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## WEIGHTED ERGODIC AVERAGES

*M.D. Ha*

### Abstract

Let  $(X, \mathcal{F}, \lambda)$  be the unit circle  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$  with the usual  $\sigma$ -algebra  $\mathcal{F}$  of Lebesgue measurable subsets and the normalized Lebesgue measure  $\lambda$ . Consider a sequence  $\nu_n : \mathbb{N} \rightarrow \mathbb{R}$ ,  $\nu_n(k) \geq 0$ ,  $\sum_{k=1}^{\infty} \nu_n(k) = 1$ . For any measure-preserving  $\tau : X \rightarrow X$ , this sequence induces a sequence  $(T_n)_1^{\infty}$  of bounded, linear operators on  $L^p(X)$ ,  $1 \leq p \leq \infty$ , by defining

$$T_n f = \sum_{k=1}^{\infty} \nu_n(k) f \circ \tau^k, \quad n = 1, 2, \dots$$

We shall prove that under suitable conditions imposed on  $\tau$  and  $(\nu_n)_1^{\infty}$ , there exists a large collection of measurable characteristic functions  $f$  for which  $\limsup_{n \rightarrow \infty} T_n f - \liminf_{n \rightarrow \infty} T_n f = 1$  a.e on  $X$ .

**Keywords:** Weights, weighted averages, Fourier transforms.

### 1. Introduction

The classical ergodic theorem of Birkhoff states that if  $\tau$  is an ergodic transformation on a probability space  $(X, \mathcal{F}, \mu)$ , then for each  $f \in L^1(X)$ , we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f \circ \tau^k = \int_X f d\mu$$

almost everywhere (on  $X$ ). So, the sequence

$$\left\{ \frac{1}{n} \sum_{k=1}^n f \circ \tau^k \right\}_{n=1}^{\infty}$$

of averages of  $f$  under iterates of  $\tau$  converges a.e. for all  $f$  in  $L^1(X)$ .

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A natural question that arises from the above is that instead of considering the sequence of usual average of  $f$  under iterates of  $\tau$  as above, what would happen if we consider sequence of weighted averages, i.e., averages of the form

$$\sum_{k=1}^{\infty} \nu_n(k) f \circ \tau^k,$$

where  $\nu_n(k) \geq 0$  for all  $n, k$  and

$$\sum_{k=1}^{\infty} \nu_n(k) = 1, \quad n = 1, 2, \dots,$$

and investigate its behaviour with respect to convergence. This question has been considered by several authors, including those given in [1] and [4] of the references. For our present purpose, we shall refer to any sequence

$$\nu : \mathbb{N} \rightarrow \mathbb{R}, \quad \nu(k) \geq 0, \quad \sum_{k=1}^{\infty} \nu(k) = 1$$

as weights.

Now, let  $(\nu_n)_1^\infty$  be a sequence of weights. For any probability space  $(X, \mathcal{F}, \mu)$  and any ergodic  $\tau : X \rightarrow X$ , this sequence of weights induces a sequence of operators

$$(T_n)_1^\infty : L^1(X, \mathcal{F}, \mu) \rightarrow L^1(X, \mathcal{F}, \mu),$$

where each  $T_n$  is defined by the following formula:

$$T_n f = \sum_{k=1}^{\infty} \nu_n(k) f \circ \tau^k, \quad n = 1, 2, \dots$$

If we assume that the measure algebra of the space  $X$  under consideration is normalized, non-atomic, and separable, it is a standard result in ergodic theory that in the study of such averages, it is enough to consider the special case where  $(X, \mathcal{F}, \lambda)$  is the unit circle  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$  with the usual  $\sigma$ -algebra  $\mathcal{F}$  of Lebesgue measurable subsets and the normalized Lebesgue measure  $\lambda$ , and where  $\tau$  is an ergodic rotation  $\tau(z) = z_* z$  on  $\mathbb{S}^1$  for some fixed  $z_*$  in  $\mathbb{S}^1$ .

The purpose of this note is to give an elementary proof of the following result. For many other related results, the readers should consult [1].

**Theorem.** *Let  $(\nu_n)_1^\infty$  and  $(T_n)_1^\infty$  be as above. Assume that  $z_* \in \mathbb{S}^1$ , which defines  $\tau$ , has the following properties:*

*Given any arc  $J$  on the unit circle  $\mathbb{S}^1$ , there exists  $\Lambda(J) \subseteq \mathbb{N}$  such that*

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i)  $D = \{z_*^m : m \in \Lambda(J)\}$  is dense in  $\mathbb{S}^1$

ii)  $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \nu_n(k) z^k$  exists and belong to  $J$  for all  $z$  in  $D$ .

Then, given  $\epsilon > 0$ ,  $N \in \mathbb{N}$ , there exists a set  $B \in \mathcal{F}$ ,  $\lambda(B) < \epsilon$  such that

$$\lambda \left( \sup_{n \geq N} T_n \chi_B > 1 - \epsilon \right) = 1.$$

Consequently, there exists a dense subset  $\mathcal{R}$  of  $\mathcal{F}$  such that if  $A \in \mathcal{R}$ ,

$$\limsup_{n \rightarrow \infty} T_n \chi_A = 1 \quad a.e \quad \text{and} \quad \liminf_{n \rightarrow \infty} T_n \chi_A = 0 \quad a.e$$

(Recall that if  $(X, \mathcal{F}, \mu)$  is a probability space and if we define  $\rho(A, B) = \mu(A \Delta B)$  for  $A, B$  in  $\mathcal{F}$ , then this  $\rho$  induces a metric on  $\mathcal{F}$  if we identify sets that differ from each other by a set of measure zero.)

## 2. Some Lemmas

The proof of our theorem follows from the following two simple lemmas. To simplify notation, let us define for  $n_j \in \mathbb{N}$ ,  $\ell = 1, 2, \dots, \infty$ ,  $\omega \in \mathbb{S}^1$ ,

$$S_{n_j}^\ell = \sum_{k=1}^{\ell} \nu_{n_j}(k) \quad \text{and} \quad S_{n_j}^\ell(\omega) = \sum_{k=1}^{\ell} \nu_{n_j}(k) \omega^k.$$

We also let  $\{t\}$  denote the fractional part of a real number  $t$ , and  $\bar{\gamma}$  is the complex conjugate of  $\gamma \in \mathbb{C}$ .

**Lemma 1.** Assume i) and ii) of the previous theorem hold. Let  $0 < \epsilon, \eta$  and  $N \in \mathbb{N}$  be given. There exists an integer  $K > N$  and  $K$  points  $z_1, \dots, z_K \in \mathbb{S}^1$ , integers  $n_1, n_2, \dots, n_K$ , each  $n_j > K$ , and an integer  $m \in \mathbb{N}$  such that

- (i)  $|S_{n_j}^\infty(z_*^m) - z_j| \leq \eta$  for all  $j = 1, 2, \dots, K$
- (ii) Any arc of length  $\epsilon$  contains at least 3 points among  $z_1, \dots, z_K$ .

**Proof.** (i) Let  $0 < \epsilon, \eta$  and  $N \in \mathbb{N}$  be given. Choose  $K > N$ ,  $K \in \mathbb{N}$  such that  $\frac{8\pi}{K} < \epsilon$ . Fix some  $\tilde{\eta} > 0$  with  $\tilde{\eta} < \min(\eta, \frac{1}{2K})$ .

Let  $I_1 = \{e^{2\pi i x} : x \in (-\tilde{\eta}, \tilde{\eta})\}$  and for  $j = 2, 3, \dots, K$ , let

$$I_j = \left\{ e^{2\pi i x} : x \in \left( \frac{j-1}{K} - \tilde{\eta}, \frac{j-1}{K} + \tilde{\eta} \right) \right\}$$

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be  $K$  arcs on  $\mathbb{S}^1$  of equal arc lengths. By hypothesis, there exists  $m_1 \in \mathbb{N}$  and large integers  $n_1, h_1 > K$  and some  $z_1 \in I_1$  such that

$$|S_{n_1}^{h_1}(z_*^{m_1}) - z_1| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_1}^{h_1} > 1 - \frac{\tilde{\eta}}{2}. \quad (1)$$

Again, by assumption, there exists a dense set  $D_2 \subseteq \mathbb{S}^1$  such that  $\lim_{n \rightarrow \infty} S_n^\infty(z)$  exists and belongs to  $I_2$  for each  $z \in D_2$ , and moreover, each element of  $D_2$  is of the form  $z_*^n$  for some  $n \in \mathbb{N}$ . Choose  $z_*^{m_2} \in D_2$  with  $|z_*^{m_2} - z_*^{m_1}|$  so small (so that  $|z_*^{m_2 k} - z_*^{m_1 k}|$  is very small for each  $k = 1, 2, \dots, h_1$ ) so that  $|S_{n_1}^{h_1}(z_*^{m_2}) - z_1| < \frac{\tilde{\eta}}{2}$ . Since  $\lim_{n \rightarrow \infty} S_n^\infty(z_*^{m_2})$  exists and belongs to  $I_2$ , there exists  $h_2, n_2 > K$  and  $z_2 \in I_2$  with

$$|S_{n_2}^{h_2}(z_*^{m_2}) - z_2| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_2}^{h_2} > 1 - \frac{\tilde{\eta}}{2}. \quad (2)$$

Again, there exists a dense set  $D_3 \subseteq \mathbb{S}^1$  such that  $\lim_{n \rightarrow \infty} S_n^\infty(z)$  exists and belong to  $I_3$  and each element  $z$  of  $D_3$  is of the form  $z_*^n$  for some  $n \in \mathbb{N}$ . Choose  $z_*^{m_3} \in D_3$  with  $|z_*^{m_3} - z_*^{m_j}|$  very small for  $j = 1, 2$ , so that  $|S_{n_1}^{h_1}(z_*^{m_3}) - z_1| < \frac{\tilde{\eta}}{2}$  and  $|S_{n_2}^{h_2}(z_*^{m_3}) - z_2| < \frac{\tilde{\eta}}{2}$ . Also, because  $\lim_{n \rightarrow \infty} S_n^\infty(z_*^{m_3})$  exists and belong to  $I_3$ , there exists  $z_3 \in I_3$  and integers  $h_3, n_3 > K$  such that  $|S_{n_3}^{h_3}(z_*^{m_3}) - z_3| < \frac{\tilde{\eta}}{2}$  and  $S_{n_3}^{h_3} > 1 - \frac{\tilde{\eta}}{2}$ . Continuing this process to the  $K^{\text{th}}$  stage, we obtain an integer  $m = m_K$  and  $z_1, \dots, z_K$ ,  $z_j \in I_j$ , integers  $h_j, n_j > K$  for  $j = 1, 2, \dots, K$  such that

$$|S_{n_j}^{h_j} - z_j| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_j}^{h_j} > 1 - \frac{\tilde{\eta}}{2} \quad \text{for all } j = 1, 2, \dots, K.$$

Hence, it follows that for each  $j = 1, 2, \dots, K$ ,

$$|S_{n_j}^\infty(z_*^m) - z_j| \leq |S_{n_j}^{h_j}(z_*^m) - z_j| + \left| \sum_{k=h_j+1}^{\infty} \nu_{n_j}(k) z_*^{mk} \right| < \frac{\tilde{\eta}}{2} + \frac{\tilde{\eta}}{2} = \tilde{\eta} < \eta.$$

This completes the proof of (i).

(ii) Each arc  $I_j$  has arc-length  $2\pi(2\tilde{\eta}) = 4\pi\tilde{\eta}$  and the gap between 2 adjacent arcs is  $2\pi(\frac{1}{K} - 2\tilde{\eta})$ . So any arc of length

$$4(4\pi\tilde{\eta}) + 3 \left[ 2\pi \left( \frac{1}{K} - 2\tilde{\eta} \right) \right] < \pi \left( \frac{2}{K} + \frac{6}{K} \right) = \frac{8\pi}{K} < \epsilon$$

must contain 3 points among those of  $z_1, z_2, \dots, z_K$ .  $\square$

We shall need the following lemma

**Lemma 2.** ([1]) Let  $\epsilon, \rho > 0$  be given. Then there exists  $0 < \eta = \eta(\epsilon, \rho)$  satisfying the following.

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For any probability measure  $\mu$  on  $\mathbb{R}$ ,

$$|\hat{\mu}(1) - 1| < \eta \Rightarrow \mu(G_\epsilon) < \rho.$$

Here,  $G_\epsilon = \cup_{k=-\infty}^{\infty} (\epsilon + k, 1 - \epsilon + k)$  and  $\hat{\mu} : \mathbb{R} \rightarrow \mathbb{C}$  is the Fourier transform of  $\mu$  defined by

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{2\pi i x t} d\mu(x), \quad t \in \mathbb{R}.$$

Using the above, we will show the following:

**Lemma 3.** Let  $\delta, \rho > 0$  be given. Then there exists  $0 < \eta = \eta(\delta, \rho)$  such that

$$\left| \sum_{k=1}^{\infty} \nu(k) z_k - 1 \right| < \eta \Rightarrow \sum_{|z_k - 1| > \delta} \nu(k) < \rho$$

whenever  $\nu(k) \geq 0$ ,  $\sum_{k=1}^{\infty} \nu(k) = 1$  and  $(z_k)_1^{\infty} \subseteq \mathbb{S}^1$ .

**Proof.** Let  $\delta, \rho > 0$  be given. Choose  $\epsilon > 0$  such that whenever  $t \in \mathbb{R}$ ,  $\{t\} \in (\epsilon, 1 - \epsilon) \Leftrightarrow |e^{2\pi i t} - 1| > \delta$ . Let  $0 < \eta = \eta(\epsilon, \rho)$  be as in Lemma 2. Note that  $\epsilon = \epsilon(\delta)$  so that  $\eta = \eta(\delta, \rho)$ . Suppose  $\nu(k) \geq 0$ ,  $\sum_{k=1}^{\infty} \nu(k) = 1$  and  $(z_k)_1^{\infty} \subseteq \mathbb{S}^1$ . Write  $z_k$  as  $z_k = e^{2\pi i t_k}$ ,  $t_k \in \mathbb{R}$ ,  $k = 1, 2, \dots$ . Define a probability measure  $\mu$  on  $\mathbb{R}$  by  $\mu(t_k) = \nu(k)$ , so that  $\mu$  has support contained in  $\{t_1, t_2, \dots\}$ . We then have

$$\hat{\mu}(1) = \int_{\mathbb{R}} e^{2\pi i x} d\mu(x) = \sum_{k=1}^{\infty} e^{2\pi i t_k} \mu(t_k) = \sum_{k=1}^{\infty} \nu(k) z_k.$$

By Lemma 2. above, we get

$$|\hat{\mu}(1) - 1| < \eta \Rightarrow \mu(G_\epsilon) < \rho,$$

and this completes the proof, because

$$\mu(G_\epsilon) = \sum_{k=1}^{\infty} \mu(\epsilon + k, 1 - \epsilon + k) = \sum_{\{t_k\} \in (\epsilon, 1 - \epsilon)} \mu(t_k) = \sum_{|z_k - 1| > \delta} \nu(k).$$

□

**Lemma 4.** Given  $\epsilon_0 > 0$ , there exists  $\eta_0 > 0$  such that

$$\left| \sum_{k=1}^{\infty} \nu_n(k) \gamma_k - \gamma \right| < \eta_0 \Rightarrow \sum_{|\gamma_k - \gamma| < \epsilon_0} \nu_n(k) > 1 - \epsilon_0$$

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for all sequences  $(\nu_n)_1^\infty$  of weights, and all  $(\gamma_k)_1^\infty \subseteq \mathbb{S}^1$ ,  $\gamma \in \mathbb{S}^1$ .

**Proof.** Given  $\epsilon_0 > 0$ , let  $\delta = \rho = \frac{\epsilon_0}{2}$ . Choose  $\eta = \eta(\delta, \rho)$  as in Lemma 3. Now, suppose  $(\nu_n)_1^\infty$  is any sequence of weights and let  $(\gamma_k)_1^\infty \subseteq \mathbb{S}^1$ ,  $\gamma \in \mathbb{S}^1$  be arbitrary. Then, by Lemma 3, for every  $n$ ,

$$\left| \sum_{k=1}^{\infty} \nu_n(k) \gamma_k - \bar{\gamma} \right| < \eta_0 \Rightarrow \sum_{|\gamma_k - \bar{\gamma}| \leq \delta} \nu_n(k) \geq 1 - \rho.$$

Thus, if we let  $\eta_0 = \eta$ , we then have, for every  $n$ ,

$$\left| \sum_{k=1}^{\infty} \nu_n(k) \gamma_k - \gamma \right| < \eta_0 \Rightarrow \sum_{|\gamma_k - \gamma| < \epsilon_0} \nu_n(k) > 1 - \epsilon_0.$$

□

### 3. Proof of the Theorem

Let  $\epsilon > 0$ ,  $N \in \mathbb{N}$  be given. Let  $\epsilon_0 = \frac{\epsilon}{10}$ . Choose  $\eta_0 > 0$  as in Lemma 4. Then by Lemma 1, there exists  $K > N$  and integers  $n_1, \dots, n_K$ , each  $n_j > K > N$  and points  $z_1, \dots, z_K$  in  $\mathbb{S}^1$  and an integer  $m$  satisfying (i) and (ii) with  $\epsilon_0, \eta_0$  in place of  $\epsilon, \eta$  there. Let  $J$  be any arc on  $\mathbb{S}^1$  of length  $\epsilon$ . Let  $B = P_m^{-1}(J)$ , where  $P_m(z) = z^m$  for all  $z$  in  $\mathbb{S}^1$ . Suppose  $z \in \mathbb{S}^1$ . Let  $J_z = \bar{z}^m J$  so that  $J_z$  has arc length  $= \epsilon$ . Let  $\mathbf{p}$  be the middle point of  $J_z$  and let  $\tilde{J}_z$  be the arc centered at  $\mathbf{p}$  with length  $\epsilon_0$ . By Lemma 1,  $\tilde{J}_z$  must contain at least three adjacent points among those of  $z_1, \dots, z_K$ . Let  $z_q$  be the middle point of these three. Then  $|S_{n_q}^\infty(z_*^m) - z_q| < \eta_0$ , hence, by Lemma 4

$$\sum_{|z_*^{m_k} - z_q| < \epsilon_0} \nu_{n_q}(k) \geq 1 - \epsilon_0.$$

Now,  $|z_*^{m_k} - z_q| < \epsilon \Rightarrow z_*^{m_k} \in J_z$  (since  $J_z$  contains  $\tilde{J}_z$  in the middle and  $\tilde{J}_z$  has arc-length very small compare to that of  $J_z$ ).

Thus we have

$$\sum_{|z_*^{m_k} - z_q| < \epsilon_0} \nu_{n_q}(k) > 1 - \epsilon_0 \Rightarrow \sum_{z_*^{m_k} \in J_z} \nu_{n_q}(k) > 1 - \epsilon_0.$$

Now,  $z_*^{m_k} \in J_z \Leftrightarrow (zz_*^k)^m \in J \Leftrightarrow zz_*^k \in B$ , hence,

$$\begin{aligned} \sup_{n \geq N} T_n \chi_B(z) &= \sup_{n \geq N} \sum_{k=1}^{\infty} \nu_n(k) \chi_B(zz_*^k) = \sup_{n \geq N} \sum_{zz_*^k \in B} \nu_n(k) \\ &\geq \sum_{zz_*^k \in B} \nu_{n_q}(k) = \sum_{z_*^{m_k} \in J_z} \nu_{n_q}(k) > 1 - \epsilon_0 > 1 - \epsilon. \end{aligned}$$

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Since,  $z \in \mathbb{S}^1$  was arbitrary, this clearly implies

$$\lambda \left( \left\{ \sup_{n \geq N} T_n \chi_B > 1 - \epsilon \right\} \right) = 1,$$

as to be shown.

Finally to see the other conclusions of the theorem, we let

$$G_n A = T_n \chi_A, \quad n = 1, 2, \dots; \quad A \in \mathcal{M}.$$

Then if  $A$  and  $B$  are disjoint

$$\begin{aligned} G_n(AUB) &= T_n \chi_{AUB} = \sum_{k=1}^{\infty} \nu_n(k) \chi_{AUB} \circ \tau^k \\ &= \sum_{k=1}^{\infty} \nu_n(k) \chi_A \circ \tau^k + \sum_{k=1}^{\infty} \nu_n(k) \chi_B \circ \tau^k \\ &= T_n \chi_A + T_n \chi_B = G_n A + G_n B. \end{aligned}$$

Also,  $(G_n)^\infty$  is a sequence of monotone linear maps which are continuous in measure with

$$G_n(\mathbb{S}^1) = \sum_{k=1}^{\infty} \nu_n(k) \chi_{\mathbb{S}^1} \circ \tau^k = \sum_{k=1}^{\infty} \nu_n(k) = 1$$

and such that given  $\epsilon > 0$ ,  $N \in \mathbb{N}$ , there exists  $B \in \mathcal{F}$ ,  $\lambda(B) < \epsilon$  with

$$\lambda \left( \left\{ \sup_{n \geq N} G_n B > 1 - \epsilon \right\} \right) = \lambda \left( \left\{ \sup_{n \geq N} T_n \chi_B > 1 - \epsilon \right\} \right) = 1.$$

Hence, by Theorem 1.3 of [3], the conclusion of our theorem follows.



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