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ON CERTAIN VARIETIES OF SEMIGROUPS

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Abstract

In this paper we generalize the class of completely regular semigroups (unions of groups) to the class of local monoids, that is the class of all semigroups where the local subsemigroups aSa are local submonoids. The sublattice of this variety $(\mathbf{L}(\mathcal{L}(\mathcal{M}))$ covers another lattice isomorphic to the lattice of all bands $([x^2 = x])$. Every bundvariety $\mathcal{U}$ has as image the variety $\Phi - \mathcal{U}$, which is the class of all semigroups, where Φ is a $\mathcal{U}$-congruence $(a\Phi b \Leftrightarrow aSa = bSb)$. It is shown how one can find the laws for $\Phi - \mathcal{U}$ for a given bandvariety $\mathcal{U}$. The laws for $\Phi - \mathcal{B}$ are given and it is shown that $\Phi - \mathcal{RB} - \mathcal{L}(\mathcal{G})\mathcal{L}(\mathcal{V}) := \{S : aSa \in \mathcal{V}\forall a \in S\})$.

1. Preliminary

In 1940 Clifford ([3]) generalized the concept of groups by introducing relative inverses. This class of semigroups became one of the most studied class of semigroups. Semigroups which admit relative inverses are a union of groups and they are also known to be completely regular. Regularity was introduced by von Neumann in ringtheory. A ring is regular if the multiplication is such that one can find for every element a and element x with $a = axa$. A semigroup is regular if all its elements are regular. If in addition, there is an element x which commutes with a, that is $a = axa$ and $ax = xa$, then the element is completely regular and if all elements are completely regular, then the semigroup is called completely regular.

Note that there are many examples of completely regular semigroups. Of course, groups are completely regular as are idempotent semigroups, that is all semigroups with $a^2 = a$ for every element a (lattices, semilattices, rectangular bands,...). The latter are called bands and will play an important role in this paper.

Since 1935, when Birkhoff ([1]) proved his famous result, namely that varieties are equational classes, there has been the aim to find equations for a given veriety of algebras and to show the relation between varieties by ordering them. This leads to sublattices of varieties of a given variety.

It is clear, that some subclasses of semigroups are not varieties. For example, the class of groups is not a variety of semigroups, because it is not closed under taking subsemigroups (e.g. $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$ but not a group). Therefore a unary operation (inversion) was introduced and the class of all completely regular semigroups is described as a variety of type 2-1. This variety is denoted by $\mathcal{CR}$ and its lattice of subvarieties is denoted by $\mathbf{L}(\mathcal{CR})$.

If a semigroup is complete, then of course every element lies in a subgroup of the given semigroup. This property has been generalized to groupbond, which means that the power of each element is in a subgroup. Here, we introduce a concept somewhat in between. Using the fact, that for every element a there is an idemponent a^0 (the identity of the maximal subgroup a lies in), with $aa^0 = a^0a = a$, it is easy to see, that the subsemigroups of the form aSa are monoids with a^0 as its identity. In [4] I showed that the class of all semigroups with this property (aSa being a monoid for every a) is a variety of algebras of type 2-1. (For general information about universal algebra see [2].) Now we pull our attention to this class and we will describe a sublattice of subvarieties of this variety.

The class of all monoids is denoted by $\mathcal{M}$, that of all groups by $\mathcal{G}$. $\mathcal{B}$ denotes the class of bands, $\mathcal{RB}$ all rectangular bands.

If ρ is a congruence of a semigroup S then we say that ρ is a $\mathcal{V}$-congruence if the factorsemigroup S/ρ is in $\mathcal{V}$.

For a given variety $\mathcal{V}\,\mathcal{L}(\mathcal{V})$ is the class of all semigroups such taht the local subsemigroups aSa are in $\mathcal{V}$ and $\mathcal{U}\,\mathcal{L}(\mathcal{V})$ denotes the class of semigroup S $\bigcup_{a\in S} aSa$ is in $\mathcal{V}$.

On any semigroup S there is an equivalence relation Φ given by

$$a\Phi b \Leftrightarrow aSa = bSb.$$

Not, that Φ is not a congruence in general.

In the following 'RESULT' marks an outcome which is found in [?].

Semigroups in which Φ is a Bandcongruence

As we mentioned above, Φ is not a congruence in general. But there are semigroups where this is the case. By [4] Φ seperates idemponents, teherfore Φ is $\epsilon(a\epsilon b \Leftrightarrow a = b)$ on bands and hence a congruence. If we demand Φ to be a bandcongruence (or $\mathcal{B}$-congruence) on a semigroup S, then a and a^2 lie in the same Φ-class and so by [?] (Fundamental lemma) all local subsemigroups aSa are monoids. As a conclusion we find that all semigroups in which Φ is a $\mathcal{B}$-congruence are members of $\mathcal{L}(\mathcal{M})$. Therefore the following laws hold in such semigroups:

RESULT 1 *A semigroup S is a member of $\mathcal{L}(\mathcal{M})$ if and only if there is an unary operation '*' on S, such that the following laws hold in S:*

$$x^* = x^*xx^* \tag{1}$$

$$xx^* = x^*x (=: x^0) \tag{2}$$

$$xyx = x^0xyxx^0 \tag{3}$$

The following lemma shows a property of Φ, which holds in semigroups in $\mathcal{L}(\mathcal{M})$.

LEMMA 1 *Let S be a member of $\mathcal{L}(\mathcal{M})$. Then a, a^* and a° are all in one Φ class.*

Proof. Let $u \in aSa$. Therefore we find an element s, with $u = asa$. By (3) $u = asa = a^0asaa^0 \in a^0Sa^0$. If $u \in a^0Sa^0$, then we have $u \in aa^*Sa^0a \subset aSa$. Because of (1) and (2) we have $a^*Sa^* = a^0Sa^0$, which completes the proof. □

Before we state the theorem, which describes the variety of semigroups with Φ as bandcongruence ($\Phi - \mathcal{B}$), we will look for another classs. In this class Φ is the trivial congruence, that is Φ is a $\mathcal{T}$-congruence, where $\mathcal{T} = [x = y]$. Of course this is the case if $\Phi = (a\omega b \forall a, b \in S)$. We have the following results:

RESULT 2 *For a semigroup S the following conditions are equivalent:*

1. $\Phi = \omega$
2. there is an unary operation '*' on S such that
   (a) $x^* = x^* x x^*$
   (b) $xx^* = x^* x (=: x^0)$
   (c) $(xyx)^{**} = xyx$
   (d) $x^0 = y^0$

Now we can reformulate this result. If we denote the class of all semigroups with Φ as $\mathcal{U}$-congruence by $\Phi - \mathcal{U}$, then we have:

COROLLARY 1 $\mathcal{UL}(\mathcal{G}) = \Phi - \mathcal{T}$

Now we return to the main question, namly to describe $\Phi - \mathcal{B}$. We know, that this class is a subclass of $\mathcal{L}(\mathcal{M})$. We have the following theorem:

THEOREM 1 *A semigroup S is a member of $\Phi - \mathcal{B}$ if and only if there is an unary operation '*' on S, such that*

1. $S \in \mathcal{L}(\mathcal{M})$
2. $(a^0 b^0) = (ab)^0$ *holds is S.*

Proof. [$\Rightarrow$] We saw, that a semigroup in $\Phi - \mathcal{B}$ must be a member of $\mathcal{L}(\mathcal{M})$. So it remains to show that law (2) holds in S. By lemma 1 we know, that $a \Phi a^* \Phi a^0$ holds. Therefore we have:

$$(ab)^0 \Phi ab \Phi a^0 b^0 \Phi (a^0 b^0)^0.$$

Here, we also make use of the condition that Φ is a congruence ($a \Phi a^0, b \Phi b^0 \Rightarrow ab\Phi, a^0 b^0$). But the elements $(ab)^0$ and $(a^0 b^0)^0$ are idempotent elements; therefore they are equal.

[$\Leftarrow$] If a semigroup S is a member of $\mathcal{L}(\mathcal{M})$, then of course all local subsemigroups are monoids and therefore we have $a \Phi a^2$. We have to show, that under the assumption of (2) Φ is a congruence. To do this we choose $a, b \in S$ with $a \Phi b$. Note that $a^0 = b^0$. (Using lemma 1 again we have $a^0 \Phi a \Phi b \Phi b^0$ and Φ seperates idempotents.) Let c be an arbitrary elements of S, then we compute:

$$\begin{aligned} acSac &= (ac)^0 \cdot acSac \cdot (ac)^0 = \\ &\overset{(2)}{=} (a^0c^0)^0 \cdot acSac \cdot (a^0c^0)^0 = \\ &\overset{a\Phi b)}{=} (b^0c^0)^0 \cdot acSac \cdot (b^0c^0)^0 = \\ &\overset{(2)}{=} (bc)^0 \cdot acSac \cdot (bc)^0 \subset bcSbc. \end{aligned}$$

If we replace a by b and b by a, we find $bcSbc \subset acSac$. In a similar way, we find $caSca = cbScb$. Therefore Φ is a congruence and together with $a \, \Phi \, a^2 \forall a \in S$ we conclude Φ is a band congruence. □

3. A copy of $\mathcal{L}(\mathcal{M})$

mmIt is clear that $\mathbf{L}(\mathcal{B})$ is a sublattice $\mathbf{L}(\mathcal{L}(\mathcal{M}))$. We will show that there is a second sublattice isomorphic to $\mathbf{L}(\mathcal{B})$. We already know two members of the sublattice: $\Phi - \mathcal{T}$ the smallest and $\Phi - \mathcal{B}$. the greatest element. Note that $\Phi - \mathcal{U}$ is allways larger than $\mathcal{U}$, if $\mathcal{U} \subset \mathcal{B}$. We will define an injective function from $\mathbf{L}(\mathcal{B})$ into $[\Phi - \mathcal{T}, \Phi - \mathcal{B}]$, the lattice intervall between $\Phi - \mathcal{T}$ and $\Phi - \mathcal{B}$.

If $\mathcal{V}$ is a variety, let $\Sigma(\mathcal{V})$ be the set of identities, which are valid in $\mathcal{V}$. Conversly, if Σ is a set of identities, let $[\Sigma]$ be the corresponding variety, that is the variety of all semigroups in which the given identities hold.

With the following theorem we solve two questions. First we introduce the injective function $\mathbf{L}(\mathcal{B})$ into $[\Phi - \mathcal{T}, \Phi - \mathcal{B}]$, which shows, that there is an isomorphic copy of $\mathbf{L}(\mathcal{B})$ in $\mathbf{L}(\mathcal{L}(\mathcal{M}))$, then we describe how laws for $\Phi - \mathcal{U}$ can be found for a given bandvariety $\mathcal{U}$.

THEOREM 2 *Let $\mathcal{U}$ be a bandvariety with $\Sigma(\mathcal{U})$ as set of identities. For every identity $u \approx v$ in $\Sigma(\mathcal{U})$ build $u^0 \approx v^0$ and denote by $\Sigma(\mathcal{U})^0$ the set of all such identities. Then*

$[\Sigma(\Phi - \mathcal{B}) \cup \Sigma(\mathcal{U})^0]$ *defines the variety* $\Phi - \mathcal{U}$*. Moreover,*

$$\Theta : \mathbf{L}(\mathcal{B}) \mapsto [\Phi - \mathcal{T}, \Phi - \mathcal{B}] : \mathcal{U}\Theta = [\Sigma(\Phi - \mathcal{B}) \cup \Sigma(\mathcal{U})^0]$$

defines an injective function of $\mathbf{L}(\mathcal{B})$ *into* $[\Phi - \mathcal{T}, \Phi - \mathcal{B}]$.

Proof. Let $\mathcal{U}$ be a variety of bands. S is a member of $\mathcal{U}\Theta$. Of course, Φ is a band congruence and the factor S/Φ satisfies the laws of $\mathcal{U}$, because $u\,\Phi\,u^0 \approx v^0\,\Phi\,v$. Therefore $\mathcal{U}\Phi \subset \Phi - \mathcal{U}$.

Let $u \approx v$ be in $\Sigma(\mathcal{U})$ and S a member of $\Phi - \mathcal{U}$. Then we have $u\,\Phi\,v$ and with lemma 1 we get $u^0\,\Phi u\,\Phi v\,\Phi\,v^0$. So we find $u^0 \approx v^0$ in $\Phi - \mathcal{U}$. That is, $\Phi - \mathcal{U} \subset \mathcal{U}\Theta$. We conclude, that $\Phi - \mathcal{U} = \mathcal{U}\Theta$.

If $\mathcal{U}\Theta = \mathcal{V}\Theta$, then we know that $\Phi - \mathcal{U} = \Phi - \mathcal{V}$. But then every band in $\mathcal{U}$ must also be in $\mathcal{V}$ and vice versa. So $u = \mathcal{V}$. This shows that Θ is injective. In addition, $(\mathcal{U} \vee \mathcal{V})\Theta = \mathcal{U}\Theta \wedge \mathcal{V}\Theta$ and $(\mathcal{U} \wedge \mathcal{V})\Theta = \mathcal{U}\Theta \wedge \mathcal{V}\Theta$. This completes the proof. □

Now we want to apply this result to a special class of bands, the rectangular bands. They are given by the identity $a \approx aba$. They are given by the identity $a \approx aba$. With Θ we get a set of identies for $\Phi - \mathcal{RB}$:

$$[\Phi - \mathcal{RB}] = \Sigma(\Phi - \mathcal{B}] \cup \{a^0 \approx (aba)^0\}.$$

It turns out that $\Phi - \mathcal{RB}$ can be described with other identities as well, which I found in [4] in a different context.

THEOREM 3 $\Phi - \mathcal{RB} = \mathcal{L}(\mathcal{G})$

Proof. In [4] we defined the laws for $\mathcal{L}(\mathcal{G})$ by

1. $x^* = x^*xx^*$
2. $xx^* = x^*x$
3. $(xyx)^{**} = xyx$
4. $x^0 = (xyx)^0$

so these sets of identities differ only in the laws $(xyx)^{**} = xyx$ and $xyx = x^0xyxx^0$.

We show first that $\mathcal{L}(\mathcal{G}) \subset \Phi - \mathcal{RB}$:

$$xyx \overset{(3)}{=} (xyx)^{**} \overset{(1)}{=} (xyx)^{**}(xyx)^*(xyx)^{**} \overset{(1)}{=} xyx(xyx)^*xyx$$

Therefore

$$xyx = (xyx)^0 xyx(xyx)^0 = x^0 xyxx^0$$

Note that $xyx(xyx)^* = (xyx)^* xyx = (xyx)^0$

Now we show that $\Phi - \mathcal{RB} \subset \mathcal{L}(\mathcal{G})$: New know that

$$((xyx)^*)^0 \Phi\, xyx\, \Phi (xyx)^0.$$

Therefore we get $((xyx)^*)^0 = (xyx)^0$. We conclude:

$$\begin{aligned}(xyx)^{**} &= (xyx)^{**}(xyx)^*(xyx)^{**} = \\ &= (xyx)^{**}(xyx)^* xyx(xyx)^*(xyx)^{**} = \\ &= ((xyx)^*)^0 xyx((xyx)^*)^0 = \\ &= (xyx)^0 xyx(xyx)^0 = x^0 xyxx^0 = \\ &= xyx\end{aligned}$$

This completes the proof. □

We finish with some remarks on the Φ-classes of semigroups in $\Phi - \mathcal{B}$.)

THEOREM 4 *Let S be a member of $\Phi - \mathcal{B}$. Then the*

$$G_a := [a]_\Phi \cap aSa$$

is a group and $G_a = a[a]_\Phi a$.

Proof. Let class $[a]_\Phi$ and aSa are subsemigroups. Their intersection is not emty because at least a^3 is in it. If b is in this, then b is regular, because $b \in aSa = bSb$. Moreover, we can find an element $x \in G_a$ with $b = bxb$. This is easy to see recalling $bSb = b^2Sb^2$. Hence G_a is a regular semigroup, which contains only one idempokut, since Φ seperates idempotents. Therefore G_a is a group.

To prove that $a[a]_\Phi a = G_a$ we have only to show, that G_a is a subset of $a[a]_\Phi a$ because the converse is obvious. Choose an element from G_a, say u. Since $u \in aSa$ too, we can multiply it with a^0 on both sides. We find

$$u = asa = a^0 asaa^0 = a \cdot a^0 sa^0 \cdot a.$$

The element $a^0 sa^0 is in aSa$ and

$$a\Phi u = asa\Phi a^0 sa^0.$$

Hence $G_a = a[a]_\Phi a$. □

COROLLARY 2 *If S is a member of $\Phi - \mathcal{B}$, then the Φ-classes are in $\Phi - \mathcal{T}$.*

Proof. We know that $a[a]_\Phi a = aSa \cap [a]_\Phi$. Therefore if $a\Phi b$ we have $aSa = bSb$ and $[a]_\Phi = [b]_\Phi$ and hence $a[a]_\Phi b$. □

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