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Abstract

The main purpose of this paper is to investigate, using the Tachibana operator, transfer of the complete lifts of affinor structures along the cross-sections of the tangent and cotangent bundles.

1. Introduction

Let A_m be an associative commutative unital algebra of finite dimension m over the field $\mathbb{R}$ of real numbers and $z = x^\alpha e_\alpha$, $\alpha = 1, \ldots, m$, a variable in the algebra A_m, where e_α and x^α denote the basic units of A_m and real variables, respectively. Then, $w = f^\alpha(x^1, \ldots, x^m)e_\alpha$ is an algebraic function of z, where $f^\alpha(x^1, \ldots, x^m)$ are real functions of all x^α. We now define the differentials in A_m by

$$dw = df^\alpha e_\alpha = (\partial_\beta f^\alpha) dx^\beta e_\alpha, \quad dz = dx^\alpha e_\alpha.$$

If, for A-functions $w = w(z)$, the differential dw can be represented in the form $dw = w'(z)dz$, then f is said to be A-holomorphic ([1] p.85, [2]), and the A-function $w'(z)$ is called the derivative.

The necessary and sufficient condition for an A-function $w = w(z)$ to be A-holomorphic is that

$$S_\alpha \mathcal{D} = \mathcal{D} S_\alpha, \tag{1.1}$$

where $S_\alpha = (C^\gamma_{\alpha\beta})$, $\mathcal{D} = \left(\frac{\partial f^\alpha}{\partial x^\beta}\right)$ and $C^\gamma_{\alpha\beta}$ are the structure constants of the algebra A_m. The conditions (1.1) will be called the Scheffers conditions [3]. In particular, in case of the algebra of complex numbers $A_2 = \mathbb{C}(i)$, $i^2 = -1$, the Scheffers conditions coincide with the Cauchy-Riemann conditions.

On a differentiable manifold M_n of class $\mathcal{C}^\infty$ we consider a polyaffinor structure $\Pi = \{\underset{\alpha}{\varphi}{}^i_j\}$-a collection of tensor fields of type (1, 1) that represents the algebra A_m isomorphically, that is

$$\underset{\alpha}{\varphi}{}^m_i \underset{\beta}{\varphi}{}^j_m = C^\gamma_{\alpha\beta} \underset{\gamma}{\varphi}{}^j_i$$

where we indicate by $\underset{\alpha}{\varphi}{}^i_j$ $(\alpha = 1, \ldots, m)$ the affinors of Π-structure corresponded elements e_α $(\alpha = 1, \ldots, m)$ of the base of A_m under the isomorphism. If M_n admits a smooth atlas of local charts such that all the affinors of the Π-structure have constant components in any chart of this atlas, then the Π-structure is said to be integrable. Let Π-structure defined with the Frobenius algebra A_m be a r-regular structure $\underset{\alpha}{\varphi}{}^j_i = \delta^u_v \mathcal{C}^\gamma_{\alpha\beta}$, $i, j = 1, \ldots, n$; $\alpha, \beta = 1, \ldots, m$; $u, v = 1, \ldots, r$; and δ^u_v as the Kronecker symbol (for example, almost complex structure) [3]. If the structure is integrable, then it can be shown that the manifold M_n is transformed to the holomorphic manifold $X_r(A_m)$ over the algebra A_m, where the atlas determined the holomorphic manifold $X_r(A_m)$ is one for which every pair of charts is A-holomorphic related. In particular, if $A_m = \mathbb{C}(i)$, $i^2 = -1$ $(m = 2)$ then $X_r(\mathbb{C})$ is an analytic complex manifold [3, 6, 7].

We define the Tachibana operators $\Phi_{\underset{\alpha}{\varphi}} g, \Phi_{\underset{\alpha}{\varphi}} t, \Phi_{\underset{\alpha}{\varphi}} w$ ([4], see also [5]) associated with an algebraic structure $\Pi = \{\underset{\alpha}{\varphi}\}$ and an arbitrary $X \in \mathcal{T}^1_0(M_n)$, and we apply to the arbitrary tensor fields $g \in \mathcal{T}^0_2(M_n)$, $t \in \mathcal{T}^1_0(M_n)$, $w \in \mathcal{T}^0_1(M_n)$ as follows:

$$\begin{aligned}(\Phi_{\underset{\alpha}{\varphi}} g)(X, Z_1, Z_2) &= L_{\underset{\alpha}{\varphi} X}\, g - L_X(go\underset{\alpha}{\varphi}))(Z_1, Z_2) + g(Z_1, \underset{\alpha}{\varphi}(L_X Z_2)) \\ &\quad - g(\underset{\alpha}{\varphi} Z_1, L_X Z_2)\end{aligned} \tag{1.2}$$

$$(\Phi_{\underset{\alpha}{\varphi}} t)(X) = -(L_t \underset{\alpha}{\varphi} t)(X), \tag{1.3}$$

$$(\Phi_{\underset{\alpha}{\varphi}} w)(X, Y) = (L_{\underset{\alpha}{\varphi} x}\, w - L_X(w \circ \underset{\alpha}{\varphi})(Y), \tag{1.4}$$

where L_X denotes the operator of Lie derivation with respect to X and

$$(go\underset{\alpha}{\varphi})(Z_1, Z_2) = g(\underset{\alpha}{\varphi} Z_1, Z_2),$$

$$(wo\underset{\alpha}{\varphi})(Y) = w(\underset{\alpha}{\varphi} Y).$$

The expression (2) define the tensor fields $\Phi_{\underset{\alpha}{\varphi}} g \in \mathcal{T}^0_3(M_n)$, if and only if g a pure tensor field [5], that is,

$$g(\underset{\alpha}{\varphi} Z_1, Z_2) = g(Z_1, \underset{\alpha}{\varphi} Z_2), \tag{$*$}$$

for all $Z_1, Z_2 \in \mathcal{T}^1_0(M_n)$, $\underset{\alpha}{\varphi} \in \Pi$. The expressions (3) and (4) always defines the tensor fields $\Phi_{\underset{\alpha}{\varphi}} t \in \mathcal{T}^1_1(M_n)$ and $\Phi_{\underset{\alpha}{\varphi}} w \in \mathcal{T}^0_2(M_n)$, respectively. The equality (*) is

$$g_{mj} \underset{\alpha}{\varphi}{}^m_i = g_{im} \underset{\alpha}{\varphi}{}^m_j, \ \ \forall\ \underset{\alpha}{\varphi}{}^i_j \in \Pi$$

with respect to a natural coordinate system in M_n. A tensor field $t^{i_1 \cdots m}_{j_1 \cdots j_q}$ is said to be pure with respect to the Π-structure if

$$t^{i_1 \cdots i_p}_{m j_2 \cdots j_q} \underset{\alpha}{\varphi}{}^m_{j_1} = \cdots = t^{i_1 \cdots i_p}_{j_1 j_2 \cdots m} \underset{\alpha}{\varphi}{}^m_{j_q} = t^{m i_1 \cdots i_p}_{j_1 \cdots j_q} \underset{\alpha}{\varphi}{}^{i_1}_m = \cdots = t^{i_1 \cdots m}_{j_1 \cdots j_q} \underset{\alpha}{\varphi}{}^{i_p}_m, \quad \forall\, t^{i_1 \cdots i_p}_{m j_2 \cdots j_q} \underset{\alpha}{\varphi}{}^i_j \in \Pi.$$

We consider for convenience the tensor fields of type (1, 0) and (0, 1) as pure tensor fields [12].

The tensors $\underset{\alpha}{\Phi_\varphi} g \underset{\alpha}{\Phi_\varphi} t$ and $\underset{\alpha}{\Phi_\varphi} w$ have, respectively, components

$$\underset{\alpha}{\Phi}{}_k g_{ij} = \underset{\alpha}{\varphi}{}_k^m \partial_m g_{ij} - \partial_k (g_{mj} \underset{\alpha}{\varphi}{}_i^m) + g_{im} \partial_j \underset{\alpha}{\varphi}{}_k^m + g_{mj} \partial_i \underset{\alpha}{\varphi}{}_k^m, \tag{1.5}$$

$$\underset{\alpha}{\Phi}{}_k t^i = -L_t \underset{\alpha}{\varphi}{}_k^i = -t^m \partial_m \underset{\alpha}{\varphi}{}_k^i + \underset{\alpha}{\varphi}{}_k^m \partial_m t^i - \underset{\alpha}{\varphi}{}_m^i \partial_k t^m, \tag{1.6}$$

$$\underset{\alpha}{\Phi}{}_k w_i = \underset{\alpha}{\varphi}{}_k^m \partial_m w_i - \underset{\alpha}{\varphi}{}_i^m \partial_k w_m - w_m (\partial_k \underset{\alpha}{\varphi}{}_i^m - \partial_i \underset{\alpha}{\varphi}{}_k^m) \tag{1.7}$$

with respect to a natural coordinate system in M_n.

When

$$(\Phi_{\underset{\alpha}{\varphi}} g)(X, Z_1, Z_2) = 0 \tag{1.8}$$

for a pure tensor g and $X, Z_1, Z_2 \in \mathcal{T}_0^1(M_n), M_n$ being a manifold with integrable algebraic Frobenius r-regular Π-structure, g is said to be A-holomorphic. Actually, in case of the tensor $g^*_{uv} = g_{uv\sigma} e^\sigma$ in $X_r(A_m)$ corresponding the pure tensor g satisfies the A-holomorphic condition

$$\mathcal{C}^\mu_{\alpha\gamma} \partial_{w\mu} g_{uv\sigma} = \mathcal{C}^\mu_{\alpha\sigma} \partial_{w\gamma} g_{uv\mu}$$

(see [3]). If Π-structure is non-integrable, then the pure tensor g satisfying the equality (1.8) is called almost A-holomorphic [3] [4].

2. Complete Lifts on the Cross-Section

Let us consider the tensor bundle of $T_q^p(M_n)$ with a natural projection $\pi : T_q^p(M_n) \to M_n$. If a differentiable mapping $\sigma : M_n \to T_q^p(M_n)$ which satisfies $\pi o \sigma = id_{M_n}$, then σ is called a cross-section of $T_q^p(M_n)$, where id_{M_n} is the identity mapping on M_n. It is obvious that the cross-section of $T_q^p(M_n)$ on M_n defines a tensor field $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ of type (p, q). Since the rank of the differential of the mapping σ is n and σ injective, the cross-section of $T_q^p(M_n)$ is a submanifold of $T_q^p(M_n)$ with respect to induced topology, which is diffeomorphic to M_n. We will investigate the complete lift of a tensor φ_j^i along a pure submanifold defined by the pure cross-section (i.e., the pure tensor field $t^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ of type (p, q)).

The complete lift of a vector field $V = (v^i) \in \mathcal{T}_0^1(M_n)$ to the tensor bundle $T_q^p(M_n)$ with respect to the coordinate neighborhood $\pi^{-1}(U) \subset T_q^p(M_n)$ was defined in [6] as

$${}^cV = ({}^cV^i, {}^cV^{\bar{i}}) = (v^i, L_V \alpha), \tag{2.9}$$

$\forall\ \alpha \in T_p^q(U);\quad i = 1, \ldots, n;\ \bar{i} = n+1, \ldots, n + n^{p+q}$, where α can be considered as a differentiable function on the space $T_q^p(M_n)$ in the usual way by contraction $\alpha = \alpha(t)$.

In particularly, if we get $\alpha = -t^{i_1\cdots i_p}_{j_1\cdots j_q}$, then the complete lift of V to $T^p_q(M_n)$ in the coordinate neighborhood $\pi^{-1}(U)$ with respect to the natural frame $\{\partial_j, \partial_{\overline{j}}\}$, $x^{\overline{j}} = t^{i_1\cdots i_p}_{j_1\cdots j_q}$ is of the form

$$ {}^cV = ({}^cV^j,\ {}^cV^{\bar{j}}) = \left(v^j, \sum_{\lambda=1}^{p} t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^{q} t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m \right). \qquad (2.10)$$

Let us consider the cross-section of $T^p_q(M_n)$ defined by the tensor field $t^{i_1\cdots i_p}_{j_1\cdots j_q}(x^i)$. This cross-section equation is written as

$$\overline{x}^J = \overline{x}^J(x^j), \quad J = 1, \ldots, n + n^{p+q}$$

or

$$\left.\begin{array}{rcl} \overline{x}^j &=& x^j \\ \overline{x}^{\overline{j}} &=& t^{i_1\cdots i_p}_{j_1\cdots j_q}(x^j). \end{array}\right\}$$

It is obvious that the system

$$\left.\begin{array}{rcl} B_i &=& \{\partial_i \overline{x}^A\} = \{B^h_i, B^{\overline{h}}_i\} = \{\delta^h_i, \partial_i t^{i_1\cdots i_p}_{j_1\cdots j_q}\} = \delta^h_i \partial_h + \partial_i t^{i_1\cdots i_p}_{j_1\cdots j_q} \partial_{\overline{h}} \\ C_{\overline{i}} &=& \{\partial_{\overline{i}} \overline{x}^A\} = (C^h_{\overline{i}}, C^{\overline{h}}_{\overline{i}}) = (0, \delta^{\ell_1}_{j_1} \cdots \delta^{\ell_q}_{j_q} \delta^{i_1}_{h_1} \cdots \delta^{i_p}_{h_p}) = \delta^{\ell_1}_{j_1} \cdots \delta^{\ell_q}_{j_q} \delta^{i_1}_{h_1} \cdots \delta^{i_p}_{h_p} \partial_{\overline{h}} \end{array}\right\}$$

defined a frame along the cross-section. B_i and $C_{\overline{i}}$, $i = 1, \ldots, n$; $\overline{i} = n+1, \ldots, n+n^{p+q}$ span the tangent plane of $T^p_q(M_n)$ and are tangent to the cross-section and the fibre, respectively.

Using (2.10) and ${}^cV^A = \tilde{V}^i B^A_i + \tilde{V}^{\overline{i}} C^A_{\overline{i}}$, we have

$$\left.\begin{array}{rcl} v^i \partial_i x^{\overline{h}} &+& \left(\sum_{\lambda=1}^p t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^q t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m \right) \partial_{\bar{i}} x^{\bar{h}} = \tilde{V}^i B^{\overline{h}}_i + \tilde{V}^{\overline{i}} C^{\overline{h}}_{\overline{i}} \\ v^i \partial_i x^h &+& \left(\sum_{\lambda=1}^p t^{i_1\cdots m\cdots i_p}_{(j)} \partial_m v^{i_\lambda} - \sum_{\mu=1}^q t^{(i)}_{j_1\cdots m\cdots j_q} \partial_{j_\mu} v^m \right) \partial_{\bar{i}} x^{\bar{h}} = \tilde{V}^i B^h_i + \tilde{V}^{\overline{i}} C^h_{\overline{i}} \end{array}\right\}.$$

Therefore, we obtain

$$\begin{array}{rcl} \tilde{V}^i &=& v^i \\ \tilde{V}^{\overline{i}} &=& -L_V t^{i_1\cdots i_p}_{j_1\cdots j_q}, \end{array}$$

that is, the complete lift ${}^cV$ of V with respect to the frame (B, C) along the cross-section $t^{i_1\cdots i_p}_{j_1\cdots j_q}$, is written as

$$ {}^cV = ({}^cV^j,\ {}^cV^{\overline{j}}) = (v^j, -L_V t^{i_1\cdots i_p}_{j_1\cdots j_q}). \qquad (2.11)$$

2.1. Complete Lifts of the Affinor to $T_0^1(M_n)$ Along a Pure Cross-Section

We will find a formula for a complete lift of affinor field φ_j^i along the pure cross-section $t \in \mathcal{T}_0^1(M_n)$ of tangent bundle $T_0^1(M_n)$.

We define the complete lift ${}^c\varphi$ of a tensor field $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section $t \in \mathcal{T}_0^1(M_n)$ of $T_0^1(M_n)$ by

$$ {}^c(\varphi(V)) = {}^c\varphi({}^cV), \quad \forall\, V \in \mathcal{T}_0^1(M_n), \tag{2.12} $$

where ${}^cV$ is in the form (2.11). The equality (2.12) can be written as

$$ {}^c(\varphi(V))^K = {}^c\varphi_L^K\, {}^cV^L, \tag{2.13} $$

by using coordinates. If we take $K = k$ in (2.13), we have

$$ \varphi_\ell^k v^\ell = (\varphi(V))^k = {}^c\varphi_L^k\, {}^cV^L = {}^c\varphi_\ell^k\, {}^cV^\ell + {}^c\varphi_{\overline{\ell}}^k\, {}^cV^{\overline{\ell}}. $$

Then, we obtain

$$ {}^c\varphi_\ell^k = \varphi_\ell^k, \qquad {}^c\varphi_{\overline{\ell}}^k = 0. \tag{2.14} $$

If we take $K = \overline{k}$ in the equality (2.13), we have

$$ {}^c(\varphi(V))^{\overline{k}} = {}^c\varphi_L^{\overline{k}}\, {}^cV^L = {}^c\varphi_\ell^{\overline{k}}\, {}^cV^\ell + {}^c\varphi_{\overline{\ell}}^{\overline{k}}\, {}^cV^{\overline{\ell}}. \tag{2.15} $$

Now, let us find solutions which are ${}^c\varphi_\ell^{\overline{k}}$ and ${}^c\varphi_{\overline{\ell}}^{\overline{k}}$ in equation (2.15). For this purpose, taking account of (1.6), we have

$$ L_{\varphi V} t^k = v^\ell \Phi_\ell t^k + \varphi_\ell^k L_V t^\ell. \tag{2.16} $$

From (2.11) and (2.16), we get

$$ -\,{}^c(\varphi(V))^{\overline{k}} = {}^cV^\ell \Phi_\ell t^k - \varphi_\ell^k\, {}^cV^{\overline{\ell}}. \tag{2.17} $$

Then from (2.15) and (2.17) we obtain

$$ {}^c\varphi_\ell^{\overline{k}} = -\Phi_\ell t^k, \qquad {}^c\varphi_{\overline{\ell}}^{\overline{k}} = \varphi_i^k, \quad (x^{\overline{k}} = t^k). \tag{2.18} $$

Thus (2.14) and (2.18) are the complete lift of the tensor structure $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section of $T_0^1(M_n)$. As a special case, this lift was obtained with respect to the natural frame $\{\partial_i, \partial_{\overline{i}}\}$ in [7] (see also [8]).

2.2. Complete Lifts of the Affinor to $T_1^0(M_n)$ Along a Pure Cross-Section

We will find a formula for complete lift of affinor field φ_j^i along the pure cross-section $w \in \mathcal{T}_1^0(M_n)$ of cotangent bundle $T_1^0(M_n)$.

If the Tachibana operator Φ is applied to the pure tensor field $w \in \mathcal{T}_1^0(M_n)$, then from (1.7) we have

$$v^j \Phi_j w_i = L_{\varphi V} w_i - \varphi_i^j L_V w_j - w_j L_V \varphi_i^j. \tag{2.19}$$

We define a complete lift ${}^c\varphi$ of the tensor $\varphi \in \mathcal{T}_1^1(M_n)$ along the pure cross-section w of $T_1^0(M_n)$ by

$${}^c(\varphi(V)) + {}^v(L_V\varphi) = {}^c\varphi({}^cV)$$

or

$${}^c(\varphi(V))^I + {}^v(L_V\varphi)^I = {}^c\varphi_J^I \, {}^cV^J \tag{2.20}$$

by using the coordinates, where ${}^v(L_V\varphi)$ denotes the vertical lift of Lie derivative.

In the equality (2.20), let $I = i$. Then we have ${}^v(L_V\varphi)^i = 0$ by the definition of the vertical lift. In this case, the equality (2.20) can be written

$${}^c(\varphi(V))^i = {}^c\varphi_j^i \, {}^cV^j + {}^c\varphi_{\overline{j}}^i \, {}^cV^{\overline{j}}. \tag{2.21}$$

Thus, from (2.21), we see that

$${}^c\varphi_j^i = \varphi_j^i, \qquad {}^c\varphi_{\overline{j}}^i = 0. \tag{2.22}$$

Now, let $I = \overline{i}$. From the definition of the vertical lift, we have ${}^v(L_V\varphi)^{\overline{i}} = w_j L_V \varphi_i^j$. Taking account of (2.20), we have

$${}^c(\varphi(V))^{\overline{i}} = {}^c\varphi_j^{\overline{i}} \, {}^cV^j + {}^c\varphi_{\overline{j}}^{\overline{i}} \, {}^cV^{\overline{j}} - w_j L_V \varphi_i^j. \tag{2.23}$$

From (2.11) and (2.19), we see that

$$\begin{aligned} L_{\varphi V} w_i &= v^j \Phi_j w_i + \varphi_i^j L_V w_j + w_j L_V \varphi_i^j, \\ -\,{}^c(\varphi(V))^{\overline{i}} &= {}^cV^j \Phi_j w_i - \varphi_i^j \, {}^cV^{\overline{j}} + w_j L_V \varphi_i^j. \end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we have

$${}^c\varphi_j^{\overline{i}} = -\Phi_j w_i, \qquad {}^c\varphi_{\overline{j}}^{\overline{i}} = \varphi_i^j, \quad (x^{\overline{i}} = w_i).$$

3. Transfer of the Complete Lift of the Affinor Structure

Let M_n be *a* paracompact manifold with a Riemannian metric. We shall mean by the Riemannian metric a symmetric covariant tensor field g of degree 2 which is nondegenerate. If g is a pure tensor, then a manifold M_n with an algebraic Π-structure is called an almost B-manifold [1, p.31] and this will be denoted V_n.

Suppose that $T_0^1(V_n)$ and $T_1^0(V_n)$ are the tensor bundle of type (1, 0) and (0, 1) over V_n, respectively. Clearly $\dim T_0^1(V_n) = \dim T_1^0(V_n) = 2n$.

Let the diffeomorphism $f : T_0^1(V_n) \to T_1^0(V_n),\ \ y^I = y^I(x^J),\ I, J = 1, \ldots, 2n$ be defined by a local expression such that

$$\begin{aligned} y^i &= x^i \\ y^{\overline{i}} &= w_i = g_{im}t^m. \end{aligned} \tag{3.25}$$

Since

$$\begin{aligned} x^{\overline{k}} &= t^k, \\ \frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} &= \frac{\partial}{\partial x^{\overline{k}}}(w_i) = \frac{\partial}{\partial x^{\overline{k}}}(g_{im}t^m) = \frac{\partial}{\partial x^{\overline{k}}}(g_{ik}t^k) = g_{ik}, \\ 0 &= \frac{\partial y^{\overline{i}}}{\partial x^k} = \frac{\partial w_i}{\partial x^k} = \frac{\partial}{\partial x^k}(g_{im}t^m) = (\partial_k g_{im})t^m, \end{aligned}$$

we have

$$A = \left(\frac{\partial y^I}{\partial x^K}\right) = \begin{pmatrix} \frac{\partial y^i}{\partial x^k} & \frac{\partial y^i}{\partial x^{\overline{k}}} \\ \frac{\partial y^{\overline{i}}}{\partial x^k} & \frac{\partial y^{\overline{i}}}{\partial x^{\overline{k}}} \end{pmatrix} = \begin{pmatrix} \delta_k^i & 0 \\ 0 & g_{ik} \end{pmatrix}.$$

The inverse of the mapping f is written as

$$x^\ell = y^\ell, \quad x^{\overline{\ell}} = t^\ell = g^{\ell m} w_m.$$

Suppose that $y^{\overline{j}} = w_j$, we have

$$A^{-1} = \left(\frac{\partial x^L}{\partial y^J}\right) = \begin{pmatrix} \delta_j^\ell & 0 \\ 0 & g^{\ell j} \end{pmatrix},$$

which is the Jacobian matrix of inverse mapping f^{-1}.

Theorem 3.1. *Suppose that* ${}^c\overset{1}{\varphi}$ *and* ${}^c\overset{2}{\varphi}$ *denote the complete lift of the affinor* φ *of the* Π*-structure to* $T_0^1(V_n)$ *and* $T_1^0(V_n)$ *along the pure cross-sections* t^i *and* w_i*, respectively. If* $\Phi_\varphi(g) = 0$*, then* ${}^c\overset{2}{\varphi}$ *is transferred from* ${}^c\overset{1}{\varphi}$ *by means of the diffeomorphism* f*, where* Φ_φ *denotes the Tachibana operator.*

Proof. Suppose that $\Phi_\varphi(g)=0$. Then, if we write ${}^{c}\overset{2}{\varphi}$ along the pure cross-section $w_i(y)$, we obtain

$$\begin{aligned}
{}^{c}\overset{2}{\varphi} &= \left({}^{c^2}\varphi^I_J\right)\\
&= \begin{pmatrix} \varphi^i_j & 0\\ -\underset{\varphi}{\Phi}{}_j w_i & \varphi^j_i \end{pmatrix}\\
&= \begin{pmatrix} \varphi^i_j & 0\\ -g_{im}\Phi_j t^m-(\Phi_j g_{im})t^m & \varphi^j_i \end{pmatrix}\\
&= \begin{pmatrix} \varphi^i_j & 0\\ -g_{im}\Phi_j t^m & \varphi^j_i \end{pmatrix}\\
&= \begin{pmatrix} \delta^i_k & 0\\ 0 & g_{ik} \end{pmatrix}\begin{pmatrix} \varphi^k_\ell & 0\\ -\Phi_\ell t^k & \varphi^k_\ell \end{pmatrix}\begin{pmatrix} \delta^\ell_j & 0\\ 0 & g^{\ell j} \end{pmatrix}\\
&= A\,{}^{c}\overset{1}{\varphi}\,A^{-1}. \qquad (3.26)
\end{aligned}$$

To show (3.26), we have taken account of

$$g_{ik}\varphi^k_\ell g^{\ell j}=g_{k\ell}\varphi^k_i g^{\ell j}=\varphi^k_i\delta^j_k=\varphi^j_i$$

and used that g_{ij} is the pure tensor field. □

We introduce in some coordinate neighborhood $U\subset M_n$ a connection in which all the affinors of the Π-structure are covariantly constant. Such connections are called Π-connection. A Π-structure will be said to be almost integrable [3] if in a coordinate neighborhood of each point $x\in M_n$ there exists at least one Π-connection without torsion. The Π-structure is almost integrable on the Riemann connection if and only if $\Phi_{\underset{\alpha}{\varphi}}(g)=0$, for all $\underset{\alpha}{\varphi}\in\Pi$ [9] (see also [10]). Further, it has been shown that if the algebraic Π-structure is almost integrable, then the structure ${}^{c}\Pi=\{{}^{c}\underset{\alpha}{\varphi}\}$ determines the algebraic structure along the pure subbundle of the tensor bundle $T^p_q(M_n)$ [11]. Using these facts, we have the following result:

Theorem 3.2. *If the metric g of the B-manifold is almost A-holomorphic, then the algebraic*

$\Pi_2=\{{}^{c}\overset{2}{\underset{\alpha}{\varphi}}\}$ *-structure is transferred from the algebraic* $\Pi_1=\{{}^{c}\overset{1}{\underset{\alpha}{\varphi}}\}$ *-structure by means of the diffeomorphism* f.

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