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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. Aouf & Nak Eun Cho

Abstract

The object of the present paper is to derive several interesting properties of the class $T_n(\lambda, \alpha)$ consisting of analytic and univalent functions with negative coefficients. Coefficient inequalities, distortion theorems and closure theorems of functions in the class $T_n(\lambda, \alpha)$ are determined. Also radii of close-to-convexity, starlikeness and convexity are determined. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $T_n(\lambda, \alpha)$ are studied here.

Key words and phrases. Analytic, univalent, modified Hadamard product.

1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in S , we define

$$D^0 f(z) = f(z), \quad (1.2)$$

$$D^1 f(z) = Df(z) = zf'(z), \quad (1.3)$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, \dots\}). \quad (1.4)$$

*AMS (1991) Subject Classification. 30C45.

The differential operator D^n was introduced by Salagean [3]. With the help of the differential operator D^n , we say that a function $f(z)$ belonging to S is in the class $S_n(\lambda, \alpha)$ if and only if

$$Re \left\{ \frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}f(z)}{D^n f(z)} + (1-\lambda)} \right\} > \alpha \quad (n \in N_0 = N \cup \{0\}) \quad (1.5)$$

for some $\alpha(0 \leq \alpha < 1), \lambda(0 \leq \lambda < 1)$ and for all $z \in U$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \quad (1.6)$$

Further, we define the class $T_n(\lambda, \alpha)$ by

$$T_n(\lambda, \alpha) = S_n(\lambda, \alpha) \cap T. \quad (1.7)$$

We note that by specializing the parameters n, λ , and α , we obtain the following subclasses studied by various authors:

- (i) $T_0(\lambda, \alpha) = T(\lambda, \alpha)$ and $T_1(\lambda, \alpha) = C(\lambda, \alpha)$ (Altintas and Owa [1]);
- (ii) $T_0(0, \alpha) = T^*(\alpha)$ and $T_1(0, \alpha) = C(\alpha)$ (Silverman [5]);
- (iii) $T_n(0, \alpha) = T(n, \alpha)$ (Hur and Oh [2]).

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_n(\lambda, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} a_k \leq 1 - \alpha. \quad (2.1)$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\begin{aligned} \left| \frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}f(z)}{D^n f(z)} + (1-\lambda)} - 1 \right| &= \left| \frac{\sum_{k=2}^{\infty} (1-\lambda)k^n(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n[1 + \lambda(k-1)]a_k z^{k-1}} \right| \\ &\leq \frac{\sum_{k=2}^{\infty} (1-\lambda)k^n(k-1)a_k}{1 - \sum_{k=2}^{\infty} k^n[1 + \lambda(k-1)]a_k} \leq 1 - \alpha \end{aligned} \quad (2.2)$$

This shows that the values of $\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}f(z)}{D^n f(z)} + (1-\lambda)}$ lies in a circle centered at $w = 1$ whose radius is $1 - \alpha$. Hence $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ defined by (1.6) is in the class $T_n(\lambda, \alpha)$. Then

$$\operatorname{Re} \left\{ \frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}f(z)}{D^n f(z)} + (1-\lambda)} \right\} = \operatorname{Re} \left\{ \frac{1 - \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k-1)] a_k z^{k-1}} \right\} \quad (2.3)$$

$$> \alpha$$

for $z \in U$. Choose values of z on the real axis so that $\frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}f(z)}{D^n f(z)} + (1-\lambda)}$ is real. Upon clearing the denominator in (2.3) and letting $z \rightarrow 1^-$ through real values, we obtain

$$1 - \sum_{k=2}^{\infty} k^{n+1} a_k \geq \alpha \left\{ 1 - \sum_{k=2}^{\infty} k^n [1 + \lambda(k-1)] a_k \right\} \quad (2.4)$$

which gives (2.1). Finally the result is sharp with the extremal function $f(z)$ given by

$$f(z) = z - \frac{1 - \alpha}{k^n \{k^n - \alpha[1 + \lambda(k-1)]\}} z^k \quad (k \geq 2). \quad (2.5)$$

□

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have*

$$a_k \leq \frac{1 - \alpha}{k^n \{k^n - \alpha[1 + \lambda(k-1)]\}} \quad (k \geq 2). \quad (2.6)$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

3. Some Properties of the Class $T_n(\lambda, \alpha)$

Theorem 2. Let $0 \leq \alpha < 1, 0 \leq \lambda_1 \leq \lambda_2$ and $n \in N_0$. Then

$$T_n(\lambda_1, \alpha) \subseteq T_n(\lambda_2, \alpha)$$

Proof. It follows from Theorem 1 that

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda_2(k - 1)]\} a_k \leq \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda_1(k - 1)]\} a_k \leq 1 - \alpha$$

for $f(z) \in T_n(\lambda_1, \alpha)$. Hence $f(z) \in T_n(\lambda_2, \alpha)$. □

Theorem 3. Let $0 \leq \alpha < 1, 0 \leq \lambda < 1$ and $n \in N_0$. Then

$$T_{n+1}(\lambda, \alpha) \subset T_n(\lambda, \alpha).$$

The proof follows immediately from Theorem 1.

4. Distortion Theorems

Theorem 4. Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have

$$|D^i f(z)| \geq |z| - \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} |z|^2 \quad (4.1)$$

and

$$|D^i f(z)| \leq |z| + \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} |z|^2 \quad (4.2)$$

for $z \in U$, where $0 \leq i \leq n$. Then equalities in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$D^i f(z) = z - \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} z^2. \quad (4.3)$$

Proof. Note that $f(z) \in T_n(\lambda, \alpha)$ if and only if $D^i f(z) \in T_{n-i}(\lambda, \alpha)$, where

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \quad (4.4)$$

Using Theorem 1, we know that

$$z^{n-1}[2 - \alpha(1 + \lambda)] \sum_{k=2}^{\infty} k a_k \leq \sum_{k=2}^{\infty} k^n [k - \alpha(1 + \lambda(k - 1))] a_k \leq 1 - \alpha, \quad (4.5)$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]}. \quad (4.6)$$

It follows from (4.4) and (4.6) that

$$|D^i f(z)| \geq |z| - |z|^2 \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} k^i a_k \geq |z| - \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} |z|^2 \quad (4.7)$$

and

$$|D^i f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} k^i a_k \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]} |z|^2. \quad (4.8)$$

This completes the proof of Theorem 4. \square

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have*

$$|f(z)| \geq |z| - \frac{1 - \alpha}{2^n [2 - \alpha(1 + \lambda)]} |z|^2 \quad (4.9)$$

and

$$|f(z)| \leq |z| + \frac{1 - \alpha}{2^n [2 - \alpha(1 + \lambda)]} |z|^2 \quad (4.10)$$

for $z \in U$. Then equalities in (4.9) and (4.10) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1 - \alpha}{2^n [2 - \alpha(1 + \lambda)]} z^2. \quad (4.11)$$

Proof. Taking $i = 0$ in Theorem 4, we can easily show (4.9) and (4.10). \square

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then we have*

$$|f'(z)| \geq 1 - \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]}|z| \quad (4.12)$$

and

$$|f'(z)| \leq 1 + \frac{1 - \alpha}{2^{n-1}[2 - \alpha(1 + \lambda)]}|z| \quad (4.13)$$

for $z \in U$. The equalities in (4.12) and (4.13) are attained for the function $f(z)$ given by (4.11).

Proof. Note that $D(f(z) = zf'(z))$. Hence taking $i = 1$ in Theorem 4, we have the corollary. \square

Corollary 4. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{2^n[2 - \alpha(1 + \lambda)] - (1 - \alpha)}{2^n[2 - \alpha(1 + \lambda)]}. \quad (4.14)$$

The result is sharp with the extremal function $f(z)$ given by (4.11).

5. Closure Theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \dots, m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0) \quad (5.1)$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $T_n(\lambda, \alpha)$.

Theorem 5. *Let the functions $f_j(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$ for every $j = 1, 2, \dots, m$. Then the functions $h(z)$ defined by*

$$h(z) = \sum_{j=1}^m c_j f_j(z) \quad (c_j \geq 0) \quad (5.2)$$

is also in the same class $T_n(\lambda, \alpha)$ where

$$\sum_{j=1}^m c_j = 1. \quad (5.3)$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^m c_j a_{k,j} \right) z^k. \quad (5.4)$$

Further, since $f_j(z)$ are in $T_n(\lambda, \alpha)$ for every $j = 1, 2, \dots, m$ we get

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k - 1)]\} a_{k,j} \leq 1 - \alpha \quad (5.5)$$

for every $j = 1, 2, \dots, m$. Hence we can see that

$$\begin{aligned} & \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k - 1)]\} \left(\sum_{j=1}^m c_j a_{k,j} \right) \\ & \sum_{j=1}^m c_j \left(\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k - 1)]\} a_{k,j} \right) \\ & \leq \left(\sum_{j=1}^m c_j \right) (1 - \alpha) = 1 - \alpha, \end{aligned} \quad (5.6)$$

which implies that $h(z) \in T_n(\lambda, \alpha)$. Thus we have the theorem. \square

Corollary 5. *The class $T_n(\lambda, \alpha)$ is closed under convex linear combination.*

Proof. Let the function $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $T_n(\lambda, \alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1 - \mu) f_2(z) \quad (0 \leq \mu \leq 1) \quad (5.7)$$

is in the class $T_n(\lambda, \alpha)$. But, taking $m = 2, c_1 = \mu$, and $c_2 = 1 - \mu$ in Theorem 5, we have the corollary.

As a consequence of Corollary 5, there exists the extreme points of the class $T_n(\lambda, \alpha)$. \square

Theorem 6. Let $f_1(z) = z$ and

$$f_k(z) = z - \frac{1 - \alpha}{k^n \{k - \alpha[1 + \lambda(k - 1)]\}} z^k \quad (k \geq 2) \quad (5.8)$$

for $0 \leq \alpha < 1, 0 \leq \lambda < 1$ and $n \in N_0$. Then $f(z)$ is in the class $T_n(\lambda, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z) \quad (5.9)$$

where $\mu_k \geq 0 (k \geq 1)$ and $\sum_{k=2}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=2}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1 - \alpha}{k^n \{k - \alpha[1 + \lambda(k - 1)]\}} \mu_k z^k. \quad (5.10)$$

Then it follows that

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{1 - \alpha} \frac{(1 - \alpha) \mu_k}{k^n \{k - \alpha[1 + \lambda(k - 1)]\}} \\ &= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned} \quad (5.11)$$

So by Theorem 1, $f(z) \in T_n(\lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_n(\lambda, \alpha)$. Then

$$a_k \leq \frac{(1 - \alpha) \mu_k}{k^n \{k - \alpha[1 + \lambda(k - 1)]\}} \quad (k \geq 2). \quad (5.12)$$

Setting

$$\mu_k = \frac{k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{1 - \alpha} a_k \quad (k \geq 2), \quad (5.13)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k, \quad (5.14)$$

we can see that $f(z)$ can be expressed in the form (5.9). This completes the proof of Theorem 6. \square

Corollary 6. *The extreme points of the class $T_n(\lambda, \alpha)$ are the functions $f_k(z)$ ($k \geq 1$) given by Theorem 6.*

6. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 7. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then $f(z)$ is close-to-convex of order δ ($0 \leq \delta < 1$) in $|z| < r_1(n, \lambda, \alpha, \delta)$, where*

$$r_1(n, \lambda, \alpha, \delta) = \inf_k \left[\frac{(1-\delta)k^{n-1}\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (6.1)$$

The result is sharp with the extremal function $f(z)$ given by (2.5)

Proof. It is sufficient to show that $|f'(z) - 1| \leq 1 - \delta$ ($0 \leq \delta < 1$) $|z| < r_1(n, \lambda, \alpha, \delta)$. We have

$$|f'(z) - 1| = \left| \sum_{k=2}^{\infty} k a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\delta} \right) a_k |z|^{k-1} \leq 1. \quad (6.2)$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} a_k \leq 1. \quad (6.3)$$

Hence (6.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\delta)} \leq \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\delta)k^{n-1}\{k - \alpha[1 + \lambda(k-1)]\}}{1-\alpha} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (6.4)$$

The theorem follows easily from (6.4). \square

Theorem 8. Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then $f(z)$ is starlike of order δ ($0 \leq \delta < 1$) in $|z| < r_2(n, \lambda, \alpha, \delta)$, where

$$r_2(n, \lambda, \alpha, \delta) = \inf_k \left[\frac{(1 - \delta)k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{(k - \delta)(1 - \alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (6.5)$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof. We must show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$ ($0 \leq \delta < 1$) for $|z| < r_n(n\lambda, \alpha, \delta)$. We have

Thus $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$ if

$$\sum_{k=2}^{\infty} \frac{(k - \delta)a_k |z|^{k-1}}{1 - \delta} \leq 1. \quad (6.6)$$

Hence, by using (6.3), (6.6) will be true if

$$\frac{(k - \delta)|z|^{k-1}}{1 - \delta} \leq \frac{k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{1 - \alpha}$$

or if

$$|z| \leq \left[\frac{(1 - \delta)k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{(k - \delta)(1 - \alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (6.7)$$

□

Corollary 7. Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$. Then $f(z)$ is convex of order δ ($0 \leq \delta < 1$) in $|z| < r_3(n, \lambda, \alpha, \delta)$, where

$$r_3(n, \lambda, \alpha, \delta) = \inf_k \left[\frac{(1 - \delta)k^n \{k - \alpha[1 + \lambda(k - 1)]\}}{(k - \delta)(1 - \alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (6.8)$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

7. Integral Operators

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_n(\lambda, \alpha)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (7.1)$$

also belongs to the class $T_n(\lambda, \alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad (7.2)$$

where

$$b_k = \left(\frac{c+1}{c+k} \right) a_k. \quad (7.3)$$

therefore,

$$\begin{aligned} \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} b_k &= \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} \left(\frac{c+1}{c+k} \right) \\ &\leq \sum_{k=2}^{\infty} k^n \{k - [1 + \lambda(k-1)]\} a_k \leq 1 - a, \end{aligned} \quad (7.4)$$

since $f(z) \in T_n(\lambda, \alpha)$. Hence, by Theorem 1, $F(z) \in T_n(\lambda, \alpha)$. \square

Theorem 10. *Let c be a real number such that $c > -1$. If $F(z) \in T_n(\lambda, \alpha)$, then the function $f(z)$ defined by (7.1) is univalent in $|z| < R^*$, where*

$$R^2 = \inf_k \left[\frac{(c+1)k^{n-1} \{k - \alpha[1 + \lambda(k-1)]\}}{(c+k)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (7.5)$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ ($a_k \geq 0$). It follows from (7.1) that

$$f(z) = \frac{z^{1-c} [z^c F(z)]'}{(c+1)} = z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1} \right) a_k z^k \quad (c > -1). \quad (7.6)$$

In order to obtain the required result, it suffices to show that $|f'(z) - 1| < 1$ in $|z| < R^*$.
Now

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \quad (7.7)$$

Hence, by using (6.3), (7.7) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{c+1} \leq \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} \quad (k \geq 2)$$

or if

$$|z| \leq \left[\frac{(c+1)k^{n-1} \{k - \alpha[1 + \lambda(k-1)]\}}{(c+k)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \quad (7.8)$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n \{k - \alpha[1 + \lambda(k-1)]\}(c+1)} z^k \quad (k \geq 2). \quad (7.9)$$

□

8. Modified Hadamard Products

Let the functions $f_j(z)$ ($j = 1, 2$) be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined here by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \quad (8.1)$$

Theorem 11. *Let the functions $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $T_n(\lambda, \alpha)$. Then $f_1 * f_2(z)$ belongs to the class $T_n(\lambda, \beta(n, \lambda, \alpha))$ where*

$$\beta(n, \lambda, \alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n \{2 - \alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}. \quad (8.2)$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(n, \lambda, \alpha)$ such that

$$\sum_{k=2}^{\infty} \frac{k^n \{k - \beta[1 + \lambda(k-1)]\}}{1 - \beta} a_{k,1} a_{k,2} \leq 1 \quad (8.3)$$

Since

$$\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} a_{k,1} \leq 1 \quad (8.4)$$

and

$$\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} a_{k,2} \leq 1, \quad (8.5)$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} a \sqrt{a_{k,1} a_{k,2}} \leq 1. \quad (8.6)$$

Thus it is sufficient to show that

$$\begin{aligned} & \frac{k^n \{k - \beta[1 + \lambda(k-1)]\}}{1 - \beta} a_{k,1} a_{k,2} \\ & \leq \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq 2), \end{aligned} \quad (8.7)$$

that is, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(1 - \beta) \{k - \alpha[1 + \lambda(k-1)]\}}{(1 - \alpha) \{k - \beta[1 + \lambda(k-1)]\}}. \quad (8.8)$$

Not that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{1 - \alpha}{k^n \{k - \alpha[1 + \lambda(k-1)]\}} \quad (k \geq 2). \quad (8.9)$$

Consequently, we need only to prove that

$$\frac{1 - \alpha}{k^n \{k - \alpha[1 + \lambda(k-1)]\}} \leq \frac{(1 - \beta) \{k - \alpha[1 + \lambda(k-1)]\}}{(1 - \alpha) \{k - \beta[1 + \lambda(k-1)]\}} \quad (k \geq 2), \quad (8.10)$$

or, equivalently, that

$$\beta \leq -1 \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n \{k - \alpha[1 + \lambda(k-1)]\}^2 - [1 + \lambda(k-1)](1-\alpha)^2} (k \geq 2). \quad (8.11)$$

Since

$$A(k) = 1 - \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n \{k - \alpha[1 + \lambda(k-1)]\}^2 - [1 + \lambda(k-1)](1-\alpha)^2} \quad (8.12)$$

is an increasing function of k ($k \geq 2$), letting $k = 2$ in (8.12), we obtain

$$\beta \leq A(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n \{2 - \alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}, \quad (8.13)$$

which completes the proof of Theorem 11.

Finally, by taking the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{1-\alpha}{2^n [2 - \alpha(1+\lambda)]} z^2 \quad (j = 1, 2), \quad (8.14)$$

we can see that the result is sharp. \square

Corollary 8. For $f_1(z)$ and $f_2(z)$ as in Theorem 11, the function

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1} a_{k,2}} z^k \quad (8.15)$$

belongs to the class $T_n(\lambda, \alpha)$.

This result follows from the Cauchy-Schwarz inequality (8.6). It is sharp for the same functions as in Theorem 11.

Theorem 12. Let the function $f_1(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$ and the function $f_2(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha)$. Then $f_1 * f_2(z)$ belongs to the class $T_n \eta(n, \lambda, \alpha, \gamma)$, where

$$\eta(n, \lambda, \alpha, \gamma) = 1 - \frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^n \{2 - \alpha(1+\lambda)\} \{2 - \gamma(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\gamma)}. \quad (8.16)$$

The result is best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{2^n [2 - \alpha(1+\lambda)]} z^2 \quad (8.17)$$

and

$$f_g(z) = z - \frac{1-\gamma}{2^n[2-\gamma(1+\lambda)]}z^2. \quad (8.18)$$

Proof. Proceeding as in the proof of Theorem 11, we get

$$\eta \leq B(k) = 1 - \frac{(k-1)(1-\lambda)(1-\alpha)(1-\gamma)}{k^n\{k-\alpha[1+\lambda(k-1)]\}\{k-\gamma[1+\lambda(k-1)]\} - [1+\lambda(k-1)](1-\alpha)(1-\gamma)} \quad (k \geq 2) \quad (8.19)$$

Since the function $B(k)$ is an increasing function of $k(k \geq 2)$, setting $k = 2$ in (8.19), we get

$$\eta \geq B(2) = 1 - \frac{(1-\lambda)(1-\alpha)(1-\gamma)}{2^n\{2-\alpha(1+\lambda)\}\{2-\gamma(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\gamma)}. \quad (8.20)$$

This completes the proof of Theorem 12. \square

Corollary 9. Let the functions $f_j(z)$ ($j = 1, 2, 3$) defined by (5.1) be in the class $T_n(\lambda, \alpha)$. Then $f_1 * f_2 * f_3(z)$ belongs to the class $T_n(\lambda, \zeta(n, \lambda, \alpha))$, where

$$\zeta(n, \lambda, \alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^3}{4^n\{2-\alpha(1+\lambda)\}^3 - (1+\lambda)(1-\alpha)^3}. \quad (8.21)$$

The result is best possible for the functions

$$f_j(z) = z - \frac{1-\alpha}{2^n[2-\alpha(1+\lambda)]}z^2 \quad (j = 1, 2, 3). \quad (8.22)$$

Proof. From Theorem 11, we have $f_1 * f_2 * f_3(z) \in T_n(\lambda, \beta(n, \lambda, \alpha))$, where β is given by (8.2). By using Theorem 12, we get $f_1 * f_2 * f_3 * (z) \in T_n(\lambda, \zeta(n, \lambda, \alpha))$, where

$$\begin{aligned} \zeta(n, \lambda, \alpha) &= 1 - \frac{(1-\lambda)(1-\alpha)(1-\beta)}{2^n\{2-\alpha(1+\lambda)\}\{2-\beta(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\beta)} \\ &= \frac{(1-\lambda)(1-\alpha)^3}{4^n\{2-\alpha(1+\lambda)\}^3 - (1+\lambda)(1-\alpha)^3} \end{aligned}$$

This completes the proof of Corollary 9. \square

Theorem 13. *Let the functions $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $T_n(\lambda, \alpha)$. Then the function*

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \quad (8.23)$$

belong to the class $T_n(\phi(n, \lambda, \alpha))$ where

$$\phi(n, \lambda, \alpha) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n-1}\{2-\alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}. \quad (8.24)$$

The result is sharp for the functions $f_j(z)$ ($j = 1, 2$) defined by (8.14).

Proof. By virtue of Theorem 1, we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \left[\frac{k^n \{k^n \{k - \alpha[1 + \lambda(k-1)]\} 1 - \alpha\}}{1 - \alpha} \right]^2 a_{k,1}^2 \\ & \leq \left[\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} a_{k,1}^2 \right] \leq 1 \end{aligned} \quad (8.25)$$

and

$$\begin{aligned} & \sum_{k=2}^{\infty} \left[\frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} \right]^2 a_{k,2}^2 \\ & \leq \left[\sum_{k=2}^{\infty} \frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} a_{k,2}^2 \right] \leq 1. \end{aligned} \quad (8.26)$$

It follows from (8.25) and (8.26) that

$$\left| su \frac{1}{2} \left[\frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} \right]^2 (a_{k,1}^2 + a_{k,2}^2) \right| \leq 1. \quad (8.27)$$

Therefore, we need to find the largest $\phi = \phi(n, \lambda, \alpha)$ such that

$$\frac{k^n \{k - \phi[1 + \lambda(k-1)]\}}{1 - \phi} \leq \frac{1}{2} \left[\frac{k^n \{k - \alpha[1 + \lambda(k-1)]\}}{1 - \alpha} \right]^2 \quad (k \geq 2), \quad (8.28)$$

that is,

$$\phi \leq 1 - \frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^n \{k - \alpha[1 + \lambda(k-1)]\}^2 - 2[1 + \lambda(k-1)](1-\alpha)^2} \quad (k \geq 2). \quad (8.29)$$

Since

$$D(k) = 1 - \frac{2(k-1)(1-\lambda)(1-\alpha)^2}{k^n \{k - \alpha[1 + \lambda(k-1)]\}^2 - 2[1 + \lambda(k-1)](1-\alpha)^2} \quad (8.30)$$

is an increasing function of $k(k \geq 2)$, we readily have

$$\phi \leq D(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n-1} \{2 - \alpha(1+\lambda)\}^2 - (1+\lambda)(1-\alpha)^2}, \quad (8.31)$$

and Theorem 13 follows at once. \square

Theorem 14. *Let the function $f_1(z) = z - \sum_{k=2}^{\infty} a_{k,1} z^k$ ($a_{k,1} \geq 0$) be in the class $T_n(\lambda, \alpha)$ and $f_2(z) = z - \sum_{k=2}^{\infty} |a_{k,2}| z^k$, with $|a_{k,2}| \leq 1, k = 2, 3, \dots$. Then $f_1 * f_2(z) \in T_n(\lambda, \alpha)$.*

Proof. Since

$$\begin{aligned} \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} |a_{k,1} a_{k,2}| &= \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} a_{k,1} |a_{k,2}| \\ &\leq \sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} a_{k,1} \\ &\leq 1 - \alpha, \end{aligned}$$

by Theorem 1, it follows that $f_1 * f_2(z) \in T_n(\lambda, \alpha)$. \square

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