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A CHARACTERIZATION OF $\overline{NC} - p$-GROUPS

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Abstract

In this work, presented is a partial characterization of a perfect locally nilpotent p-group in which every proper subgroup is nilpotent-by-Chernikov.

1. Introduction

Let G be a locally finite group. If every proper subgroup of G is nilpotent-by-finite (NF-group) while G is not nilpotent-by-finite, then G is called a **minimal non nilpotent-by-finite group ($\overline{NF}$-group)**. If in the above definition "finite" is replaced by "Chernikov", then one obtains NC-groups and $\overline{NC}$-groups, respectively. Bruno in [4] showed, among other things, that if G is a non-perfect $\overline{NF}$-group for a prime p, then $G/G' \cong C_{p\infty}, G'$ is nilpotent and G' is not properly supplemented in G. The first example of such a group was constructed by Heineken and Mohamed in [8]. For this reason groups of this type are called HM-groups. Later, Meldrum in [9] and Hartley in [6] gave similar constructions. Bruno and Phillips in [5] constructed an HM-group with a metabelian derived subgroup. Recently, Menegazzo in [10] constructed HM-groups with derived subgroups either abelian of infinite exponent or solvable of arbitrary derived length. Thus the structure of nonperfect $\overline{NF} - p$-groups is known. However it is not known yet whether or not perfect $\overline{NF} - p$-groups exist.

Otal and Peña in [12] extended some of the results of [4] to $\overline{NC}$-groups. Later $\overline{NC} - p$-groups were considered in [1], [2] and [3]. Theorem A of [2] (see also the end of this section) shows that an $\overline{NC} - p$-group must be perfect. Another consequence of the same theorem is that a perfect $\overline{NF} - p$-group has a proper epimorphic image in which every proper subgroup is nilpotent of finite exponent. Of course, the determination of $\overline{NC} - p$-groups will be useful in the investigation of perfect locally nilpotent p-groups, about which not much is known yet. In this work the first question is considered. The main results of this work are stated below.

Theorem. *Let G be an $\overline{NC} - p$-group. Then G contains a proper normal subgroup K such that every proper subgroup of G/K is a Chernikov extension of a nilpotent subgroup of finite exponent and one of the following holds:*

(i) *G/K is an $\overline{NF} - p$-group.*

(ii) *$G/K = \bigcup_{i=1}^{\infty}(T_i/K)'$, where for each $i \geq 1$, T_i/K is an HM^*-subgroup of G/K such that $(T_i/K)/(T_i/K)' \cong C_{p^\infty}$ and $T_i/K \leq T_{i+1}/K$.*

(iii) *G/K is generated by its normal HM^*-subgroups T/K such that $(T/K)/(T/K)' \cong C_{p^\infty}$.*

(The definition of an HM^*-group is given below)

Corollary. *Let G be an $\overline{NC} - p$-group satisfying the normalizer condition. Then G contains a proper normal subgroup K such that every proper subgroup of G/K is a Chernikov extension of a nilpotent subgroup of finite exponent and (i) or (iii) of the theorem holds.*

It is not known whether or not (ii) and (iii) are necessary in the above results.

In the study of $\overline{NC}-p$-groups certain subgroups, similar to the Heineken-Mohamed group, are unavoidable. To single them out, they were called HM^*-groups in [2]. By definition a locally nilpotent p-group $X \neq 1$ is called an **HM^*-group** if X' is nilpotent and

$$X/X' \cong C_{p^\infty}^{(n)} = C_{p^\infty} \times \cdots \times C_{p^\infty}$$

is n copies for some $n \geq 1$. If $n = 1$, $X' \neq 1$ and every proper subgroup of X is subnormal in X (in which case X' is not properly suplemented in X by Lemma 2.2 (iii)) then X is called a **group of Heineken-Mohamed type** or an **HM-group** for brevity.

Elementary properties of HM^*-groups were collected in [2] and [3] but they will be restated here (see §2) in order to make this work self-contained.

For any group X, X° denotes the unique maximal normal radicable abelian subgroup of X, whenever it exits. We end this section by stating Theorem A of [2].

Theorem A. *Let G be locally nilpotent p-group which does not have any proper subgroup of finite index. Suppose that every proper subgroup of G is an NC-group. Then the following hold.*

(i) *If $G = G'$, then G is the union of an ascending chain of normal nilpotent subgroups. Furthermore, G has a normal nilpotent subgroup N such that in G/N every proper subgroup is a Chernikov extension of a nilpotent subgroup of finite exponent and, for every normal nilpotent subgroup K of $G, KN/N$ has finite exponent.*

(ii) *If $G' \neq G$, then G is an NC-group.*

2. Properties of HM^*-Groups

Lemma 2.1. Let A be a nilpotent p-group and B be a normal subgroup of A of finite exponent m such that $A/B \cong C_{p^\infty}^{(n)} = C_{p^\infty} \times \cdots \times C_{p^\infty}$, n factors for some $n \geq 1$. Then $A = TB$, where $T = A^\circ$.

Proof. We may use induction on the nilpotency class c of A. First suppose that $c = 1$. Then A is abelian. Define $f : A \to A$ by $f(a) = a^m$. Then f is a homomorphism with kernel $K = \{a : a^m = 1\}$. Also, $B \leq K$. Therefore $f(A) \cong A/K \cong C_{p^\infty}^{(n)}$ since K has finite exponent. So if we put $T = f(A)$ then $TB/B \cong C_{p^\infty}^{(n)} \cong A/B$ which implies that $TB = A$.

Next suppose that $c > 1$ and the assertion holds for $c - 1$. Let $R = K_c(A)$, the cth term of the lower central series of A and put $\overline{A} = A/R$. Let $\overline{U} = \overline{A}^\circ$. By induction hypothesis $\overline{A} = \overline{UB}$. Also, since $\overline{U} \cong C_{p^\infty}^{(n)}$ and $R \leq Z(U)$, it is easy to see that U is abelian and so $U = VR$, where $V = U^\circ$ by the first paragraph. Substituting this in $A = UB$ gives that $A = VB$. Also $V = A^\circ$ since B has finite exponent. This completes the proof of the lemma. □

Lemma 2.2. Let X be an HM^*-group for a prime p. Then the following hold.

(i) $X' = [X, X']$.

(ii) There does not exist any proper normal subgroup N of X satisfying $X = NX'$. In particular, X/N cannot have finite exponent.

(iii) If X satisfies the normalizer condition then X' is not properly supplemented in X.

Proof. (i) Let $\overline{X} = X/[X, X']$. Then $\overline{X}' \leq Z(\overline{X})$ and so $\overline{X}$ is nilpotent and $\overline{X}/\overline{X}'$ is radicable abelian which implies that $\overline{X}$ is abelian and, hence, $X' \leq [X, X']$ by Theorem 9.23 of [13].

(ii) Assume that $X = NX'$ for some proper normal subgroup N of X. Then

$$\begin{aligned} X' = [X', NX'] &= [X', N]X'' \\ &= [X', N] \\ &\leq N \end{aligned}$$

and so $X = N$ by (i) and by Lemma 2.22 of [13], which is a contradiction. Also, if X/N has finite exponent then, as X/NX' is radicable abelian and has finite exponent, it follows that $X = NX'$ and so $X = N$ by the first part of (ii), which is another contradiction.

(iii) Assume that $X = CX'$ for some $C < X$. Let $D = C \cap X'$ and put $\overline{X} = X/DX''$. Then $\overline{X} = \overline{CX}'$ and $\overline{C}$ is radicable abelian and Chernikov. Let

$\overline{Y} = N_{\overline{X}}(\overline{C})$. Then $\overline{Y} = \overline{C}(\overline{Y} \cap \overline{X}')$ which implies that $\overline{Y}$ is nilpotent since $\overline{C}$ and $\overline{Y} \cap \overline{X}'$ are normal nilpotent subgroups of $\overline{Y}$. In particular, $\overline{C} \leq Z(\overline{Y})$ by Lemma 3.13. of [13]. Assume that $\overline{Y} < \overline{X}$ and put $\overline{V} = N_{\overline{X}}(\overline{Y})$. Then $\overline{Y} < \overline{V}$ by hypothesis. But since $\overline{V} = \overline{C}(\overline{V} \cap \overline{X}')$ it follows as in the first case that $\overline{C} \leq Z(\overline{V})$ and so $\overline{V} \leq \overline{Y}$ which is a contradiction. Consequently, it follows that $\overline{Y} = \overline{X}$ and so $\overline{C} \leq Z(\overline{X})$. But now

$$\begin{aligned} \overline{X}' = [\overline{X}', \overline{X}] &= [\overline{X}', \overline{C}\overline{X}'] \\ &= [\overline{X}', \overline{X}'] \\ &= \overline{X}'' \end{aligned}$$

which is possible only if $\overline{X}' = 1$ and so $X' \leq DX'' \leq D \leq C$ by Lemma 2.22 of [13] since X' is nilpotent. This a contradiction since $C < X$. □

Lemma 2.3. Let X be an $NC-p$-group and T be an HM^*-subgroup of X. If N is a normal subgroup of X such that X/N is Chernikov then $T' \leq N$.

Proof. Let $D = T' \cap N$ and put $\overline{T} = T/D$. Then $\overline{T}'$ is nilpotent and Chernikov so the Corollary to Theorem 3.29 (2) of [13] gives that $\overline{T}/C_{\overline{T}}(\overline{T}')$ is finite which implies that $\overline{T} = C_{\overline{T}}(\overline{T}')$ and hence $\overline{T}' = [\overline{T}', \overline{T}] = 1$ by (ii) and (i) of Lemma 2.2. This means that $T' \leq D \leq N$, which was to be shown. □

Lemma 2.4. Let X be an $NC-p$-group and N be a normal nilpotent subgroup of finite exponent of X such that X/N is infinite Chernikov. Then X contains a maximal normal HM^*-subgroup T such that X/T has finite exponent. Furthermore, T is unique.

Proof. We use induction on the nilpotency class c of N. If $c = 0$ then $N = 1$ and X is Chernikov so, then, we may let $T = X^\circ$. Now suppose that $c \geq 1$ and the assertion holds for $c - 1$. Let $Z = Z(N)$ and put $\overline{X} = X/Z$. By induction hypothesis $\overline{X}$ contains a maximal HM^*-subgroup $\overline{T}$ such that $\overline{X}/\overline{T}$ and, hence, also X/T has finite exponent. Also $T'Z = [T', T]Z$ since $\overline{T}' = [\overline{T}', \overline{T}]$. Moreover, $T' \leq N$ by Lemma 2.3.

□

Next put $\overline{T} = T/[T', T]$. Then $\overline{T}' \leq Z(\overline{T})$ and so $\overline{T}$ is nilpotent. Also, $\overline{T}' = \overline{Z}$ by the preceding paragraph which implies that $\overline{T}/\overline{Z} \cong C_{p^\infty}^{(n)}$ for some $n \geq 1$. Therefore, $\overline{T} = \overline{S}\overline{Z}$ by Lemma 2.1 since $\overline{Z}$ has finite exponent, where $\overline{S}$ is the maximal radicable abelian subgroup of $\overline{T}$. In fact $\overline{S} \cong C_{p^\infty}^{(n)}$ since $\overline{Z}$ has finite exponent. Furthermore, as $\overline{T}$ is nilpotent, $\overline{S} \leq Z(\overline{T})$ by Lemma 3.13 of [13], which yields that $\overline{T}$ is abelian and, hence, $T' = [T', T]$. Consequently it follows that $S/T' \cong C_{p^\infty}^{(n)}$.

We claim that S is a normal HM^*-subgroup of X such that X/S has finite exponent. Since S is characteristic in T and T is normal in X, it follows that S is normal in X. Moreover, T/S has finite exponent as does X/T, since $T = SZ$ and Z has finite exponent. Therefore, X/S has finite exponent. Now put $\overline{T} = T/S'$. Then $\overline{T} = \overline{SZ}$ and so $\overline{T}$ is nilpotent. Also, by Lemma 2.1 $\overline{S} = \overline{V}\,\overline{T}'$, where $\overline{V}$ is the maximal radicable abelian subgroup of $\overline{S}$ since $T' \leq N$ by Lemma 2.3 and N has finite exponent. Substituting this above yields that $\overline{T} = \overline{VT'Z}$, which yields as before that $\overline{T} = \overline{VZ}$ is abelian and hence $T' = S'$. Thus it follows that S is an HM^*-subgroup of X.

Now let U be any HM^*-subgroup of X. Then $U \cap S$ is a normal subgroup of U such that $U/U \cap S$ has finite exponent which yields that $U = U \cap S \leq S$ by Lemma 2.2 (ii). Therefore S is the unique maximal HM^*-subgroup of X such that X/S has finite exponent.

Lemma 2.5. Let X be an HM^*-group for a prime p such that X' has finite exponent. Then X is a product of a finite number of normal HM^*-subgroups T such that $T/T' \cong C_{p^\infty}$.

Proof. There exists an $n \geq 1$ such that

$$X/X' = Y_1/X' \times \cdots \times Y_n/X'$$

and $Y_i/X' \cong C_{p^\infty}$ for $i = 1, \ldots, n$. By Lemma 2.4 each Y_i contains a unique normal HM^*-subgroup T_i such that Y_i/T_i has finite exponent and $T_i/T_i' \cong C_{p^\infty}$, since X' has finite exponent. Evidently, $T_1T_2\cdots T_n$ is a normal subgroup of X such that $X/T_1T_2\cdots T_n$ has finite exponent which implies that

$$X = T_1T_2\cdots T_n$$

by Lemma 2.2(ii), as claimed. □

Proof of the Theorem

Lemma 3.1. Let G be an $\overline{NC} - p$-group. Then G is countably infinite.

Proof. By hypothesis G is infinite and every proper subgroup of it, being an NC-group, is solvable. However G is not solvable since it is perfect by Theorem A(ii) (see the end of §1). Therefore, for each $n \geq 1$ we can find a finite subgroup F_n of G of derived length equal to n. Let $F = \langle F_n : n \geq 1 \rangle$. Then F is countably infinite but not solvable which implies that $F = G$, since every proper subgroup of G is solvable.

□

Lemma 3.2. Let G be a locally nilpotent p-group such that every proper subgroup of G is an NC-subgroup. If every finite subgroup of G is subnormal in G, then F^G is nilpotent of finite exponent for every finite subgroup F of G.

Proof. Assume that every finite subgroup of G is subnormal in G. Let F be a finite subgroup of G and put $H = F^G$. First suppose that H is nilpotent. Since H/H', being abelian, has finite exponent it follows from the Corollary to Theorem 2.26 of [13] that H has finite exponent. Thus to complete the proof we must show that H is nilpotent.

If G is perfect, then by Theorem A(i) of [2] G contains a normal nilpotent subgroup N such that $F \leq N$. Then $H = F^G \leq N$, and so H is nilpotent. So suppose that G is not perfect. Then G is an NC-group by Theorem A(ii), that is, G contains a normal nilpotent subgroup K such that G/K is Chernikov. Since $HK/K = F^GK/K = (FK)^G/K$ is Chernikov, F is finite and subnormal, it is easy to see that HK/K is finite. Thus $HK = LK$ for some finite subgroup L of H. Moreover, LK is nilpotent by (1) Lemma of [11] since L is subnormal, which implies that H is nilpotent.

□

In a locally nilpotent p-group G in which every proper subgroup is an NC-group, let $W(G)$ be the set of all HM^*-subgroups T of G such that $T/T' \cong C_{p^\infty}$ and let K be the subgroup of G which is generated by all the maximal elements of $W(G)$. Of course, $W(G)$ might be empty or it may not have maximal elements, in which case K is not defined.

Lemma 3.3. Let G be a locally nilpotent p-group such that every proper subgroup of G is a Chernikov extension of a nilpotent subgroup of finite exponent. Suppose that every finite subgroup of G is subnormal in G. Then every maximal element of $W(G)$ is normal in G.

Proof. Let T be a maximal element of $W(G)$. Without loss of generality we may suppose that $T \neq G$. Then T' has finite exponent by hypothesis and by Lemma 2.3. Let $a \in G$ and put $H = a^G$. By Lemma 3.2 H is nilpotent of finite exponent. Put $L = HT$. Since HT' has finite exponent, it must be nilpotent by hypothesis. Thus by Lemma 2.4. L contains a unique maximal HM^*-subgroup R. Then $T \leq R$ and $HT = HR$. Hence

$$\begin{aligned} R/R \cap HT' \cong RH/HT' &= TH/HT' \\ &\cong T/T \cap HT' \\ &\cong C_{p^\infty}, \end{aligned}$$

which yields that $R/R' \cong C_{p^\infty}$, since HT' has finite exponent. Clearly then $R = T$ by the maximality of T and hence L normalizes T. In particular, a normalizes T since $a \in L$. Since a is any element of G, it follows that G normalizes T.

□

Lemma 3.4. Let G be an $\overline{NC}-p$-group such that every proper subgroup of G is a Chernikov extension of a nilpotent subgroup of finite exponent. Then one of the following holds.

(i) G has an epimorphic image which is an $\overline{NF}$-group.

(ii) $G = \bigcup_{i=1}^{\infty} T_i$,
where for each $i \geq 1$, $T_i \in W(G)$ and $T_i \leq T_{i+1}$.

(iii) $G = K$.

proof Assume that (i) and (ii) do not hold. Partially, order $W(G)$ by set inclusion. Let $\{T_i : i \geq 1\}$ be a chain in $W(G)$ and put

$$E = \bigcup_{i=1}^{\infty} T_i.$$

(It sufficies to consider only the chains of the above form since G is countable by Lemma 3.1). By assumption $E \neq G$, so by Lemma 2.4 E contains a unique maximal HM^*-subgroup Y. Then $E = Y$, since $T_i \leq Y$ for all $i \geq 1$. Thus E is an HM^*-subgroup of G. Also E' has finite exponent by hypothesis and by Lemma 2.3 and, for the some reason, $T_i' \leq E'$ for all $i \geq 1$. Hence it follows that

$$E'T_i = E'T_{i+1}$$

for all $i \geq 1$, since $T_i \leq T_{i+1}$ and $T_i/T_i' \cong C_{p^\infty}$. Clearly this yields that $E = E'T_1$ and so $E/E' \cong C_{p^\infty}$, that is $E \in W(G)$. Thus by Zorn's Lemma $W(G)$ contains maximal elements and so K is defined.

By Theorem A(i) every finite subgroup of G is subnormal in G. Therefore every maximal element of $W(G)$ is normal in G by Lemma 3.3. This means that K is normal in G.

Suppose that $K \neq G$. Put $\overline{G} = G/K$. Since $\overline{G}$ is not an $\overline{NF}-p$-group, it contains a proper subgroup $\overline{X}$ such that $\overline{X}$ is not nilpotent. Then X is also not nilpotent. Also, X contains a normal nilpotent subgroup U of finite exponent such that X/U is Chernikov. Evidently X/U has infinite exponent since X is not nilpotent. Thus by Lemma 2.4 X contains a normal HM^*-subgroup Y such X/Y has finite exponent. But, since $Y \leq K$ by Lemma 2.5, it follows that $\overline{X}$ has finite exponent, which is a contradiction.

Proof of the Theorem. By hypothesis and by Theorem A(i) G contains a proper normal subgroup N such that every proper subgroup of G/N is a Chernikov extension of a nilpotent subgroup of finite exponent. Thus G/N satisfies one of (i), (ii) or (iii) of

Lemma 3.4. Assume that (i) and (iii) are not satisfied. Then G/N satisfies (ii). Put $\overline{G} = G/N$. Then

$$\overline{G} = \bigcup_{i=1}^{\infty} \overline{T}_i,$$

where for each $i \geq 1$, $\overline{T}_i \in W(\overline{G})$ and $\overline{T}_i \leq \overline{T}_{i+1}$. Put

$$\overline{H} = \bigcup_{i=1}^{\infty} \overline{T}'_i.$$

It is easy to see that $\overline{H}$ is normal in $\overline{G}$ and $\overline{G}/\overline{H}$ is abelian since each $\overline{T}_i\overline{H}/\overline{H}$ is abelian. This implies that $\overline{G}/\overline{H} = 1$ and hence $\overline{G} = \overline{H}$ since G is perfect. This completes the proof of the theorem.

Proof of the Corollary. By the theorem G contains a proper normal subgroup K such that every proper subgroup of G/K is a Chernikov extension of a nilpotent subgroup of finite exponent and one of (i), (ii) or (iii) of the Theorem is satisfied. Assume that (ii) is satisfied. Without loss of generality $K = 1$. Then

$$G = \bigcup_{i=1}^{\infty} T_i,$$

where for each $i \geq 1$, $T_i \in W(G)$ and $T_i \leq T_{i+1}$. Also, $T_1T'_i = T_i$ for all $i \geq 1$, since $T_i/T'_i \cong C_{p^\infty}$. But this implies that $T_1 = T_i$ for all $i \geq 1$ and hence $G = T_1$ by Lemma 2.2 (iii) which is a contradiction.

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