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ON A CERTAIN FAMILY OF LINEAR POSITIVE OPERATORS

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Abstract

In this study, a family of linear positive operators, which includes the sequence of linear positive operators built in a paper of A. D. Gadjiev and I. I. Ibragimov and later investigated by B. Wood and P. Radatz, is defined and using the some inequalities proved by P. J. Davis, results related to approximation properties of this family are obtained.

1. Introduction

Let $C([a,b], x^2)$ be the space of functions defined on the entire line, continuous in the interval $[a,b]$, continuous from the right at the point $x = b$, and from the left at the point $x = a$, and increasing to infinity not more rapidly than x^2. Let us assume also that L_n, $n = 1, 2, ...$, are linear positive operators, defined on the set $C([a,b], x^2)$.

The following theorem [5] is well known (concerning functions bounded in growth, see [2]).

Theorem of P. P. Korovkin. *If* $L_n(t^k; x) \longrightarrow x^k$, $k = 0, 1, 2$, *uniformly on the entire interval* $[a,b]$, *then* $L_n(f; x) \longrightarrow f(x)$ *uniformly on this interval for all functions* $f(x) \in C([a,b], x^2)$.

In [4], a general sequence of linear positive operators, while satisfies the conditions of this theorem, is defined and it is shown that this sequence of operators, in special case, consists of the well known Bernstein (see [6]), Szas (see [8]), Bernstein-Chlodovsky (see [6]) and V. A. Baskakov [1] operators. Some new properties of the operators defined in [4] are investigated in [7].

In this study, we define and investigate a generalization of the linear positive operators defined in [4].

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2. Generalized Linear Positive Operators

Let λ and A be positive real numbers, $\{\varphi_\lambda(t)\}$ and $\{\psi_\lambda(t)\}$ be the family of functions in $C\,[0,A]$ such that $\varphi_\lambda(0)=0,\ \psi_\lambda(t)>0,$ for each $t\in[0,A]$.

Let also $\{\alpha_\lambda\}$ be a family of positive numbers such that

$$\lim_{\lambda\to\infty}\frac{\alpha_\lambda}{\lambda}=1,\quad \lim_{\lambda\to\infty}\frac{1}{\lambda^2\psi_\lambda(0)}=0\,.$$

Assume that a family of functions of three variables $\{K_\lambda(x,t,u)\}$ $(x,t\in[0,A]$, $-\infty<u<\infty,\ \lambda\geq 0)$ satisfies the following conditions:

1^0. Each function of this family is an entire analytic function with respect to u for fixed x and t of the segment $[0,A]$.

2^0. $K_\lambda(x,0,0)=1$ for any $x\in[0,A]$ and for any $\lambda\geq 0$.

3^0. $\left\{(-1)^v\frac{\partial^v}{\partial u^v}\,K_\lambda(x,t,u)|_{\substack{u=u_1\\t=0}}\right\}\geq 0$ $(\forall x\in[0,A],\quad \lambda\geq 0,\quad v=0,1,...)$.

4^0. $\frac{\partial^v}{\partial u^v}\,K_\lambda(x,t,u)|_{\substack{u=u_1\\t=0}}=-\lambda x\left[\frac{\partial^{v-1}}{\partial u^{v-1}}K_{h(\lambda)}(x,t,u)\right]\Big|_{\substack{u=u_1\\t=0}}$

$(\forall x\in[0,A],\quad \lambda\in R^+,\ v=1,2,...)$ where $h(\lambda)$ is a nonnegative function satisfying the condition $\lim_{\lambda\to\infty}\frac{h(\lambda)}{\lambda}=1$.

Consider the family of linear operators;

$$L_\lambda(f;x)=\sum_{v=0}^{\infty}f(\frac{v}{\lambda^2\psi_\lambda(0)})\left\{\left[\frac{\partial^v}{\partial u^v}K_\lambda(x,t,u)\right]_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^v}{v!}\,, \tag{1}$$

where $f\in C([0,A],x^2)$.

Note that for $\lambda=n$ and $h(\lambda)=m+n$ $(m+n=0,1,2,...)$, the operators defined by (1) are reduced to the operators defined in [4].

First we prove the following theorem.

Theorem 1. *Uniformly on* $[0,A]$

$$\lim_{\lambda\to\infty}L_\lambda(f;x)=f(x)$$

for every function $f\in C([0,A],x^2)$.

Proof. It is easily seen that by 3^0 the operators $L_\lambda(f;x)$ defined in (1) are linear positive operators. Thus, it is sufficient to verify the conditions of P. P. Korovkin's theorem.

Since $\{K_\lambda(x,t,u)\}$ is entire analytic function, we can expand it to Taylor series of any arbitrary point $u=u_1$. Thus for each $\lambda\geq 0$,

$$K_\lambda(x,t,u)=\sum_{v=0}^{\infty}\left\{\left[\frac{\partial^v}{\partial u^v}K_\lambda(x,t,u)\right]_{u=u_1}\right\}\frac{[u-u_1]^v}{v!}\,.$$

Setting $u = \varphi_\lambda(t)$ and $u_1 = \alpha_\lambda \psi_\lambda(t)$, we have

$$K_\lambda(x,t,\varphi_\lambda(t)) = \sum_{v=0}^{\infty} \left\{ \left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u) \right]_{u=\alpha_\lambda \psi_\lambda(t)} \right\} \frac{[\varphi_\lambda(t) - \alpha_\lambda \psi_\lambda(t)]^v}{v!} .$$

Since $\varphi_\lambda(0) = 0$ and $K_\lambda(x,0,0) = 1$, by setting $t = 0$ we get,

$$\sum_{v=0}^{\infty} \left\{ \left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u) \right]_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^v}{v!} = 1 .$$

This means that $L_\lambda(1;x) = 1$ for any $\lambda \geq 0$ and the first condition of Korovkin theorem is satisfied.

On the other hand, by using 4^0 in the equality

$$L_\lambda(t;x) = \sum_{v=1}^{\infty} \frac{v}{\lambda^2 \psi_\lambda(0)} \left\{ \left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u) \right]_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^v}{v!}$$

we obtain after simplification,

$$\begin{aligned} L_\lambda(t;x) &= \sum_{v=1}^{\infty} \frac{[-\alpha_\lambda \psi_\lambda(0)]\,(-\lambda x)}{\lambda^2 \psi_\lambda(0)} \left\{ \left[\frac{\partial^{v-1}}{\partial u^{v-1}} K_{h(\lambda)}(x,t,u) \right]_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^{v-1}}{(v-1)!} \\ &= \frac{\alpha_\lambda x}{\lambda} \sum_{v=0}^{\infty} \left\{ \left[\frac{\partial^v}{\partial u^v} K_{h(\lambda)}(x,t,u) \right]_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^v}{v!} \\ &= \frac{\alpha_\lambda x}{\lambda} L_{h(\lambda)}(1;x). \end{aligned}$$

Since $L_{h(\lambda)}(1;x) = 1$, we get

$$L_\lambda(t;x) = \frac{\alpha_\lambda x}{\lambda},$$

and since $\lim_{\lambda\to\infty} \frac{\alpha_\lambda}{\lambda} = 1$ uniformly on $[0,A]$,

$$\lim_{\lambda\to\infty} L_\lambda(t;x) = x.$$

Finally, using

$$L_\lambda(t^2;x) = \sum_{v=1}^{\infty} \left(\frac{v}{\lambda^2 \psi_\lambda(0)} \right)^2 \left\{ \left[\frac{\partial^v}{\partial u^v} K_\lambda(x,t,u) \right]_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} \right\} \frac{[-\alpha_\lambda \psi_\lambda(0)]^v}{v!},$$

and making simplifications, we get

$$\begin{aligned}
L_\lambda(t^2;x) &= \frac{\alpha_\lambda x}{\lambda}\sum_{v=1}^{\infty}\frac{v}{\lambda^2\psi_\lambda(0)}\left\{\left[\frac{\partial^{v-1}}{\partial u^{v-1}}K_{h(\lambda)}(x,t,u)\right]_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^{v-1}}{(v-1)!}\\
&= \left(\frac{\alpha_\lambda x}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}\sum_{v=2}^{\infty}\left\{\left[\frac{\partial^{v-2}}{\partial u^{v-2}}K_{h(h(\lambda))}(x,t,u)\right]_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^{v-2}}{(v-2)!}\\
&\quad+\frac{\alpha_\lambda x}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}\sum_{v=1}^{\infty}\left\{\left[\frac{\partial^{v-1}}{\partial u^{v-1}}K_{h(\lambda)}(x,t,u)\right]_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\right\}\frac{[-\alpha_\lambda\psi_\lambda(0)]^{v-1}}{(v-1)!}\\
&= \left(\frac{\alpha_\lambda x}{\lambda}\right)^2\frac{h(\lambda)}{\lambda}+\frac{\alpha_\lambda x}{\lambda}\frac{1}{\lambda^2\psi_\lambda(0)}.
\end{aligned}$$

Since

$$\lim_{\lambda\to\infty}\frac{h(\lambda)}{\lambda}=1,\quad \lim_{\lambda\to\infty}\frac{\alpha_\lambda}{\lambda}=1 \quad\text{and}\quad \lim_{\lambda\to\infty}\frac{1}{\lambda^2\psi_\lambda(0)}=0\ ,$$

we see that all conditions of Korovkin's theorem hold.

The proof is complete. □

Remark. By choosing, $\lambda = n \quad (n=1,2,...)$ in $\{K_\lambda(x,t,u)\}$ we obtain, as in [4], some known sequences of linear positive operators. Some of them follows:

By choosing

$$K_n(x,t,u)=\left[1-\frac{ux}{1+\ t}\right]^n,\ \alpha_n=n\ ,\ \psi_n(0)=\frac{1}{n}\ ,$$

we have $h(n)=n-1$ and the operators defined by (1) are transformed into Bernstein polynomials.

For

$$\alpha_n=n\ ,\ \psi_n(0)=\frac{1}{nb_n}\ \left(\lim_{n\to\infty}b_n=\infty\ ,\ \lim_{n\to\infty}\frac{b_n}{n}=0\ \right),$$

we obtain Bernstein-Chlodovsky polynomials.

By choosing

$$K_n(x,t,u)=e^{-n(t+ux)},\ \alpha_n=n\ ,\ \psi_n(0)=\frac{1}{n}\ ,$$

we have $h(n)=n$ and get Sasz operators.

If $K_n(z)$ is entire analytic function and

$$K_n(x,t,u)=K_n(t+ux),\ \alpha_n=n\ ,\ \psi_n(0)=\frac{1}{n}\ ,$$

then we obtain Baskakov [1] operators, and for

$$\alpha_n = n \ , \ \psi_n(0) = \frac{1}{\alpha_n} \quad \text{and} \quad \frac{n^2}{\alpha_n} = \beta_n \ ,$$

we get Baskakov [2] operators.

3. The Properties of The Operators Family (1)

Let us assume that $\{K_\lambda(x,t,u)\}$, in addition to $1^0, 2^0, 3^0, 4^0$, satisfies the condition

5^0. $\frac{d}{dx} K_\lambda(x,t,u)\big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} = -\lambda\alpha_\lambda\psi_\lambda(0)\ K_{h(\lambda)}(x,t,u)\big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}$.

It can be easily shown that for $\lambda = n$ $(n = 1, 2, ...)$ the sequences of functions $\{K_n(x,t,u)\}$ in special case mentioned above also satisfy 5^0.

Now let us remember the definition of the divided difference of a function g;

Assume that $x_0, x_1, ..., x_n$ be any points in the domain of g. Denote

$$[x_0; g] = g(x_0)$$
$$[x_0, x_1; g] = \frac{g(x_0)}{x_0 - x_1} + \frac{g(x_1)}{x_1 - x_0}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$[x_0, x_1, ..., x_n; g] = \frac{g(x_0)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} + ... + \frac{g(x_n)}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})}.$$

The left hand side of this equalities are called, respectively, the zeros, first, second, ... ,$(n-1)$-th divided differences of the function g.

Theorem 2. *Let the family of the functions $\{K_\lambda(x,t,u)\}$ satisfies conditions $1^0 - 5^0$. Then for each integer $p > 0$,*

$$\frac{d^p}{dx^p} L_\lambda(f;x) = \frac{\alpha_\lambda^p\, p!}{\lambda^{2p}} \sum_{v=0}^{\infty} \left[\frac{v}{\lambda^2\varphi_\lambda(0)}, \frac{v+1}{\lambda^2\varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2\varphi_\lambda(0)}; f \right]$$
$$\times\ \frac{\lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v+p-1)}(\lambda)}{v!} (x\alpha_\lambda\psi_\lambda(0))^v\ K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} , \qquad (2)$$

where

$$h_{(v+p)}(\lambda) = \underbrace{h(h(...h}_{v+p-\ times}\ (\lambda)))$$

and

$$\left[\frac{v}{\lambda^2\varphi_\lambda(0)}, \frac{v+1}{\lambda^2\varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2\varphi_\lambda(0)}; f \right]$$

is the divided difference of f at the points $\frac{v}{\lambda^2\varphi_\lambda(0)}, \frac{v+1}{\lambda^2\varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2\varphi_\lambda(0)}$.

Proof. First, consider the equality

$$\frac{\partial^{\upsilon}}{\partial u^{\upsilon}}\, K_{\lambda}(x,t,u)|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} = (-1)^{\upsilon}x^{\upsilon}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon-1)}(\lambda)\; K_{h_{(\upsilon)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} .$$

Taking the derivative, we obtain by 5^0,

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{\partial^{\upsilon}}{\partial u^{\upsilon}}\, K_{\lambda}(x,t,u)|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}}\right] \\
&= (-1)^{\upsilon}\upsilon x^{\upsilon-1}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon-1)}(\lambda) \\
&\quad \times K_{h_{(\upsilon)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} - (-1)^{\upsilon}x^{\upsilon}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon)}(\lambda) \\
&\quad \times K_{h_{(\upsilon+1)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} \alpha_{\lambda}\psi_{\lambda}(0).
\end{aligned}$$

From this, we get

$$\begin{aligned}
\frac{d}{dx}L_{\lambda}(f;x) &= \sum_{\upsilon=0}^{\infty} f(\frac{\upsilon}{\lambda^2\psi_{\lambda}(0)})\frac{d}{dx}\left\{\left[\frac{\partial^{\upsilon}}{\partial u^{\upsilon}}K_{\lambda}(x,t,u)\right]_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}}\right\}\frac{[-\alpha_{\lambda}\psi_{\lambda}(0)]^{\upsilon}}{\upsilon!} \\
&= \sum_{\upsilon=0}^{\infty}\left[f(\frac{\upsilon+1}{\lambda^2\psi_{\lambda}(0)}) - f(\frac{\upsilon}{\lambda^2\psi_{\lambda}(0)})\right] x^{\upsilon}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon)}(\lambda) \\
&\times \quad K_{h_{(\upsilon+1)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} \frac{[\alpha_{\lambda}\psi_{\lambda}(0)]^{\upsilon+1}}{\upsilon!}.
\end{aligned}$$

Similarly, since

$$\begin{aligned}
&\frac{d^2}{dx^2}\left[\frac{\partial^{\upsilon}}{\partial u^{\upsilon}}\, K_{\lambda}(x,t,u)|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}}\right] \\
&= (-1)^{\upsilon}\upsilon(\upsilon-1)x^{\upsilon-2}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon-1)}(\lambda)\; K_{h_{(\upsilon)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} \\
&\quad -2(-1)^{\upsilon}\upsilon x^{\upsilon-1}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon)}(\lambda)\; K_{h_{(\upsilon+1)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} \\
&\quad +(-1)^{\upsilon}x^{\upsilon}\lambda\, h(\lambda)\, h_{(2)}(\lambda)^{\cdots}h_{(\upsilon+1)}(\lambda)\; K_{h_{(\upsilon+2)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_{\lambda}\psi_{\lambda}(t)\\ t=0}} (\alpha_{\lambda}\psi_{\lambda}(0))^2,
\end{aligned}$$

we have

$$\frac{d^2}{dx^2}L_{\lambda}(f;x) = \sum_{\upsilon=0}^{\infty}\left[f(\frac{\upsilon+2}{\lambda^2\psi_{\lambda}(0)}) - 2f(\frac{\upsilon+1}{\lambda^2\psi_{\lambda}(0)}) + f(\frac{\upsilon}{\lambda^2\psi_{\lambda}(0)})\right]$$

$$\times\ x^{v}\lambda\, h(\lambda)\, h_{(2)}(\lambda)\cdots h_{(v+1)}(\lambda)\ K_{h_{(v+2)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{[\alpha_\lambda\psi_\lambda(0)]^{v+2}}{v!}\ .$$

In a similar way, we obtain

$$\begin{aligned}\frac{d^3}{dx^3}L_\lambda(f;x) &= \sum_{v=0}^{\infty}\left[f(\frac{v+3}{\lambda^2\psi_\lambda(0)})-3f(\frac{v+2}{\lambda^2\psi_\lambda(0)})+3f(\frac{v+1}{\lambda^2\psi_\lambda(0)})-f(\frac{v}{\lambda^2\psi_\lambda(0)})\right]\\ &\times\ x^{v}\lambda\, h(\lambda)\, h_{(2)}(\lambda)\cdots h_{(v+2)}(\lambda)\ K_{h_{(v+3)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{[\alpha_\lambda\psi_\lambda(0)]^{v+3}}{v!}.\end{aligned}$$

Thus, in general one can have

$$\begin{aligned}\frac{d^p}{dx^p}L_\lambda(f;x) &= \sum_{v=0}^{\infty}\left(\sum_{i=0}^{p}(-1)^{p-i}C_p^i f(\frac{v+i}{\lambda^2\psi_\lambda(0)})\right) x^{v}\lambda\, h(\lambda)\, h_{(2)}(\lambda)\cdots h_{(v+p-1)}(\lambda)\\ &\times\ K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{[\alpha_\lambda\psi_\lambda(0)]^{v+p}}{v!}. \qquad (3)\end{aligned}$$

On the other hand, in [3, p.65] Davis shows that for $[x_0, x_1, ..., x_p; f]$ as being the divided difference of the function f at the points $x_0, x_1, ..., x_p$, and for

$$\triangle^p f(x_0)=\sum_{i=0}^{p}(-1)^{p-i}C_p^i f(x_0+ib)\ ;\quad b=x_{k+1}-x_k\ ,\quad 0<k<p-1\,,$$

we have the equality

$$[x_0, x_1, ..., x_p; f]=\frac{1}{b^p p!}\triangle^p f(x_0)\ .$$

Thus, for the points $x_0=\frac{v}{\lambda^2\psi_\lambda(0)}$, $x_1=\frac{v+1}{\lambda^2\psi_\lambda(0)}, ...,$ $x_p=\frac{v+p}{\lambda^2\psi_\lambda(0)}$ we have $b=\frac{1}{\lambda^2\psi_\lambda(0)}$. Therefore

$$\triangle^p f(\frac{v}{\lambda^2\psi_\lambda(0)})=\sum_{i=0}^{p}(-1)^{p-i}C_p^i f(\frac{v+i}{\lambda^2\psi_\lambda(0)})\,,$$

and

$$\left[\frac{v}{\lambda^2\varphi_\lambda(0)},\frac{v+1}{\lambda^2\varphi_\lambda(0)},...,\frac{v+p}{\lambda^2\varphi_\lambda(0)};f\right]=\frac{1}{(\frac{1}{\lambda^2\psi_\lambda(0)})^p p!}\triangle^p f(\frac{v}{\lambda^2\psi_\lambda(0)})\ .$$

Using these in (1), we obtain

$$\begin{aligned}\frac{d^p}{dx^p}L_\lambda(f;x) &= \sum_{v=0}^{\infty}\frac{1}{(\lambda^2\psi_\lambda(0))^p}\ p!\ \left[\frac{v}{\lambda^2\varphi_\lambda(0)},\frac{v+1}{\lambda^2\varphi_\lambda(0)},...,\frac{v+p}{\lambda^2\varphi_\lambda(0)};f\right]\\ &\times\ x^{v}\lambda\, h(\lambda)\, h_{(2)}(\lambda)\cdots h_{(v+p-1)}(\lambda)\ K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{[\alpha_\lambda\psi_\lambda(0)]^{v+p}}{v!}\end{aligned}$$

$$= \frac{\alpha_\lambda^p p!}{n^{2p}} \sum_{v=0}^{\infty} \left[\frac{v}{\lambda^2 \varphi_\lambda(0)}, \frac{v+1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2 \varphi_\lambda(0)}; f \right]$$

$$\times \quad \frac{\lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v+p-1)}(\lambda)}{v!} (x \alpha_\lambda \psi_\lambda(0))^v \; K_{h_{(v+p)}(\lambda)}(x,t,u) \Big|_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} .$$

Hence the proof is complete. □

Theorem 3. *Under the assumptions of Theorem 2,*

$L_\lambda(f;x)$

$$= \sum_{v=0}^{\infty} \frac{(\alpha_\lambda x)^v \lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v-1)}(\lambda)}{\lambda^{2v}} \left[0, \frac{1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v}{\lambda^2 \varphi_\lambda(0)}; f \right] K_{h_{(v)}(\lambda)}(0,t,u) \Big|_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}}$$

where $\left[0, \frac{1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v}{\lambda^2 \varphi_\lambda(0)}; f \right]$ *is the divided difference of* f *at the points* $0, \frac{1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v}{\lambda^2 \varphi_\lambda(0)}$.

Proof. The Taylor expansion of $L_\lambda(f;x)$ about $x=0$ is

$$L_\lambda(f;x) = \sum_{v=0}^{\infty} \frac{\partial^v}{\partial x^v} \, L_\lambda(f;x)|_{x=0} \, \frac{x^v}{v!}. \tag{4}$$

On the other hand, if we rewrite the expression (2) in Theorem 2, for $x=0$, we get

$$\frac{d^v}{dx^v} L_\lambda(f;x) \Big|_{x=0}$$

$$= \frac{\alpha_\lambda^v \; v!}{\lambda^{2v}} \left[0, \frac{1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v}{\lambda^2 \varphi_\lambda(0)}; f \right] \lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v-1)}(\lambda) \; K_{h_{(v)}(\lambda)}(0,t,u) \Big|_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} .$$

Using this in (4) gives
$L_\lambda(f;x) =$

$$\sum_{v=0}^{\infty} \frac{(\alpha_\lambda x)^v \lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v-1)}(\lambda)}{\lambda^{2v}} \left[0, \frac{1}{\lambda^2 \varphi_\lambda(0)}, ..., \frac{v}{\lambda^2 \varphi_\lambda(0)}; f \right] \; K_{h_{(v)}(\lambda)}(0,t,u) \Big|_{\substack{u=\alpha_\lambda \psi_\lambda(t) \\ t=0}} ,$$

and the proof is complete. □

Theorem 4. *Under the assumptions of Theorem 2,*

$$\lim_{\lambda\to\infty}\frac{d^p}{dx^p}L_\lambda(t^{p+n};x)=\frac{(p+n)!x^n}{n!},$$

where n is a natural number.

Proof. We give the proof by induction.

First, for $n=0$, we show that

$$\lim_{\lambda\to\infty}\frac{d^p}{dx^p}L_\lambda(t^p;x)=p!\,. \tag{5}$$

For this, in (2), let $f(t)=t^p$. Thus,

$$\frac{d^p}{dx^p}L_\lambda(t^p;x)=\frac{\alpha_\lambda^p\,p!}{\lambda^{2p}}\sum_{v=0}^{\infty}\left[\frac{v}{\lambda^2\varphi_\lambda(0)},\frac{v+1}{\lambda^2\varphi_\lambda(0)},\dots,\frac{v+p}{\lambda^2\varphi_\lambda(0)};t^p\right]$$

$$\times\frac{\lambda h(\lambda)h_{(2)}(\lambda)\cdots h_{(v+p-1)}(\lambda)}{v!}(x\alpha_\lambda\psi_\lambda(0))^v\,K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\,. \tag{6}$$

On the other hand, if f is p-th order differentiable, then the divided difference of f satisfies the equality

$$[x_0,x_1,\dots,x_p;f]=\frac{f^{(p)}(x)}{p!},$$

where $x_0<x<x_p$ (see [3, p.65]).

Taking

$$f(x)=x^p\,,\ \ x_0=\frac{v}{\lambda^2\varphi_\lambda(0)}\,,\ \ x_1=\frac{v+1}{\lambda^2\varphi_\lambda(0)}\,,\dots,x_p=\frac{v+p}{\lambda^2\varphi_\lambda(0)},$$

we have $f^{(p)}(x)=p!$, and

$$\left[\frac{v}{\lambda^2\varphi_\lambda(0)},\frac{v+1}{\lambda^2\varphi_\lambda(0)},\dots,\frac{v+p}{\lambda^2\varphi_\lambda(0)};x^p\right]=\frac{p!}{p!}=1\,.$$

Using this in (6) give us

$$\frac{d^p}{dx^p}L_\lambda(t^p;x)=\frac{\alpha_\lambda^p\,p!}{\lambda^{2p}}\sum_{v=0}^{\infty}\frac{\lambda\,h(\lambda)\,h_{(2)}(\lambda)\cdots h_{(v+p-1)}(\lambda)}{v!}$$

$$\times(x\alpha_\lambda\psi_\lambda(0))^v\,K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\,.$$

From the last equality, we can write

$$\frac{d^p}{dx^p}L_\lambda(t^p;x) = \frac{\alpha_\lambda^p\, p!}{\lambda^{2p}}\lambda\, h(\lambda)\, h_{(2)}(\lambda)\cdots h_{(p-1)}(\lambda) \sum_{v=0}^{\infty} \frac{h_{(p)}(\lambda)\, h_{(p+1)}(\lambda)\cdots h_{(v+p-1)}(\lambda)}{v!}$$

$$\times\, (x\alpha_\lambda \psi_\lambda(0))^v \; K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \; . \tag{7}$$

On the other hand, with $u=\varphi_\lambda(t)$, $u_1=\alpha_\lambda\psi_\lambda(t)$, the Taylor expansion of $K_{h_{(p)}(\lambda)}(x,t,u)$ about $(u-u_1)$ is

$$K_{h_{(p)}(\lambda)}(x,t,\varphi_\lambda(t)) = \sum_{v=0}^{\infty} \frac{\partial^v}{\partial u^v}\, K_{h_{(p)}(\lambda)}(x,t,u)\Big|_{u=\alpha_\lambda\psi_\lambda(t)} \frac{(\varphi_\lambda(t)-\alpha_\lambda\psi_\lambda(t))^v}{v!} \; .$$

Thus, taking $t=0$ in this equality, in view of the fact that $\varphi_\lambda(t)=0$, $K_{h_{(p)}(\lambda)}(x,0,0)=1$ we have

$$K_{h_{(p)}(\lambda)}(x,0,0) = 1 = \sum_{v=0}^{\infty} \frac{\partial^v}{\partial u^v}\, K_{h_{(p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{(-\alpha_\lambda\psi_\lambda(0))^v}{v!} \; .$$

From this, we get

$$1 = \sum_{v=0}^{\infty} h_{(p)}(\lambda) h_{(p+1)}(\lambda) \ldots h_{(v+p-1)}(\lambda)\; K_{h_{(p+v)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}} \frac{(-\alpha_\lambda\psi_\lambda(0))^v}{v!} \; .$$

Using this in (7), we obtain,

$$\begin{aligned}\frac{d^p}{dx^p}L_\lambda(t^p;x) &= \frac{\alpha_\lambda^p\, p!}{\lambda^{2p}}\lambda h(\lambda) h_{(2)}(\lambda) \ldots h_{(p-1)}(\lambda)\\ &= \left(\frac{\alpha_\lambda}{\lambda}\right)^p \frac{\lambda}{\lambda}\, \frac{h(\lambda)}{\lambda} \cdots \frac{h_{(p-1)}(\lambda)}{\lambda}\; p! \; .\end{aligned} \tag{8}$$

Since $\lim_{\lambda\to\infty} \frac{\alpha_\lambda}{\lambda}=1$, $\lim_{\lambda\to\infty}\frac{h(\lambda)}{\lambda}=1, \ldots, \lim_{\lambda\to\infty}\frac{h_{(p-1)}(\lambda)}{\lambda}=1$, from (8)

$$\lim_{\lambda\to\infty} \frac{d^p}{dx^p} L_\lambda(t^p;x) = p! \; .$$

Thus we have (5).

Finally let us assume that for $n-1$,

$$\lim_{\lambda\to\infty} \frac{d^p}{dx^p} L_\lambda(t^{p+n-1};x) = \frac{(p+n-1)!\; x^{n-1}}{(n-1)!} \; . \tag{9}$$

If we show that

$$\lim_{\lambda\to\infty} \frac{d^p}{dx^p} L_\lambda(t^{p+n};x) = \frac{(p+n)!\ x^n}{n!}, \tag{10}$$

then the proof is complete. □

Again, in (2), let $f(t) = t^{p+n}$ then

$$\frac{d^p}{dx^p} L_\lambda(t^{p+n};x) = \frac{\alpha_\lambda^p\ p!}{\lambda^{2p}} \sum_{v=0}^{\infty} \left[\frac{v}{\lambda^2\varphi_\lambda(0)}, \frac{v+1}{\lambda^2\varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2\varphi_\lambda(0)}; t^{p+n}\right]$$
$$\times \frac{\lambda\, h(\lambda)\, h_{(2)}(\lambda) \cdots h_{(v+p-1)}(\lambda)}{v!} (x\alpha_\lambda\psi_\lambda(0))^v\ K_{h_{(v+p)}(\lambda)}(x,t,u)\Big|_{\substack{u=\alpha_\lambda\psi_\lambda(t)\\ t=0}}\ . \tag{11}$$

On the other hand, since

$$f^{(p)}(x) = (p+n).(p+n-1)...(n+1)x^n$$
$$= \frac{(p+n)!\ x^n}{n!},$$

we have

$$\left[\frac{v}{\lambda^2\varphi_\lambda(0)}, \frac{v+1}{\lambda^2\varphi_\lambda(0)}, ..., \frac{v+p}{\lambda^2\varphi_\lambda(0)}; t^{p+n}\right] = \frac{(p+n)!\ x^n}{n!\ p!} = \frac{(p+n)x}{n}\ \frac{(p+n-1)!\ x^{n-1}}{(n-1)!\ p!}\ .$$

Using this in (11), we get

$$\frac{d^p}{dx^p} L_\lambda(t^{p+n};x) = \frac{(p+n)x}{n}\ \frac{d^p}{dx^p} L_\lambda(t^{p+n-1};x)\,.$$

Consequently,

$$\lim_{\lambda\to\infty} \frac{d^p}{dx^p} L_\lambda(t^{p+n};x) = \frac{(p+n)x}{n}\ \lim_{\lambda\to\infty}\ \frac{d^p}{dx^p} L_\lambda(t^{p+n-1};x)$$

and using (9) in the last equality we have (10).

Remark. In the special case $\lambda = n,\ h(\lambda) = m+n\ \ (m+n = 0,1,2,...)$, we obtain, by Theorem 2 and Theorem 3, the results proved in [4], and by Theorem 4, the results proved in [7].

Note that a system of functions $\{f_0, f_1, ..., f_n\}$ continuous on $[a,b]$ is said to be Tschebyscheff system, if the polynomial $P_n(t) = f_0(t) + f_1(t) + ... + f_n(t)$ has no more than n-zeros on $[a,b]$.

Thus for

$$f_0(x) = p!,\ f_1(x) = (p+1)!x, ..., f_n(x) = \frac{(p+n)!}{n!}x^n,$$

a system of functions $\{f_0, f_1, ..., f_n\}$ is Tschebyscheff system.

Theorem 5. *If* $\{f_0, f_1, ..., f_n\}$ *is Tschebyscheff system on* $[a, b]$ *then there exists a polynomial* $P_n(x) = f_0(x) + f_1(x) + ... + f_n(x)$ *which has the following inequality:*

$$\frac{f_n(b-a)}{2^{2n-1}} \leq \max_{a \leq x \leq b} |P_n(x)|,$$

where

$$f_0(x) = p!,\ f_1(x) = (p+1)!x, ..., f_n(x) = \frac{(p+n)!}{n!}x^n.$$

Proof. In [3, Corollary 3.3.6, p.63], Davis shows that

$$|a_0| \frac{(b-a)^n}{2^{2n-1}} \leq \max_{a \leq x \leq b} |a_0 x^n + ... + a_n|.$$

Setting

$$a_0 = \frac{(p+n)!}{n!}, ..., a_{n-1} = (p+1)!, a_n = p!$$

in the last inequality we have

$$\frac{(p+n)!}{n!} \frac{(b-a)^n}{2^{2n-1}} \leq \max_{a \leq x \leq b} \left| \frac{(p+n)!}{n!} x^n + ... + (p+1)!x + p! \right|.$$

And in this inequality letting $f_0(x) = p!$, $f_1(x) = (p+1)!x, ..., f_n(x) = \frac{(p+n)!}{n!}x^n$ and

$$f_n(b-a) = \frac{(p+n)!}{n!}(b-a)^n,$$

we have

$$\frac{f_n(b-a)}{2^{2n-1}} \leq \max_{a \leq x \leq b} |f_0(x) + f_1(x) + ... + f_n(x)|.$$

Hence the proof is complete. □

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References

[1] Baskakov, V. A., An Example of a Sequence of Linear Positive Operators in the space of Continuous Functions, DAN. 113 (1957), 249 - 251. (Russian).

[2] Baskakov, V. A., On a Construction of Converging Sequences of Linear Positive Operators, Studies of Modern Problems of Constructive Theory of Functions, Fizmatgiz, Moskow (1961), 314 - 318. (Russian) MR ♯ 1491.

[3] Davis, P. J., Interpolation and Approximation, Blaisdell, N. Y., 1963.

[4] Gadjiev, A. D. and Ibragimov, I. I.,On a Sequence of Linear Positive Operators, Soviet Math., Dokl., 11 (1970), 1092 - 1095.

[5] Korovkin, P. P., Linear Operators and Approximation Theory, Delhi, 1960.

[6] Lorentz, G. G., Bernstein Polynomials, Toronto, 1953.

[7] Radatz, P. and Wood, B., Approximating Derivatives of Functions Unbounded on The Positive Axis with Linear Operators, Rev. Roum. Math. Pures et Appl., Bucarest, Tome XXIII, No 5 (1978), 771 - 781.

[8] Szas, O., Generalization of S. Bernstein's Polynomials to The Infinite Interval, Journ. of Research of The Nat. Bureau of Stand. (1950), 45, 239 - 245.

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