right) - 2 \sin \theta \left( \sum_{j=1}^d y_j g_j \right). \end{aligned}$$

Thus  $z \in \Gamma_h(p, q, \theta)$  if and only if  $\frac{\partial L_z}{\partial p_s} = 0$ ,  $\frac{\partial L_z}{\partial q_r} = 0$ ,  $\frac{\partial L_z}{\partial \theta} = 0$  and either

$$\det(A_{st}) = 0, \tag{18}$$

or

$$\det(B_{rv}) = 0, \tag{19}$$

or

$$C = 0. \tag{20}$$

**Case (i) (13), (15), (17) and (18) hold.** (15) is linear in  $y_1, \dots, y_d$ , (17) is linear in  $x_1, \dots, x_n$ ,  $y_1, \dots, y_d$  and using (13) which is linear in  $x_1, \dots, x_n$ , (18) factorizes into linear factors in  $x_1, \dots, x_n$  since  $f$  has TRFS and  $x_1, \dots, x_n \in \Gamma_f(p)$  if and only if

$$\left( \sum_{i=1}^n x_i \frac{\partial f_i}{\partial p_s} \right) = 0 \quad \text{and} \quad \det \left( \sum_{i=1}^n x_i \frac{\partial^2 f_i}{\partial p_s \partial p_t} \right) = 0,$$

as in the proof of Theorem 1.

**Case (ii) (13), (15), (17) and (19) hold.** This is similar to case (i) but uses the fact that  $g$  has TRFS.

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**Case (i) (13), (15), (17) and (20) hold.** In this case, all equations are linear in  $x_1, \dots, x_n, y_1, \dots, y_d$ .

□

It follows that  $h$  has TRFS.

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