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## ON EXISTENCE OF A MEROMORPHIC FUNCTION WITH INFINITELY MANY OF DEFICIENT FUNCTIONS

*Halil Ardahan*

### Abstract

Let  $\{a_n(z) : \{a_n(z)\}_{n=1}^w, w \leq \infty\}$  be a finite or denumerable set of entire functions satisfying

$$T(r, a_n) \leq d_0 r^\alpha + d_n, \quad n = 1, 2, \dots,$$

where  $d_0, d_n > 0$  do not depend on  $r, 0 < \alpha < 1$ .

In this paper we show that there is a meromorphic function of order 1 and normal type having  $a_1(z), a_2(z), \dots, a_n(z), \dots$ , as deficient functions.

### 1. Introduction

Goldberg [1, 2] constructed the first examples of meromorphic functions  $f(z)$  with infinitely many deficient values in 1954. Modifications of these examples were presented by W.K. Hayman in his monograph [3].

These meromorphic functions  $f(z)$  could have any positive order and a prescribed finite or denumerable set of deficient values.

We shall prove the following theorem by using Hayman's idea.

**Theorem.** *Let  $\{a_n(z) : \{a_n(z)\}_{n=1}^w, w \leq \infty\}$  be a finite or denumerable set of entire functions satisfying the conditions*

$$T(r, a_n) \leq d_0 r^\alpha + d_n, \quad n = 1, 2, \dots, \tag{1.1}$$

*where  $d_0, d_n > 0$  do not depend on  $r$  and  $0 < \alpha < 1$ . Then there exists a meromorphic function  $f(z)$  of order 1 and normal type such that*

$$\delta(a_n, f(z)) > 0, \quad n = 1, 2, \dots$$

*This theorem seems to be of interest in connection with recent investigations related to deficient functions (see e.g. [4], p.34-41).*

**2. Proof of Theorem**

Without loss of generality we can assume  $w = \infty$ , otherwise we can consider the sequence  $a_1(z), a_2(z), \dots, a_w(z), a_w(z), a_w(z), \dots$ .

Let  $\{\eta_v\}_{v=1}^\infty$  be a decreasing sequence of positive numbers such that  $\sum_1^\infty \eta_v = 1$ . We set  $\eta_1 = \eta_0$  and for all  $1 < n < \infty$  define  $\theta_0 = 0, \theta_n = \pi \sum_{v=0}^{n-1} \eta_v, (n = 1, \text{ to } \infty)$ . We see that  $\{\theta_n\}_{n=1}^\infty$  is an increasing sequence and  $0 \leq \theta_n < \pi$ . Note that the well-known inequality,

$$\log^+ M(r, f) \leq 3T(2r, f)$$

is valid for any entire function  $f(z)$ . Therefore using (1.1), we have

$$\log^+ M(r, a_n(z)) \leq 3(d_0 2^\alpha r^\alpha + d_n), \quad n = 1, 2, \dots \tag{1.2}$$

whence

$$|a_n(z)| \leq e^{3d_n} \exp(3d_0 2^\alpha r^\alpha).$$

We now choose a sequence  $\{c_n\}_{n=1}^\infty$  of positive numbers such that

$$S_1 = \sum_{n=1}^\infty c_n < \infty, \quad S_2 = \sum_{n=1}^\infty c_n e^{3d_n} < \infty$$

and define

$$F_1(z) = \sum_{n=1}^\infty c_n a_n(z) \exp(ze^{-i\theta_n}), \quad F_2(z) = \sum_{n=1}^\infty c_n \exp(ze^{-i\theta_n}),$$

$$f(z) = F_1(z)/F_2(z).$$

We see that for  $|z| = r$ ,

$$\begin{aligned} |F_1(z)| &= \left| \sum_{n=1}^\infty c_n a_n(z) \exp(ze^{-i\theta_n}) \right| \leq \sum_{n=1}^\infty c_n |a_n(z)| e^r \\ &\leq \sum_{n=1}^\infty c_n e^r e^{3d_n} \exp(3d_0 2^\alpha r^\alpha) \\ &\leq S_2 \exp(3d_0 2^\alpha r^\alpha + r), \\ |F_2(z)| &\leq \sum_{n=1}^\infty c_n e^r < S_1 \exp(r), \end{aligned}$$

and so

$$\begin{aligned} T(r, f) &< T(r, F_1) + T(r, F_2) + O(1) \leq \log^+ M(r, F_1) + \log^+ M(r, F_2) + O(1) \\ &\leq 2r + o(r), \quad \text{as } r \rightarrow \infty. \end{aligned} \tag{1.3}$$

Suppose that  $n \geq 1$  and that

$$\theta_n - \frac{1}{3}\pi\eta_n \leq \theta \leq \theta_n + \frac{1}{3}\pi\eta_n. \quad (1.4)$$

For  $v < n$  we have the following inequalities

$$\begin{aligned} \theta &\geq \theta_n - \frac{1}{3}\pi\eta_n, & \theta_v &\leq \theta_{n-1}, & \eta_{n-1} &> \eta_n \\ \theta - \theta_v &\geq \theta_n - \theta_{n-1} - \frac{1}{3}\pi\eta_n = \pi\eta_{n-1} - \frac{1}{3}\pi\eta_n > \frac{2}{3}\pi\eta_n, \end{aligned}$$

and for  $v > n$ , we have

$$\theta_v - \theta \geq \theta_{n+1} - \theta_n - \frac{1}{3}\pi\eta_n = \pi\eta_n - \frac{1}{3}\pi\eta_n = \frac{2}{3}\pi\eta_n.$$

Hence, for all  $v \neq n$ , we have

$$\begin{aligned} \pi &\geq |\theta - \theta_v| \geq \frac{2}{3}\pi\eta_n. \\ \cos(\theta - \theta_v) &\leq \cos\left(\frac{2}{3}\pi\eta_n\right). \end{aligned}$$

We now define

$$\begin{aligned} F_{1,n}(z) &= F_1(z) - c_n a_n \exp(ze^{-i\theta_n}), \\ F_{2,n}(z) &= F_2(z) - c_n \exp(ze^{-i\theta_n}). \end{aligned}$$

Thus if  $\theta$  lies in the range (1.4) and  $z = r \exp(i\theta)$ , then we have

$$\begin{aligned} F_1(z) &= c_n a_n(z) \exp(ze^{-i\theta_n}) + F_{1,n}(z), \\ |F_{1,n}(z)| &\leq \sum_{v=1, v \neq n}^{\infty} c_v |a_v(z)| \exp(r \cos(\theta - \theta_v)) \\ &\leq S_2 \exp(3d_0 2^\alpha r^\alpha) \exp\left(r \cos \frac{2}{3}\pi\eta_n\right). \end{aligned} \quad (1.5)$$

In the similar way, for  $\theta$  lying in the range (1.4), we have

$$|F_{2,n}(z)| \leq S_1 \exp\left(r \cos \frac{2}{3}\pi\eta_n\right) \quad (1.6)$$

and so

$$\begin{aligned} |F_2(z)| &= |c_n \exp(ze^{-i\theta_n}) + F_{2,n}(z)| \geq |c_n \exp(ze^{-i\theta_n})| - |F_{2,n}(z)| \\ &\geq c_n \exp\left(r \cos \frac{\pi}{3}\eta_n\right) - S_1 \exp\left(r \cos \frac{2}{3}\pi\eta_n\right) \\ &> \frac{c_n}{2} \exp\left(r \cos \frac{\pi}{3}\eta_n\right) \end{aligned} \quad (1.7)$$

for all sufficiently large  $r$ . Tuhs, we have

$$\begin{aligned} F_1(z) - a_n(z)F_2(z) &= c_n a_n(z) \exp(ze^{-i\theta_n}) + \sum_{v \neq n}^{\infty} c_v a_v(z) \exp(ze^{-i\theta_v}) \\ &\quad - c_n a_n(z) \exp(ze^{-i\theta_n}) - a_n(z) \sum_{v \neq n}^{\infty} c_v \exp(ze^{-i\theta_v}) \\ &= F_{1,n}(z) - a_n(z)F_{2,n}(z). \end{aligned}$$

Using (1.5), (1.6), (1.2) and (1.7), we obtain

$$\begin{aligned} |f(z) - a_n(z)| &\leq \frac{|F_1(z) - a_n(z)F_2(z)|}{|F_2(z)|} \leq \frac{|F_{1,n}(z)| + |a_n(z)| |F_{2,n}(z)|}{|F_2(z)|} \\ &= \frac{2 \exp(3d_0 2^\alpha r^\alpha)}{c_n} (S_2 + S_1 e^{3d_n}) \exp(r(\cos \frac{2}{3} \pi \eta_n - \cos \frac{\pi}{3} \eta_n)) \\ &= \frac{2 \exp(3d_0 2^\alpha r^\alpha)}{c_n} (S_2 + S_1 e^{3d_n}) \exp(-2r \sin(\frac{\pi}{2} \eta_n) \sin(\frac{\pi}{6} \eta_n)). \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \log^+ \left| \frac{1}{f(z) - a_n(z)} \right| &\geq 3d_0 2^\alpha r^\alpha + 2r \sin \frac{\pi}{2} \eta_n \sin \frac{\pi}{6} \eta_n + O(1) \\ &\geq 3d_0 2^\alpha r^\alpha + \frac{2r}{3} \eta_n^2 + O(1) \\ &= \frac{2}{3} \eta_n^2 r + o(r), \quad \text{as } r \rightarrow \infty, \end{aligned}$$

and so

$$\begin{aligned} m(r, a_n, f) &\geq \frac{1}{2\pi} \int_{\theta_n - \frac{1}{3} \pi \eta_n}^{\theta_n + \frac{1}{3} \pi \eta_n} \log^+ \left| \frac{1}{f(re^{i\theta}) - a_n(re^{i\theta})} \right| d\theta \\ &\geq \frac{2r}{9} \eta_n^3 + o(r), \quad \text{as } r \rightarrow \infty. \end{aligned}$$

Thus in view of (1.3), we have

$$\delta(a_n(z), f(z)) = \liminf_{r \rightarrow \infty} \frac{m(r, a_n, f)}{T(r, f)} \geq \liminf_{r \rightarrow \infty} \frac{\frac{2r}{9} \eta_n^3 + O(r)}{2r + O(r)} = \frac{1}{9} \eta_n^3 > 0.$$

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### References

- [1] Goldberg, A.A. On the deficiencies of meromorphic functions, *Nauk SSSR (N.S)*, **98**, 893-5, Ukrain 1954.
- [2] Goldberg, A.A. On the set of deficient values of meromorphic functions of finite order, *Ukrain Mat. Zh.*, **11**, 438-43, Ukrain 1959.
- [3] Hayman, W.K. *Meromorphic functions*, Oxford University Press, London 1964.
- [4] Yang, Lo. *Value distribution theory*, Springer-Verlag, Berlin 1993.

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