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DOLD-KAN TYPE THEOREMS FOR n-TYPES OF SIMPLICITIAL COMMUTATIVE ALGEBRAS

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Abstract

A functor from simplicial algebras to crossed n-cubes is shown to be an embedding on a reflexive subcategory of the category of simplicial algebras that contains representatives for all n types.

1. Introduction

In [16] Kan showed how one might obtain the homotopy groups of a pointed connected simplicial set using only the tools of combinatorial groups theory. His methods involved use of free groups and Tietsze transformations etc. That paper and its companion [17] together provide proofs of (i) the Dold-Kan theorem for Abelian simplicial groups and (ii) the fact that the homotopy category of simplicial groups is equivalent to that of connected simplicial sets, which can be rephrased as saying that simplicial groups model all homotopy types. These two results have been crucial in the development of both algebraic topology and homological algebra in the last thirty years.

The study of n-types goes back further to work of Fox [15]. Whitehead [25] studied combinatorial homotopy in the late 1940's. In particular, he searched for algebraic models of on n-types and with MacLane in [20] found a very neat model for a 2-type of a complex. Their models are what are known as crossed modules. Whitehead in his paper "Combinatorial Hototopy II" [24] also consider "homotopy systems" (now called "crossed complexes") which model a larger class of homotopy types.

The MacLane-Whitehead result was generalised by Loday [19] to give algebraic models for all n-types for arbitrary n. These models he called cat^n-groups. cat^1-groups were known to be equivalent to crossed modules and Ellis & Steiner [14] have since shown that cat^n-groups are equivalent to crossed n-cubes (see below). The other algebraic settings such as commutative algebras, Lie algebras, Jordan algebras of this construction are due to Ellis [13].

The histroy of interactions among algebraic topology and homological algebra indicates that with each significant new model for homotopy types there should be a potential application in homological algebra. Crossed modules have appeared many times in parts of algebra other than group theory. For instance, the commutative algebra version of crossed modules has been examined by T. Porter (cf. [21] and [22]). It is known [4], [21] that simplicial algebras lead to crossed modules and crossed complexes of algebra, that free crossed modules are related to Koszul complex constructions and higher dimensional analogues have been proposed by Ellis [13] for use in homotopical and homological algebra. André [1] uses simplicial methods to investigate homological properties of commutative algebras.

In [23], T. Porter examined D. M. Kan's fundamental paper "A combinatorial definition of homotopy groups" (ref. [16]). T. Porter described a functor from the category of simplicial groups to that of crossed n-cubes of groups, based on ideas of Loday. In [6], the first author and T. Porter adapted that description to give an analogue of this functor for the algebra case.

In the present work we recall from [6] a functor from simplicial algebras to crossed n-cubes and show it to be an embedding on a reflexive subcategory of the category of simplicial algebras that contain representatives for all n-types. The construction of this functor is described using the *décalage* functor studied by Illusie [18] and Duskin [12] and is a π_0-image of a functor taking values in a category of simplicial ideal $(n+1)$-ads.

One of the aims of this series of papers is to show that what might be called 'combinatorial algebra theory', by analogy with 'combinatorial group theory', is an area with interesting structure. This may provide new methods in homological algebra.

1. Preliminaries on Simplicial Algebras

All algebras will be *commutative* and will be over the same fixed but unspecified ground ring.

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}, 0 \leq i \leq n, (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}, 0 \leq i \leq n$, satisfying the usual simplicial identities given in Andrè [1] or Illusie [18], for example. It can be completely described as a functor $\mathbf{E} : \Delta^{op} \to \mathbf{CommAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

Recall that given a simplicial algebra $\boldsymbol{E}$, the *Moore complex* $(\boldsymbol{NE}, \partial)$ of $\mathbf{E}$ is the chain complex defined by

$$(NE)_n = \bigcap_{i=0}^{n-1} \text{Ker } d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

We say that the Moore complex $\mathbf{NE}$ of a simplicial algebra is of *length* k if $NE_n = 0$ for all $n \geq k+1$ so that a Moore complex is of length k also of length r for $r \geq k$.

The n^{th} *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the n^{th} homology of the Moore complex is $\boldsymbol{E}$, i.e.,

$$\begin{aligned}\pi_n(\boldsymbol{E}) &\cong H_n(\boldsymbol{NE},\partial)\\ &= \bigcap_{i=0}^{n} \text{Ker } d_i^n / d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} \text{Ker } d_i^{n+1}\right).\end{aligned}$$

If **A** is a simplicial module, (**NA**, ∂) is a chain complex in the usual sense and **N** gives a functor from **SimpMod,** the category of simplicial modules to **ChMod** the category of (non-negatively graded) chain complexes of modules. The Dold-Kan theorem states that this functor **N: SimpMod** $\rightarrow$ **ChMod** is an equivalence of categories.

Various generalisations of the Dold-Kan theorem are known. For instance. Ashley (cf. [7]) proves an equivalence between simplicial T-complexes and crossed complexes. He ends by exploring the relation between simplicial T-complexes and simplicial groups. Conduché [11] looks at simplicial groups whose Moore complex has trivial terms in dimensions greater than 2 and links them with a notion of 2-crossed module. An important observation in Conduché's work is the existence of a semi-direct product decomposition of the groups G_n of n-simplices in a simplicial group G. These semidirect product decompositions are used in Dold-Kan theorem and have been studied in depth by Carrasco and Cegarra [10]. By encoding the multiplication of the simplicial group in terms of this decomposition, they were able to make precise the extra structure carried by the Moore complex of a simplicial group up to isomorphism. This gives the most general non-Abelian form of a Dold-Kan type theorem. Carrasco's thesis [9] contains not only the main results of [10] but also considers the case of simplicial algebras.

We will need to make use of the simidirect product decomposition several times. The basic result is the following:

Proposition 1.1. *If* **E** *is a simplicial algebra, then for any* $n \geq 0$

$$\begin{aligned}E_n \cong (\cdots(NE_n \rtimes s_{n-1}NE_{n-1}) \rtimes \cdots \rtimes s_{n-2}\cdots s_0 NE_1) \rtimes \\ (\cdots(s_{n-2}NE_{n-1} \rtimes s_{n-1}s_{n-2}NE_{n-2}) \rtimes \cdots \rtimes s_{n-1}s_{n-2}\cdots s_0 NE_0).\end{aligned}$$

Proof. This is by repeated use of the following lemma. □

Lemma 1.2. *Let* **E** *be a simplicial algebra. Then* E_n *can be decomposed as a semidirect product:*

$$E_n \cong \text{ Ker } d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}).$$

Proof. The isomorphism can be defined as follows:

$$\begin{aligned}\theta : E_n &\to \text{Ker } d_n^n \bowtie s_{n-1}^{n-1}(E_{n-1})\\ e &\mapsto (e - s_{n-1}d_ne, s_{n-1}d_ne).\end{aligned}$$

□

2. Truncations

By an *ideal chain complex* of algebras, (X, d), we mean one in which each Im d_{i+1} is an ideal of X_i. Given any ideal chain complex (X, d) of algebras and an integer n, the truncation $t_{n|}X$ of X at level n will be defined by

$$(t_{n|}X)_i = \begin{cases} X_i & \text{if} \quad i < n \\ X_i/\text{Im}d_{n+1} & \text{if} \quad i = n \\ 0 & \text{if} \quad i > n. \end{cases}$$

The differential d of $t_{n|}X$ is that of X for $i < n, d_n$ is induced from the n^{th} differential of X and all other are zero. (For more information see Illusie [18]). Truncation is of course functorial.

Proposition 2.1. *There is a truncation functor* $t_{n|}$: ***SimpAlg*** $\to$ ***SimpAlg*** *such that there is a natural isomorphism*

$$t_{n|}N \cong Nt_{n|},$$

where $\boldsymbol{N}$ *is the Moore complex functor from* ***SimpAlg*** *to the category of chain complexes of algebras.*

Proof. We first note that $d_{n+1}^{n+1}(NE_{n+1})$ is contained in E_n as an ideal and that all face maps of $\mathbf{E}$ vanish on it. We can thus take

$$(t_{n|}\boldsymbol{E})_i = \begin{cases} E_i & \text{for all } i < n \\ \frac{E_n}{d_{n+1}^{n+1}(NE_{n+1})} & \text{for } i = n, \end{cases}$$

and for $i > n$ we take the semidirect decomposition of E_i given by Proposition 1.1, delete all occurrences of NE_k for $k > n$ and replace any NE_n by $NE_n/d_{n+1}^{n+1}(NE_{n+1})$. The definition of face and degeneracy is easy as is the verification that $t_{n|}N$ and $Nt_{n|}$ are the same. □

This truncation functor has nice properties. (In the chain complex case, these are discussed in Illusie [18]).

Proposition 2.2. *Let* $T_{n|}$ *be the full subcatefory of* ***SimpAlg*** *defined by the simplicial* *algebras whose Moore complex is trivial in dimensions greater than n and let $i_n : T_{n|} \to$* ***SimpAlg*** *be the inclusion functor.*

(a) The functor $t_{n|}$ is left adjoint to i_n. (In the future, we will usually drop the i_n and so also write $t_{n|}$ for the composite functor.)

(b) The natural transformation η, the co-unit of the adjunction, is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$.

(c) For any simplicial algebra $\boldsymbol{E}$, $\pi(t_{n|}\boldsymbol{E}) = 0$ if $i > n$.

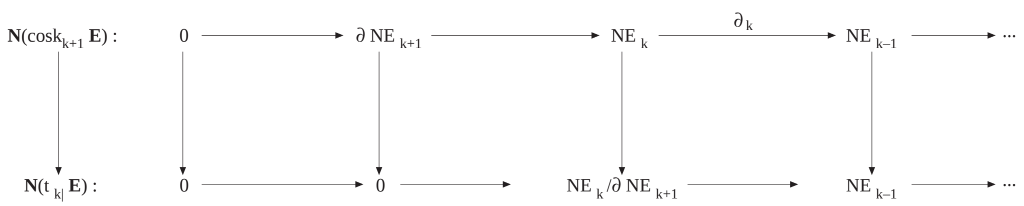
(d) To the inclusion $T_{n|} \to T_{n+1|}$, there corresponds a natural epimorphism η_n from $t_{n+1|}$ to $t_{n|}$. If $\boldsymbol{E}$ is simplicial algebra, the kernel of η_n $(\boldsymbol{E})$ is a $K(\pi_{n+1}(\boldsymbol{E}), n+1)$, i.e. has a single non-zero homotopy module in dimension $n+1$, that being π_{n+1} $(\boldsymbol{E})$.

As each statement is readily verified using the Moore complex and the semidirect product decomposition, the proof of the above will be left out.

A comparison of these properties with those of the coskeleta functors (cf. Artin and Mazur [2]) is worth making. Recall that given any integer $k \geq 0$, there is a functor cosk_n defined on the category of simplicial sets, which is the composite of a truncation functor (differently defined) and its right adjoint. The n-simplicies of $\text{cosk}_x X$ are given by Hom $(\text{sk}_k\Delta[n], X)$, the set of simplicial maps from the k-skeleton of the n-simplex $\Delta[n]$ to the simplicial set X. There is a canonical map from X to $\text{cosk}_k X$ whose homotopy fibre an Eilenberg-Maclane space of type $(\pi_k(X), k)$. This k-coskeleton is constructed using finite limits and there is an analogue in any category of simplicial objects in a category **C** provided that **C** has finite limits, thus in particular in **SimpAlg**. The first author and T. Porter (cf. [6]) have calculated the Moore complex of cosk_k **E** for a simplicial algebra **E** using a construction described in Duskin's Memoir [12]. Our results gives

$$N(\text{cosk}_k\boldsymbol{E})_\ell = 0 \text{ if } \ell > k+1$$
$$N(\text{cosk}_k\boldsymbol{E})_{k+1} = \text{Ker } (\partial_k : NE_k \to NE_{k-1})$$
$$N(\text{cosk}_k\boldsymbol{E})_\ell = NE_\ell \text{ if } \ell \leq k.$$

There is a natural epimorphism from $\text{cosk}_{k+1}\mathbf{E}$ to $t_{n|}E$ which on passing to Moore complexes gives



This epimorphism of chain complexes thus has an acyclic kernel. The epimorphism therefore induces an isomorphism on all homotopy modules and hence is a weak homotopy equivalence. We may thus use either $t_{k|}\ \boldsymbol{E}$ or $\mathrm{cosk}_{k+1}\ \boldsymbol{E}$ as a model for k-type of simplicial algebra $\mathbf{E}$.

3. Cat^n-Algebras and Crossed n-Cubes

The notion of a cat^n-algebra is defined by Ellis [13]. A cat^n-algebra $\mathcal{A}$ is an (commutative) algebra A together with $2n$ endomorphisms $s_i, t_i : A \to A (1 \leq i \leq n)$ such that

$$\begin{array}{llll} t_i s_i = s_i & s_i t_i = t_i & & \\ s_i s_j = s_j s_i & t_i t_j = t_j t_i, & s_i t_j = t_j s_i & \text{for } i \neq j \\ aa' = 0 & \text{for } a \in \mathrm{Ker} s_i, & a' \in \mathrm{Ker} t_i. & \end{array}$$

A morphism of cat^n-algebras $\phi : \mathcal{A} \to \mathcal{A}'$ is an algebra homomrphism $\phi : A \to A'$ which preserve the s_i and t_i.

T. Porter (cf. [22]) shows that a cat^1-algebra is equivalent to a crossed module and also to an internal category within the category of algebras.

Crossed n-cubes in algebraic settings such as commutative algebras Jordan algebras and Lie algebras have been defined by Ellis [13].

A crossed n-cube of commutative algebras is a family of commutative algebras M_A for $A \subseteq \langle n \rangle = \{1, \ldots, n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in \langle n \rangle$ and for $A, B \subseteq \langle n \rangle$ and functions

$$h : M_A \times M_B \to M_{A \cup B}$$

such that for all $k \in \mathbf{k}, a, a' \in M_A, b, b' \in M_B, c \in M_C, i, j \in \langle n \rangle$ and $A \subseteq B$

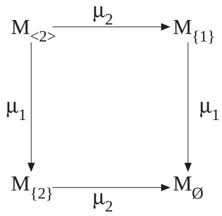
1. $\mu_i a = a$ if $i \notin A$
2. $\mu_i \mu_j a = \mu_j \mu_i a$
3. $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
4. $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
5. $h(a, a') = aa'$
6. $h(a, b) = h(b, a)$
7. $h(a + a', b) = h(a, b) + h(a', b)$
8. $h(a, b + b') = h(a, b) + h(a, b')$
9. $k \cdot h(a, b) = h(k \cdot a, b) = h(a, k \cdot b)$
10. $h(h(a, b), c) = h(a, h(b, c)) = h(b, h(b, c))$.

A morphism of crossed n-cubes is defined in the obvious way: It is a family of commutative algebra homomorphisms, where for $A \subseteq \langle n \rangle, f_A : M_A \rightarrow M'_A$ commuting with the μ_i's and h's. We thus obtain a category of crossed n-cubes denoted by $\mathbf{Crs}^n$.

Remarks.

1. In the correspondence between cat^n-algebras and crossed n-cubes (Ellis and Steiner, [14]) the cat^n-algebra corresponding to a crossed n-cube (M_A) is constructed as a repeated semidirect product of the various M_A. Within the results of algebra, the h-functions are interpreted as being multiplications. This explains the structure of the h-function axioms.

2. A crossed 1-cube is the same as a crossed module. Crossed squares, that is crossed 2-cubes, give the square



in which each μ_i is a crossed module as is $\mu_1\mu_2$, the h-functions give actions and a pairing

$$h : M_{\{1\}} \times M_{\{2\}} \rightarrow M_{\langle 2 \rangle}.$$

The maps μ_2 (or μ_1) also define a map of crossed modules. In fact, a crossed square can be thought of as a crossed module in the category of crossed modules. This generalises to higher dimensions.

Lemma 3.1. *Let $\mathcal{M} = \{M_A : A \subseteq \langle n \rangle, \{\mu_i\}, h\}$ be a crossed n-cube of algebras and let $i \in \langle n \rangle$. Let $\mathcal{M}_1$ denote the restriction of $\mathcal{M}$ to those A with $i \in A$ and $\mathcal{M}_0$, the restriction to those A with $i \notin A$. Then $\mathcal{M}_1$ and $\mathcal{M}_0$ are crossed $(n-1)$-cubes of algebras and $\mu_i : \mathcal{M}_1 \rightarrow \mathcal{M}_0$ is a morphism of crossed $(n-1)$-cubes of algebras.*

The proof is quite simple and so will be omitted. Note that as each level of μ_i is a crossed module of algebras, each Kerμ_i is a module and each Imμ_i is an ideal of the corresponding algebra of $\mathcal{M}_0$.

For convenience of notation, we will assume $i = n$ thus the crossed $(n-1)$-cube structures on $\mathcal{M}_0$ and $\mathcal{M}_1$ are given by:

$$\text{if } A \subseteq \langle n-1 \rangle, \quad M_{0,A} = M_A = M_A \text{ and } M_{1,A} = M_{A \cup \{n\}}$$

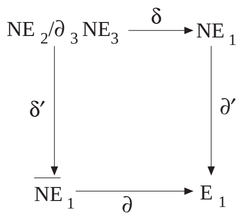
with the μ_i and the h-functions induced from those of $\mathcal{M}$. We will set $N_A = M_A/\text{Im}\mu_n$ for $A \subseteq \langle n-1 \rangle$, and note that if $i < n$, $\mu_i : M_A \rightarrow M_{A-\{i\}}$ sends Imμ_n to itself (axiom 2). This implies that μ_i induces a map $\overline{\mu_i} : N_A \to N_{A-\{i\}}$. Similarly, since $h : M_A \times M_B \to M_{A\cup B}$ satisfies $h(\mu_i a, \mu_i b) = \mu_i h(a,b)$, it induces $\bar{h} : N_A \times N_B \to N_{A\cup B}$ in the obvious way. It is routine to check that $\mathcal{N} = \{N_A : A \subseteq \langle n\rangle, \{\overline{\mu_i}\}, \bar{h}\}$ is a crossed $(n-1)$-cube which is the kernel of μ_n. (If we replaced the n^{th} direction by some other, essentially the same discussion applies but it is slightly messier). We thus obtain from $\mathcal{M}$ a crossed 2-fold extension of crossed $(n-1)$-cubes,

$$\text{Ker } \mu_i \to \mathcal{M}_1 \to \mathcal{M}_0 \to \text{ Coker } \mu_i$$

with the algebras in Ker μ_i modules.

3. Crossed n-cubes are defined algebraically, i.e. they can be specified categorically using finite products only, unlike internal categories in which the domain of the composition morphism uses a pullback. As a consequence of this, analogues of crossed n-cubes in other settings are easily found. For instance, instead of working with Sets as a base by any category of sheaves of sets on a space or site, that is in any Grothendieck topos, as all the constructions use only finite limits and colimits this would already include some quite important and interesting cases with potential links to algebraic geometry.

Example. Let **E** be a simplicial algebra. Then following diagram is a crossed square:



Here, $NE_1 = \text{Ker } d_0^1$ and $\overline{NE}_1 = \text{Ker } d_1^1$.

Since E_1 act on $NE_2\partial_3 NE_3, \overline{NE}_1$ and NE_1, there are actions of $\overline{NE}_1$ on $NE_2/\partial_3 NE_3$ and NE_1 via ∂, and NE_1 act on $NE_2/\partial_3 NE_3$ and $\overline{NE}_1$ via ∂'. As ∂ and ∂' are inclusions, all actions can be given by multiplication. The h-map is

$$\begin{aligned} NE_1 \times \overline{NE}_1 &\to NE_2/\partial_3 NE_3 \\ (x, \bar{y}) &\mapsto h(x,\bar{y}) = s_1 x(s_1 y - s_0 y) + \partial_3 NE_3, \end{aligned}$$

which is bilinear. Here, x and y are in NE_1 as there is a natural bijection between NE_1 and $\overline{NE}_1$ (by Lemma 2.1 in [5]). The element $\bar{y}$ is the image of y under this. The detailed verifications of axioms of the crossed square can be found in [6].

This example effectively introduces the functor

$$\boldsymbol{M\ (-,2) : SimpAlg \to Crs^2}.$$

The 2-dimensional case of this general construction has been examined by the first author and T. Porter [6], in which they defined the following construction:

The *décalage* functor forgets the last face operators at each level of a simplicial algebra **E** and moves everything down one level. It is denoted by Dec. Thus

$$(\text{Dec } E_n) = E_{n+1}.$$

The last face degeneracy of **E** yields a contraction of $\text{Dec}^1\ \boldsymbol{E}$ as an augmented simplicial algebra,

$$\text{Dec}^1\boldsymbol{E} \simeq \boldsymbol{K}(E_0, 0),$$

by an explicit natural homotopy equivalence (cf. [12]). The last face map will be denoted

$$\delta_0 : \ \text{Dec}\ ^1\boldsymbol{E} \to \boldsymbol{E}.$$

Iterating the Dec construction gives an augmented bisimplicial algebra

$$[\ldots\ \text{Dec}^3\boldsymbol{E} \ \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\ \text{Dec}\ ^2\boldsymbol{E} \ \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\ \text{Dec}^1\boldsymbol{E}]$$

which in expanded form is the total décalage of **E**:

$$[\ldots\ \text{Dec}^3\boldsymbol{E} \ \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\ \text{Dec}\ ^2\boldsymbol{E} \ \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\ \text{Dec}^1\boldsymbol{E}] \overset{\delta_0}{\to} \boldsymbol{E}$$

(See [12] or [18] for details.) The maps from $\text{Dec}\ ^i\ \boldsymbol{E}$ to $\text{Dec}\ ^{i-1}\ \boldsymbol{E}$ coming from the i last face maps will be labelled $\delta_0, \ldots, \delta_{i-1}$ so that $\delta_0 = d_{\text{last}}, \delta_1 = d_{\text{lastbutone}}$ and so on.

For a simplicial algebra **E** and a given n, we write ***M(E,n),*** for the crossed n-cube arising from the functor

$$\boldsymbol{M}(-,\boldsymbol{n}\ ) : \ \boldsymbol{SimpAlg} \to \boldsymbol{Crs}^n,$$

which is given by π_0 (Dec $\boldsymbol{E}$; Ker $\delta_0, \ldots,$Ker δ_{n-1}).

Proposition 3.2. *(cf. The first author and T. Porter [6]). If* **E** *be a simplicial algebra, then the crossed n-cube* ***M(E,n)*** *is determined by:*

(i) for $A \subseteq \langle n \rangle$,

$$\boldsymbol{M}\ (\boldsymbol{E},\boldsymbol{n})_A = \frac{\bigcap_{j\in A}\ Ker\ d^n_{j-1}}{d^{n+1}_{n+1}(Ker d^{n+1}_0 \bigcap \{\bigcap_{j\in A} Ker d^{n+1}_j\})};$$

(ii) the inclusion

$$\bigcap_{j\in A} Ker\ d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}} Ker\ d^n_{j-1}$$

induces the morphism

$$\mu_i : \boldsymbol{M(E,}n)_A \to \boldsymbol{M(E,}n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq \langle n\rangle$,

$$h : \boldsymbol{M(E,}n)_A \times \boldsymbol{M(E,}n)_B \to \boldsymbol{M(E,}n)_{A\cup B}$$

given by

$$h(\bar{x}, \bar{y}) = \overline{xy},$$

where an element of $\boldsymbol{M(E,}n)_A$ *is denoted by* $\bar{x}$ *with* $x \in \cap_{j\in A}\ Ker\ d^n_{j-1}$.

Proof. First some explanation of the definition $\mathcal{M}(-, n)$ as

$$\pi_0(\text{Dec } \boldsymbol{E};\ \text{Ker } \delta_0, \ldots,\ \text{Ker } \delta_{n-1}).$$

For each simplicial algebra **E** we start by looking at the canonical augmentation map δ_0 : $\text{Dec}^1\boldsymbol{E} \to \boldsymbol{E}$, which has kernel the simplicial algebra $\text{Ker}d_{\text{last}}$ mentioned above. Then take the simplicial inclusion crossed module $\text{Ker}\delta_0 \to \text{Dec}^1\ \boldsymbol{E}$ to be $\mathcal{M}(\boldsymbol{E}, 1)$ defining thus a functor

$$\mathcal{M}(\ ,1) : \boldsymbol{SimpAlg} \to \boldsymbol{Simp(IncCrs^1)}.$$

The it is easy to show that

$$\pi_0(\text{Ker } \delta_0) \to \pi_0(\text{Dec}^1\boldsymbol{E})$$

is precisely $\boldsymbol{M(E,}1)$. The higher order analogues $\boldsymbol{M(}\ \ , n)$ are as follows:

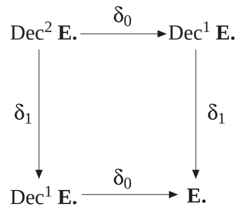
For each simplicial algebra **E** there is a functorial short exact sequence

$$\text{Ker } \delta_0 \longrightarrow \text{Dec }^1\boldsymbol{E} \xrightarrow{\delta_0} \boldsymbol{E}.$$

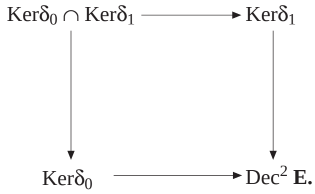
This corresponds to the 0-skeleton of the total décalage of $\boldsymbol{E}$:

$$[\ldots\ \text{Dec}^3\boldsymbol{E} \begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\ \text{Dec }^2\boldsymbol{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\ \text{Dec}^1\boldsymbol{E}] \xrightarrow{\delta_0} \boldsymbol{E}$$

For $n = 2$, the 1-skeleton of that total décalage gives the commutative diagram



Here, δ_1 is d^n_{n-1} in dimension n whilst δ_0 is d^n_n. Forming the square of kernels gives



Again, π_0 of this gives $\boldsymbol{M(E,}2)$. In general, we use the $(n-1)$-skeleton of the total décalage to form an n-cube and thus a simplicial inclusion crossed n-cube corresponding to the simplicial ideal $(n+1)$-ad

$$(\mathrm{Dec}^n \boldsymbol{E};\ \mathrm{Ker}\ \delta_{n-1}, \ldots,\ \mathrm{Ker}\ \delta_0).$$

This simplicial inclusion n-cube will be denoted by $\mathcal{M}(\boldsymbol{E}, n)$, and its associated crossed n-cube by

$$\pi_0(\mathcal{M}(\boldsymbol{E}, n)) = \boldsymbol{M(E}, n).$$

The result now follows by direct calculation on examining the construction of π_0 as the zeroth homology of the Moore complex of each term in the inclusion crossed n-cube, $\mathcal{M}(\boldsymbol{E}, n)$. □

An immediate consequence of the previous proposition is that $\boldsymbol{M(E}, n)$ contains all the information about $\pi_i(E)$ for $i < n$. This suggests that $\boldsymbol{K(E,}n)$ may contain all the information of the n-type of **E**, i.e. that on the subcategory $T_{n|}$ of n-truncated simplicial algebras. $\boldsymbol{M}(-, n)$ is an embedding. The proof of this will be our next aim; it can be considered as a form of Dold-Kan theorem.

4. A Dold-Kan Equivalence for n-Types

We recall that $T_{n|}$ is the full subcategory of **SimpAlg** given by the image of the truncation functor, $t_{n|}$.

Given a simplicial algebra ***E***, $\boldsymbol{M(E,}1)$ is the crossed module

$$\mu_1 : \frac{NE_1}{d_2(NE_2)} \to E_0$$

with μ_1 induced by d_1. Forming the cat^1-algebra associated to this gives $\frac{NE_1}{d_2(NE_2)} \rtimes E_0$ which is isomorphic to $\frac{E_1}{d_2(NE_2)}$ with source and target maps corresponding to d_1 and d_0, respectively. Thus from $\boldsymbol{M(E,}1)$, we can form the 1-skeleton of $t_{1|}\boldsymbol{E}$ and, as the higher algebras in $t_{1|}\boldsymbol{E}$ are determined by the 1-skeleton, we have constructed $t_{1|}\boldsymbol{E}$ from $\boldsymbol{M(E,}1)$. The natural map from $\mathbf{E}$ to $t_{1|}\boldsymbol{E}$ induces an isomorphism between $\boldsymbol{M(E,}1)$ and $\boldsymbol{M}\ (t_{1|}\boldsymbol{E,}1)$. Thus we have proved the following lemma:

Lemma 4.1. *On $T_{n|}$, $\boldsymbol{M}\ -,1)$ is an embedding.* □

We next recall (Lemma 1.2) that

$$E_n \cong K_{n-1} \rtimes s_{n-1}^{n-1}(E_{n-1})$$

for all $n \geq 1$ where $K_n =$ Ker d_{n+1}^{n+1}. We use this decomposition in two ways:

(i) for any $k \geq 0, n \geq 1$, we note that

$$(t_{k|}E)_n \cong (t_{k-1|}E)_{n-1} \rtimes s_{n-1}^{n-1}(t_{k|}E)$$

and

(ii) writing M_0 (resp. M_1) for the $(n-1)$-cube of those M_A with $n \notin A$ (resp. $n \in A$), we have

$$\boldsymbol{M(E,}n)_1 \cong \boldsymbol{M(K,}n-1).$$

It should now be clear how to proceed with an inductive proof. Using the inductive hypothesis, we can reconstruct from $\boldsymbol{M(E,}n)$, simplicial algebras $t_{n-1|}\boldsymbol{K}$ and $t_{n-1|}\boldsymbol{E}$. The differences between $t_{n-1|}\boldsymbol{E}$ and $t_{n|}\boldsymbol{E}$ occur only in dimensions $n-1$ and n. We write

$$d_i' : (t_{n-1|}E)_{n-1} \to E_{n-2} \qquad 0 \leq i \leq n-1$$

and note the following.

Lemma 4.2. *There is an epimorphism*

$$p : E_{n-1} \to (t_{n-1|}E)_{n-1}$$

such that p(Ker d_i^{n-1}) = Ker d_i' for each i.

In fact $p(e) = e + d_n(NE_n)$ will do. This observation allows us to use $\boldsymbol{M(E,}\ n)_{\emptyset}$ (which is E_{n-1}) and to link it to $t_{n|}\boldsymbol{E}$ so as to reconstruct the $(n-1)$-skeleton of $\mathbf{E}$. The analogous statement for a crossed n-cube not of the form $\boldsymbol{M(E,}n)$ for some $\mathbf{E}$ would seem to be false. (So, it is here that we are using properties of "special" crossed n-cubes, where by *special crossed n-cubes* we merely mean one isomorphic to some $\boldsymbol{M(E,}n)$.)

Thus we can reconstruct $\text{sk}_{n-1}E(\cong \text{sk}_{n-1}(t_{n|}\boldsymbol{E}))$ up to isomorphism. The action of $E_{n-1}(= \boldsymbol{M(E},n)_{\emptyset})$ on $\boldsymbol{M(K,}n-1)$ is given in the data specifying $\boldsymbol{M(E},n)$, so we can inductively reconstruct the semidirect product decomposition of (i) above.

If $f : \boldsymbol{M(E},n) \to \boldsymbol{M(F},n)$ is a map of crossed n-cubes, it is possible to pick the epimorphisms of the previous lemma for **E** and **F** to be compatible with f.

We thus have proved the following proposition.

Proposition 4.3. *On $T_{n|}, \boldsymbol{M}(-,n)$ is an embedding. There is a functor $\boldsymbol{L}$ defined on the full image of functor $\boldsymbol{M}(-,n)$ such that $\boldsymbol{LM(E},n)$ is the truncation functor $t_{n|}$ whilst $\boldsymbol{M(L}(\ ),n)$ is naturally isomorphic to the identity functor on the full subcategory of "special" crossed n-cubes.*

We note that $T_{n|}$ contains algebraic models for all n-types of simplicial algebras.

In [3], we give an analogue of Loday's theorem for simplicial algebras that the category of $(n+1)$-types is equivalent to a quotient category of cat^n-algebras or crossed n-cubes.

References

[1] M. André, Homologie des algébres commutatives, *Springer-Verlag,* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band **206** (1974).

[2] M. Artin and B. Mazur, Etale homotopy, *Lecture Notes in Maths,* **100**, Springer Verlag, (1968).

[3] Z. Arvasi, M. Koçak, and M. Alp, A combinatorial definition of n-types of simplicial commutative algebras, *Turkish Journal of Mathematics,* (to appear).

[4] Z. Arvasi and T. Porter, Simplicial and crossed resolutions of commutative algebras, *J. Algebra,* **181** (1996) 426-448.

[5] Z. Arvasi and T. Porter, Higher dimensional Peiffer elements in simplicial commutative algebras, *Theory and Applications of Categories,* **3,** No: 1 (1997) 1-23.

[6] Z. Arvasi and T. Porter, Freeness conditions of 2-crossed modules of commutative algebras, *Applied Categorical Structures,* Kluwer Ac.Pub. (to appear).

[7] N. Ashley, Simplicial T-complexes and crossed complexes: a non-Abelian version of a Theorem of Dold and Kan, *Diss. Math.* **265** (1988), 1-58.

[8] R. Brown and J-L. Loday, Van Kampen theorems for diagrams of spaces, *Topology* **26** (1987), 311-335.

[9] P. Carrasco, Complejos Hipercruzados Cohomologia y Extensiones, *Ph.D. Thesis,* Universidad de Granada, (1987).

[10] P. Carrosco and A.M. Cegarra, Group-theoretic algebraic models for homotopy types, *J. Pure Appl. Algebra,* **75** (1991), 195-235.

[11] D. Conduché, Modules croisés généralisés de longueur 2, *J. Pure Appl. Algebra,* **34** (1984), 155-178.

[12] J. Duskin, Simplicial methods and the interpretation of triple cohomology. *Mem. Amer. Math. Soc.,* **163** (1970) 117-136.

[13] G.J. Ellis, Higher dimensional crossed modules of algebras, J. Pure Appl. Algebra. 52 (1988) 277-282.

[14] G.J. Ellis and R. Steiner, Higher dimensional crossed modules and the homotopy groups of $n+1$-ads. *J. Pure Appl. Algebra,* **46** (1987) 117-136.

[15] R.H. Fox, On the Lusternik-Schnirelmann category, *Ann. of Math.,* **42,** (1941) 333-370.

[16] D.M. Kan, A combinatorial definition of homotopy groups. *Ann. of Math.,* **61** (1958) 288-312.

[17] D.M. Kan, Functors involving c.s.s. complexes. *Trans. Amer. Math. Soc.,* **87** (1958) 330-346.

[18] L. Illusie, Complex cotangent et deformations I, II. *Springer Lecture Notes in Math.,* **239** (1971) II **283** (1972).

[19] J.L. Loday, Spaces with finitely many non-trivial homotopy groups. *J. Pure Appl. Algebra,* **24** (1982) 179-202.

[20] S. MacLane and J.H.C. Whitehead, On the 3-type of a complex. *Proc. Nat. Acad. Sci. Washington,* **36** (1950) 41-48.

[21] T. Porter, Homology of commutative algebras and an invariant of Simis and Vasconceles, *J. Algebra,* **99** (1986), 458-465.

[22] T. Porter, Some categorial results in the theory of crossed modules in commutative algebras, *J. Algebra,* **109** (1987), 415-429.

[23] T. Porter, n-types of simplicial groups and crossed n-cubes, *Topology,* **32** (1993), 5-24.

[24] J.H.C. Whitehead, Combinatorial homotopy II, *Bull. Amer. Math. Soc.,* **55** (1949) 213-245.

[25] J.H.C. Whitehead, Algebraic homotopy theory, *Proc. Int. Cong. Math.,* (1950) 354-357.

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