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Abstract

In this paper, we will give a description of the functor from the category of crossed n-cubes to that of simplicial commutative algebras and study a commutative algebra version of Loday's theorem on n-types of simplicial groups.

Introduction

Simplicial commutative algebras are involved in homological algebra, homotopy theory, algebraic K-theory and algebraic geometry. In each theory, their own internal structure has been studied relatively little. The first author and T. Porter (cf. [5] and [4]) have recently worked on the n-types of simplicial algebras.

Combining earlier work [6] of the first author with our joint papers [7], one starts to see how a study of the links between simplicial commutative algebras and classical constructions of homological algebra can be strengthened by interposing crossed algebraic models for the homotopy types of simplicial algebras. In this paper, we continue this process using these methods to give a description of the functor from the category of crossed n-cubes to that of simplicial commutative algebras. The main result of this paper will be devoted to proving the following theorem:

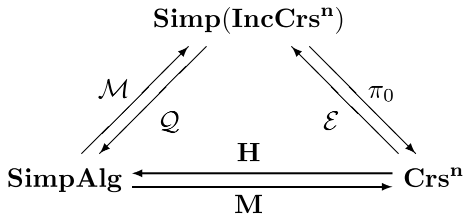
Theorem: The functor

$$\mathcal{M} : \mathbf{SimpAlg} \longrightarrow \mathbf{Simp(IncCrs^n)}$$

induces an equivalence of categories,

$$\mathrm{Ho}_n(\mathbf{SimpAlg}) \longrightarrow \mathrm{Ho}(\mathbf{Simp}(\mathbf{IncCrs^n})).$$

The situation can be summarised in the following diagram



together with the information:

(i) $T_{n]} \subset$**SimpAlg** is a reflexive subcategory with reflection $t_{n]}$ and $t_{n]}$ is an n-equivalence (Proposition 2.2 in [7]).

(ii) $\mathbf{M} = \pi_0 M$ by definition

(iii) $\mathcal{QM} \cong$Id (before Proposition 2.2)

(iv) There is a natural trivial fibration $\mathbf{HM} \to t_{n]}$.

(v) $\pi_0\mathcal{E} \cong$Id (Proposition 3.2)

(vi) $\mathcal{QE} \cong \mathbf{H}$ (Lemma 3.4).

(vii) There is a functor $\mathcal{H}$:$\mathbf{Simp(IncCrs}^n\mathbf{)} \to \mathbf{Simp(IncCrs}^n\mathbf{)}$ with natural transformations

$$\delta : \mathcal{H} \to \mathrm{Id}$$
$$\delta' : \mathcal{H} \to \mathcal{E}\pi_0$$

so that δ and δ' induce isomorphismss on π_0, (Proposition 4.3) thus $\mathcal{QHM}(E, n) \simeq E$ and $\mathcal{QHM}(\mathbf{E}, n) \simeq \mathbf{HM}(\mathbf{E}, n)$ if $E \in T_{n]}$.

The history of the interactions of algebraic topology and homological algebra indicates that with each significant new model for homotopy types, there should be a potential application in homological algebra. Crossed modules have occured many times in parts of algebra other than group theory, and their significance was always linked to non-Abelian aspects of the subject, giving finer detail than more usually used chain complexes. A problem in attempting a general non Abelian homological or homotopical algebraic version of Loday's classification theorem for n-types is his almost exlusive use of topological methods. T. Porter (cf. [22]) gave a reasonably self contained and algebraic proof of Loday's results proving that the category of $(n+1)$-types is equivalent to a quotient category of cat^n-groups or crossed n-cubes. The advantage of a purely algebraic proof is that it enables analogous results for simplicial commutative algebras. This may provide new methods in homological algebra.

1. Definitions and preliminaries

All algebras will be *commutative* and will be over the same fixed but unspecified ground ring.

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}$, $0 \le i \le n$, $(n \ne 0)$ and $s_i = s_i^n : E_n \to E_{n+1}$, $0 \le i \le n$, satisfying the usual simplicial identities given in André [1] or Illusie [18] for example. It can be completely described as a functor $\mathbf{E}\colon \Delta^{op} \to \mathbf{CommAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

Recall that given a simplicial algebra $\mathbf{E}$, *the Moore complex* $(\mathbf{NE}, \partial)$ *of* $\mathbf{E}$ is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

We say that the Moore complex $\mathbf{NE}$ of a simplicial algebra is of *length* k if $NE_n = 0$ for all $n \ge k+1$ so that a Moore complex is of length k also of length r for $r \ge k$.

The n^{th} *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the nth homology of the Moore complex of $\mathbf{E}$, i.e.,

$$\pi_n(\mathbf{E}) \quad \cong \quad H_n(\mathbf{NE}, \partial) = \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}\big(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}\big).$$

A simplicial map $f : \mathbf{E} \to \mathbf{E}'$ is called a n- *equivalence* if it induces isomorphisms

$$\pi_n(\mathbf{E}) \cong \pi_n(\mathbf{E}') \qquad \text{for } n \ge 0.$$

Two simplicial algebras $\mathbf{E}$ and $\mathbf{E}'$ are said to be have the same *n-type* if there is a chain of n-equivalences linking them. A simplicial commutative algebra $\mathbf{E}$ is an n-type if $\pi_i(\mathbf{E}) = 0$ for $i > n$.

We recall the classifying functor **B** constructed as follows:

Consider an algebra E as a category with one object. The nerve $\mathrm{Ner}(C)$ of a smal category, C, is the simplicial set consisting of all composable n-triples of arrows, so in this case $\mathrm{Ner}E$ is just E^n. the face and degeneracy maps are induced by composition and insertion of identities respectively. The classifying $\mathbf{B}E$ of E is a $K(E,0)$; namely the constant simplicial algebra of E generated by the vertices, i.e., $\pi_1(K(E,0)) = E$, $\pi_i(K(E,0)) = 0$ for $i \neq 1$.

The Semidirect Decomposition of a Simplicial Algebra

The fundamental idea behind this can be found in Conduché [12]. A detailed investigation of it for the case of a simplicial group is given in Carrasco and Cegarra [11]. The algebra case of that structure is also given in Carrasco's thesis [10].

Proposition 1.1 *If* $\mathbf{E}$ *is a simplicial algebra, then for any* $n \geq 0$

$$\begin{aligned} E_n \quad &\cong \quad (\ldots(NE_n \rtimes s_{n-1}NE_{n-1}) \rtimes \ldots \rtimes s_{n-2}\ldots s_0 NE_1) \rtimes \\ &\qquad (\ldots(s_{n-2}NE_{n-1} \rtimes s_{n-1}s_{n-2}NE_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0 NE_0). \end{aligned}$$

Proof: This is by repeated use of the following lemma. □

Lemma 1.2 *Let* $\mathbf{E}$ *be a simplicial algebra. Then* E_n *can be decomposed as a semidirect product:*

$$E_n \cong \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}).$$

Crossed Modules of Commutative Algebras

Recall from [20] the notion of a crossed module of commutative algebras. *A crossed module of commutative algebras*, (C, R, ∂), is an R-algebra C, together with an action of R on C and an R-algebra morphism $\partial : C \to R$, such that for all $c, c' \in C, r \in R$, i) $\partial(r \cdot c) = r\partial c$ and ii) $\partial c \cdot c' = cc'$. The second condition is called the *Peiffer identity.*

For example: if M is R-module, the trivial map $0 : M \to R$ that maps everting to 0 in R, is a crossed module. Conversely the kernel of any crossed module is R-module.

Truncations

By an *ideal chain complex* of algebras, (X, d) we mean one in which each $\mathrm{Im}\, d_{i+1}$ is an ideal of X_i. Given any ideal chain complex (X, d) of algebras and an integer n the truncation, $t_{n]}X$, of X at level n will be defined by

$$(t_{n]}X)_i = \begin{cases} X_i & \text{if } i < n \\ X_i/\mathrm{Im}d_{n+1} & \text{if } i = n \\ 0 & \text{if } i > n. \end{cases}$$

The differential d of $t_{n]}X$ is that of X for $i < n$, d_n is induced from the nth differential of X and all others are zero. (For more on information see Illusie [18]). Truncation is of course functorial.

The following results are due to [6].

Proposition 1.3 *There is a truncation functor* $t_{n]}$:**SimpAlg**→**SimpAlg** *such that there is a natural isomorphism*

$$t_{n]}N \cong Nt_{n]}$$

where **N** *is the Moore complex functor from* **SimpAlg** *to the category of chain complexes of algebras.*

1.1. Cat^n-algebras and crossed n-cubes

Ellis & Steiner (cf. [16]) have since shown that cat^n-groups are equivalent to crossed n-cubes. The other algebraic settings such as commutative algebras, Lie algebras, Jordan algebras of this construction are due to Ellis [15].

A cat^n-algebra $\mathcal{A}$ is an (commutative) algebra A together with $2n$ endomorphisms $s_i, t_i{:} A \to A$ $(1 \leq i \leq n)$ such that

$$\begin{array}{llll} t_i s_i = s_i & s_i t_i = t_i & & \\ s_i s_j = s_j s_i & t_i t_j = t_j t_i, & s_i t_j = t_j s_i & \text{for } i \neq j \\ aa' = 0 & \text{for } a \in \mathrm{Ker} s_i, \ a' \in \mathrm{Ker} t_i. & & \end{array}$$

A morphism of cat^n-algebras $\phi : \mathcal{A} \to \mathcal{A}'$ is an algebra homomorphism $\phi : A \to A'$ which preserve the s_i and t_i.

T. Porter (cf.[21]) shows that a cat^1-algebra is equivalent to a crossed module and also to an internal category within the category of algebras. In section 2 we will recall this equivalence.

A crossed n-cube of commutative algebras is a family of commutative algebras, M_A for $A \subseteq < n >= \{1, ..., n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in < n >$ and for $A, B \subseteq < n >$, functions

$$h : M_A \times M_B \longrightarrow M_{A \cup B}$$

such that for all $k \in \mathbf{k},\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in < n >$ and $A \subseteq B$

$$\begin{array}{lrcl}
1) & \mu_i a & = & a \quad \text{if } i \notin A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a,b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a,b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a,a') & = & aa' \\
6) & h(a,b) & = & h(b,a) \\
7) & h(a+a',b) & = & h(a,b) + h(a',b) \\
8) & h(a,b+b') & = & h(a,b) + h(a,b') \\
9) & k \cdot h(a,b) & = & h(k \cdot a, b) = h(a, k \cdot b) \\
10) & h(h(a,b),c) & = & h(a, h(b,c)) = h(b, h(b,c)).
\end{array}$$

A morphism of crossed n-cubes is defined in the obvious way: It is a family of commutative algebra homomorphisms, for $A \subseteq < n >,\quad f_A : M_A \longrightarrow M'_A$ commuting with the μ_i's and h's. We thus obtain a category of crossed n-cubes denoted by $\mathbf{Crs^n}$.

For example, crossed modules are 1-crossed cube and several examples of crossed n-cubes can be found in [6].

Lemma 1.4 *Let $\mathcal{M} = \{M_A : A \subseteq < n >, \{\mu_i\}, h\}$ be a crossed n-cubes of algebras and let $i \in < n >$. Let $\mathcal{M}_1$ denote the restriction of $\mathcal{M}$ to those A with $i \in A$ and $\mathcal{M}_0$, the restricition to those A with $i \notin A$. Then $\mathcal{M}_1$ and $\mathcal{M}_0$ are crossed $(n-1)$-cubes of algebras and $\mu_i : \mathcal{M}_1 \to \mathcal{M}_0$ is a morphism of crossed $(n-1)$-cubes of algebras.*

The proof is quite simple and so will be omitted.

By *an ideal* $(n+1)$*-ad* will be meant an algebra with n selected ideals (possibly with repeats), $(R; I_1, \ldots, I_n)$.

Let R be an algebra with ideals $I_1, \ldots, I_n$ of R. Let $M_A = \bigcap\{I_i : i \in A\}$ and $A \subseteq < n >$. If $i \in < n >$, then M_A is an ideal of $M_{A-\{i\}}$. Define $\mu_i : M_A \to M_{A-\{i\}}$ to be the inclusion. If $A, B \subseteq < n >$, then $M_{A\cup B} = M_A \cap M_B$, let $h : M_A \times M_B \to M_{A\cup B}$ given by the multiplication as $M_A M_B \subseteq M_A \cap M_B$, where $a \in M_A$, $b \in M_B$. Then $\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ is a crossed n-cube, called the *inclusion crossed n-cube* given by the ideal $(n+1)$-ad of commutative algebras $(R;\ I_1, \ldots, I_n)$.

The following result is due to [6].

Proposition 1.5 *Let* $(\mathbf{E};\ I_1, \ldots, I_n)$ *be a simplicial ideal* $(n+1)$*-ad of algebras and define for* $A \subseteq < n >$, $M_A = \pi_0(\cap_{i\in A} I_i)$ *with homomorphisms* $\mu_i : M_A \to M_{A-\{i\}}$ *and h-maps induced by the corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous example to each level. Then* $\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ *is a crossed n-cube.*

Up to isomorphism, all crossed n-cubes arise in this way. In fact any crossed n-cube can be realised (up to isomorphism) as a π_0 of a simplicial inclusion crossed n-cube coming from a simplicial ideal $(n+1)$-ad in which π_0 is a non-trivial homotopy module.

In [6], we proved that for a simplicial algebra **E,** there is a functor

$$\mathbf{M}(-,\mathbf{n}) : \mathbf{SimpAlg} \longrightarrow \mathbf{Crs^n},$$

which is given by $\pi_0(\mathrm{Dec}\mathbf{E}; \mathrm{Ker}\delta_0, \ldots, \mathrm{Ker}\delta_{n-1})$. Also it was shown that the following equality

$$\pi_0(\mathcal{M}(\mathbf{E}, n)) = \mathbf{M}(\mathbf{E}, n)$$

where

$$\mathbf{SimpAlg} \xrightarrow{\mathcal{M}} \mathbf{Simp(IncCrs^n)} \xrightarrow{\pi_0} \mathbf{Crs^n}$$

and the *décalage* functor forgets the last face operator at each level of a simplicial algebra $\mathbf{E}$ and moves everything down one level. It is denoted by Dec. Thus $(\mathrm{Dec}E)_n = E_{n+1}$. The construction of the above functors have been omitted as it was given in [6].

2. From $\mathbf{Crs}^n$ to SimpAlg

In this section our aim is to give a reasonably self contained and algebraic proof of Loday's result p roving that the category of $(n+1)$-types is equivalent to a quotient category of cat^n-algebras or crossed n-cubes. The advantage of a purely algebraic proof is that it enables analogous results for simplicial Lie algebras.

2.1. The diagonal of the multinerve

Loday's original idea in [19] of constructing a functor from spaces to cat^n-groups failed to work for technical reasons.

An important part is played in Loday's theory by a classifying space functor $\mathbf{B}$ from cat^n-groups to spaces. As we are using algebraic methods rather than topological ones, the role played by this functor $\mathbf{B}$ has to be filled by an algebraic analogue. We start as usual in low dimension for simplicial algebras.

One of the tools needed will be the multinerve of a crossed n-cube. The idea is to use the construction $\mathbf{H}(\mathcal{M})$ in the n-independent directions of the crossed n-cube thus giving us a n-simplicial algebra. We will also recall facts in this paper when they are needed.

Let $\mathcal{M} = (\partial : C \to R)$ be a crossed module (i.e. crossed 1-cube) the corresponding cat^1-alg is as follows:

We form the k-algebra $S = C \rtimes R$, the semidirect product algebra with multiplication

$$(c,r)(c',r') = (rc' + r'c + c'c, rr').$$

There are two morphisms

$$S \underset{d_1}{\overset{d_0}{\rightrightarrows}} R$$

given $d_0(c,r) = r$ and $d_1(c,r) = r + \partial c$ (cf. T. Porter [20]). Also an obvious morphism $s : R \to S$, $s(r) = (0,r)$. We think of S as being the arrows or morphism of the category, R the objects and d_0 and d_1 domain and codomain maps, then s assigns the identity map to each object. To make this into an internal category, we need to define a composition,

$\circ$, on pairs $((c,r),(c',r'))$ where $d_1(c,r) = d_0(c',r')$. This is done by setting

$$(c,r) \circ (c', r + \partial c) = (c + c', r).$$

The order of composition is illustrated by

$$r \xrightarrow{(c,r)} r + \partial c \xrightarrow{(c', r+\partial c)} r + \partial(c + c').$$

All of these structure maps are k-algebra morphisms so we have constructed an internal category in $\mathbf{Alg}_k$, the category of k-algebras. We can form the nerve of this category in the usual way. Its collection of vertices is the algebra of objects of the category, namely the algebra R, and its n-simplicies are n-tuples of composable arrows. It is easily checked that all the face and degeneracy maps are algebra homomorphisms and that this nerve is a simplicial algebra. (In other words, since **Alg** has finite limits, given any internal category in **Alg**, one can form its nerve by working the whole time within **Alg**. The result then naturally is a simplicial object in **Alg**). We denote this simplicial algebra $\mathbf{H}(\mathcal{M}) = (H_n, d_i, s_j)$. In terms of the initially given crossed module $\mathcal{M} = (C, R, \partial)$, its structure is given by

$$\begin{array}{ll} H_n = C \rtimes (C \rtimes (\ldots(C \rtimes R)\ldots)) & n \text{ copies of } C \\ d_0(c_n,\ldots,c_1,r) = (c_n,\ldots,c_2, r + \partial c_1) & \\ d_i(c_n,\ldots,c_1,r) = (c_n,\ldots,c_{i+1}, c_i,\ldots,c_1,r) & \text{if } 0 < i < n \\ d_n(c_n,\ldots,c_1,r) = (c_{n-1},\ldots,c_1,r) & \\ s_j(c_{n-1},\ldots,c_1,r) = (c_{n-1},\ldots,0,\ldots,c_1,r) & 0 \le j \le n-1, \end{array}$$

where the identity element of $\mathcal{M}$ is inserted in the $(j+1)$st position.

If $\mathcal{M}$ is a crossed n-cube, one can use the n-independent category structures of its associated cat^n-algebra to obtain a n-simplicial algebra; to obtain a simplicial algebra we use the usual process, namely the diagonal functor. We write $H^{(n)}$ (or often just $\mathbf{H}$) for the resulting functor from $\mathbf{Crs}^n$ to **SimpAlg.**

To provide some insight into the behaviour of this nerve functor, we will look at some simple examples. In each case $\mathcal{M} = (\partial : C \to R)$ is a crossed module.

(i) If $C = 0$, so that $\mathcal{M}$ is merely the algebra R, then the associated internal category is discrete and $\mathbf{H}(\mathcal{M})$ is the constant simplicial algebra, $K(R, 0)$, having R in all dimensions with all face and degeneracy maps being the identity morphism on R.

(ii) If $R = 0$, then $\mathcal{M}$ is the module C and $\mathbf{H}(\mathcal{M})$ is given in dimension n by an n-fold sum of copies of C. An inspection of the Moore complex shows that $\mathbf{H}(\mathcal{M})$ is a $K(C, 0)$, i.e. one has $\pi_1(\mathbf{H}(\mathcal{M})) \cong C$ and $\pi_1(\mathbf{H}(\mathcal{M})) = 0$ for $i \neq 1$.

Before looking at other examples we note that if

$$0 \to \mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to 0$$

is an exact sequence of crossed n-cubes, then

$$0 \to H^{(n)}(\mathcal{M}_0) \to H^{(n)}(\mathcal{M}_1) \to H^{(n)}(\mathcal{M}_2) \to 0$$

is an exact sequence of simplicial algebras. (As simplicial fibrations of simplicial algebras are precisely epimorphisms, short exact sequences and fibration sequences are just two views of the same thing.)

(iii) If $\partial : C \to R$ is the inclusion of an ideal then as

$$\begin{array}{ccccc} C & \xrightarrow{=} & C & \longrightarrow & 0 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow \\ C & \longrightarrow & R & \longrightarrow & R/C \end{array}$$

is an exact sequence, we find $\mathbf{H}(\mathcal{M})$ projects down onto a $K(R/C, 0)$ with fibre $\mathbf{H}(C, C, =)$. This latter simplicial algebra has an "extra degeneracy" given by

$$s_{-1}(c_{n-1}, \ldots, c_1, c_0) = (c_{n-1}, \ldots, c_1, c_0, 0)$$

which acts as a contraction. Thus $\mathbf{H}(\mathcal{M})$ is naturally homotopy equivalent to $K(R/C, 0)$, (i.e. $\mathbf{H}(\mathcal{M}) \to K(R/C, 0)$ is a trivial fibration).

(This example is perhaps prettiest when ∂ is a split inclusion, i.e. $R \cong C \rtimes Q$ where $Q = R/C$. In this case it is a simple matter to give a simplicial algebra homomorphism from $K(Q, 0)$ to $\mathbf{H}(\mathcal{M})$ which is a natural homotopy inverse to the quotient map (trivial fibration) going in the other direction. In general one should not expect the homotopy inverse to be a simplicial algebra morphism.)

Using these examples, we can gain information on the homotopy type of $\mathbf{H}(\mathcal{M})$ in general. We use the short exact sequence

$$\begin{array}{ccccc} \mathrm{Ker}\partial & \longrightarrow & C & \longrightarrow & \mathrm{Im}\partial \\ \big\downarrow & & \big\downarrow{\scriptstyle \partial} & & \big\downarrow \\ 0 & \longrightarrow & R & \longrightarrow & R \end{array}$$

which gives the short exact sequence

$$\mathbf{H}(\mathcal{M}_0) \to \mathbf{H}(\mathcal{M}) \to \mathbf{H}(\mathcal{M}_2)$$

where $\mathcal{M}_0 = (\mathrm{Ker}\,\partial \to 0\,)$, $\mathcal{M}_2 = (\mathrm{Im}\,\partial \to R\,)$. Our calculations tell us that $\mathbf{H}(\mathcal{M}_0) \cong K(\mathrm{Ker}\,\partial, 0\,)$ whilst $\mathbf{H}(\mathcal{M}_2) \simeq K(R/\mathrm{Im}\,\partial, 0\,)$. Both the homotopy equivalences and the isomorphisms are natural.

There are analogues for commutative algebras of examples given by Loday. To gain some insight into $H^{(2)}$, we look at crossed square

$$\mathcal{M} = \left(\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & R \end{array} \right).$$

Using the nerve functor in the two directions gives a bisimplicial algebra $(H_{p,q})$ with

$$H_{p,q} \cong (L^p \rtimes N)^q \rtimes (M^p \rtimes R)$$

where we have written L^p for $L \rtimes L \rtimes \cdots \rtimes L$ p-times, etc., the actions being via that of R on L.

This bisimplicial algebra has $H^{(2)}(\mathcal{M})$ as its diagonal. One could thus use spectral sequence techniques to investigate the homotopy modules of $H^{(2)}(\mathcal{M})$. These methods, however, are not really fine enough here and the fibration sequence/exact sequence methods that we have seen in the one dimensional case will turn out to be more useful.

We now have functors

$$\mathbf{M}(-,n) : \mathbf{SimpAlg} \to \mathbf{Crs^n} \quad \text{and} \quad \mathbf{H} : \mathbf{Crs^n} \to \mathbf{SimpAlg}.$$

We clearly would hope that these were quasi-inverse to each other or at least were so "up to n-type". The following proposition does half of this; we will have occasion to prove this result twice and also to look at a strengthened version. The second proof gives more information, but the first one is fairly direct and helps with the understanding of what **HM** looks like. (The opposite comparison will take a lot more work.)

Proposition 2.1 (First version) *If* $\mathbf{E}$ *is any simplicial algebra and any* $n \geq 0$, *then there is an epimorphism*

$$\mathbf{HM}(\mathbf{E}, n) \longrightarrow t_{n]}\mathbf{E}$$

with contractible kernel.

Proof: For $n = 0$ this is trivial, $\mathbf{HM}(\mathbf{E}, 0) \cong \pi_0(\mathbf{E}) \cong t_{0]}\mathbf{E}$.

For $n = 1$, there is again no problem

$$\mathbf{M}(\mathbf{E}, 1) = \left(\frac{\mathrm{Ker} d_0^1}{\mathrm{Ker} d_0^1 \mathrm{Ker} d_1^1} \xrightarrow{d_1^1} E_0 \right)$$

whilst $(t_{1]}\mathbf{E})_1 \cong \frac{NE_1}{d_2 NE_2} \rtimes s_0(E_0)$. (Fore More details see [4]). Thus in this case there is an isomorphism from $\mathbf{HM}(\mathbf{E}, 1)$ to $t_{1]}\mathbf{E}$, since the evident isomorphism in dimension 0 and 1 extends via the degeneracies to higher order. □

For future use we note that $\mathbf{M}(\mathbf{E}, 1)$ can be split up, as in Lemma 1.4, as a crossed module of "crossed 0-cubes", namely

$$\mu_1 : \mathbf{M}(t_{0]}K, 0) \longrightarrow \mathbf{M}(t_{0]}\mathrm{Dec}\mathbf{E}, 0).$$

Taking $H^{(0)}$ gives us back our crossed module. This is, in fact, just another way of noting that $\mathbf{M}(\mathbf{E}, 1)$ or $\mathbf{M}(\mathbf{E}, n)$ in general is $\pi_0\, \mathbf{M}(\mathbf{E}, n)$. This will be needed several times in what follows, seen each time from a slightly different viewpoint. Here we need the general case

$$\mu_n : \mathbf{M}(K, n-1) \longrightarrow \mathbf{M}(\mathrm{Dec}\mathbf{E}, n-1) \qquad (1)$$

The inductive hypothesis will therefore be the existence of a natural epimorphism

$$H^{(n-1)}\mathbf{M}(\mathbf{E}, n-1) \longrightarrow t_{n-1]}\mathbf{E}$$

having a contractible kernel and which is compatible with crossed module structures. Again for $n = 1$ this is trivial.

Applying $H^{(n-1)}$ to the situation in (1) gives us the diagram

$$\begin{array}{ccc} H^{(n-1)}\mathbf{M}(K, n-1) & \xrightarrow{\simeq} & t_{n-1]}K \\ \downarrow & & \downarrow \\ H^{(n-1)}M(\mathrm{Dec}\mathbf{E}, n-1) & \xrightarrow[\simeq]{} & t_{n-1]}\mathrm{Dec}\mathbf{E} \end{array}$$

in which the horizontal maps are level wise trivial fibrations (epimorphisms that are homotopy equivalence) whilst the vertical ones level wise crossed modules.

Next take the nerve of these crossed module structures. This gives a map of bisimplicial algebras. On the left the diagonal gives us back $E^{(n)}\mathbf{M}(\mathbf{E}, n)$. On the right the bisimplicial algebra one gets is as follows (in low dimensions)

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ K_2 \rtimes K_2 \rtimes E_3 & \rightrightarrows & K_2 \rtimes E_3 & \rightrightarrows & E_3 \\ \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\ K_1 \rtimes K_1 \rtimes E_2 & \rightrightarrows & K_1 \rtimes E_2 & \rightrightarrows & E_2 \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ K_1 \rtimes K_1 \rtimes E_1 & \rightrightarrows & K_0 \rtimes E_1 & \rightrightarrows & E_1 \end{array}$$

The decomposition $E_n \cong K_{n-1} \rtimes E_{n-1}$ may be substituted into this and on taking the diagonal of the result, one obtains

$$\cdots K_2 \rtimes K_2 \rtimes K_2 \rtimes E_2 \rightrightarrows K_1 \rtimes K_1 \rtimes E_1 \rightrightarrows K_0 \rtimes E_0$$

(again we only give this in low dimensions). It is easy to give explicit descriptions of all the d_i and s_j and to find a split epimorphism from this simplicial algebra to $t_{n]}\mathbf{E}$ (which we recall is isomorphic to $t_{n-1]}K \rtimes t_{n-1]}\,\mathrm{Dec}\,\mathbf{E}$). The kernel of this epimorphism is the subsimplicial algebra that in dimension n consists of all $(k_n, \ldots, k_0, 0)$ with all

$k_i \in K_n$. The "extra degeneracy" $s_{-1}(k_n, \ldots, k_0, 0) = (s_0 k_n, \ldots, s_0 k_0, 0, 0)$ acts as a natural contraction on this kernel.

Combining this epimorphism with the one induced from the inductive hypothesis completes the proof except to note that it clearly respects crossed module structures.

There is a similar result for the functor

$$\mathcal{M}_n : \mathbf{SimpAlg} \longrightarrow \mathbf{Simp(IncCrs^n)}.$$

In fact we noted in [6] that there is a functor

$$\mathcal{Q} : \mathbf{Simp(IncCrs^n)} \longrightarrow \mathbf{SimpAlg}$$

such that $\mathcal{Q}\mathcal{M}_n \cong \mathrm{Id}$. Using $H^{(n)} : \mathbf{Crs^n} \to \mathbf{SimpAlg}$ yields (by simplicial extension dimension) wise followed by restriction)

$$sH^{(n)} : \mathbf{Simp(IncCrs^n)} \longrightarrow \mathbf{BiSimpAlg}$$

and hence a functor $\mathcal{D}^{(n)} = \mathrm{diag}\, sH^{(n)}$ on composing with the diagonal functor from bisimplicial algebras to simplicial algebras (cf. [2]). The strengthened version of the following proposition compares $\mathcal{D}^{(n)}$ with $\mathcal{Q}$.

Proposition 2.2 *There is a natural epimorphism*

$$\mathcal{D}^{(n)} \longrightarrow \mathcal{Q}$$

such that for any simplicial inclusion of crossed n-cube $\mathcal{M}$,

$$\mathcal{D}^{(n)}\mathcal{M} \longrightarrow \mathcal{Q}\mathcal{M}$$

has a contractible kernel. In particular, $\mathcal{D}^{(n)}\mathcal{M}_n(\mathbf{E})$ is homotopy equivalent to $\mathbf{E}$ for any simplicial algebra $\mathbf{E}$.

Proof: For $n = 0$, there is nothing to prove. For $n = 1$, the main point to note is that we have already proved that if C is an ideal of R with $\partial : C \to R$ the inclusion $\mathbf{H}(C, R, \partial)$ fits into the exact sequence

$$\mathbf{H}(C, C, =) \longrightarrow \mathbf{H}(C, R, \partial) \xrightarrow{\varphi} \mathbf{H}(0 \to R/C).$$

So φ is an epimorphism with contractible kernel. Applying this to a simplicial inclusion crossed module $\mathcal{M}$ gives an epimorphism of bisimplicial algebras

$$\mathbf{H}(\mathcal{M}) \longrightarrow \mathbf{H}(0 \to \mathcal{Q}(C)).$$

As $\mathbf{H}(0 \to R/C)$ is a constant simplicial algebra with value the algebra R/C, $\mathcal{D}^{(n)}(0 \to \mathcal{Q}(C)) \cong \mathcal{Q}(C)$. The kernel of φ is constractible in one of its two directions and, therefore, yields a contractible simplicial algebra on applying "diag". The result for $n = 1$ follows.

As for weaker version of this result, we next suppose the result for $n-1$. If $\mathcal{M}$ is a simplicial inclusion crossed n-cube, then considering the n^{th} direction crossed module

$$\mu_n : \mathcal{M}_1 \longrightarrow \mathcal{M}_0$$

we can form up an exact sequence as before

$$\begin{array}{ccccc}
\mathcal{M}_1 & \xrightarrow{=} & \mathcal{M}_1 & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \mu_n} & & \downarrow \\
\mathcal{M}_1 & \xrightarrow{\mu_n} & \mathcal{M}_0 & \longrightarrow & \mathrm{Coker}\mu_n
\end{array}$$

As $H^{(n)}$ is exact, so is $\mathcal{D}^{(n)}$, so

$$\mathcal{D}^{(n)}(C) \to \mathcal{D}^{(n)}(0 \to \mathrm{Coker}\mu_n)$$

is an epimorphism with naturally contractible kernel. However

$$\mathcal{D}^{(n)}(0 \to \mathrm{Coker}\mu_n) \cong \mathcal{D}^{(n-1)}(\mathrm{Coker}\mu_n)$$

and $\mathrm{Coker}\,\mu_n$ is a simplicial inclusion crossed $(n-1)$-cube having $\mathcal{Q}(\mathcal{M})$ as its quotient. By hypothesis, the natural map from $\mathcal{D}^{(n-1)}(\mathrm{Coker}\,\mu_n)$ to $\mathcal{Q}(\mathcal{M})$ is an epimorphism with naturally contractible kernel hence the composite from $\mathcal{D}^{(n)}(\mathcal{M})$ to $\mathcal{Q}(\mathcal{M})$ is also one.

As $\mathcal{Q}(\mathcal{M}_n(\mathbf{E})) \cong \mathbf{E}$, there is noting left to prove. $\square$

3. Loday's Γ-construction.

In [6], we saw that applying π_0 to any simplicial crossed n-cube results in a crossed n-cube. One of the neatest types of crossed n-cube is that which comes from an ideal $(n+1)$-ad of algebras.

Loday in his fundamental paper [19] linked up cat^n-groups with cubes of fibrations. In [22] T.Porter interpreted some of Loday's ideas into the context of simplicial groups and crossed n-cubes. The analogue here of Loday's construction of the cube of fibrations associated with a cat^n-groups is what will be called *Loday's* Γ*-construction*, since the basic idea is already apparent in [19]. This enables us to do analogous results for commutative algebras

Suppose $\mathcal{M} = (C, R, \partial)$ is a crossed module, then there is a short exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & C & \xrightarrow{=} & C \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \partial} \\ C & \xrightarrow{\epsilon} & C \rtimes R & \xrightarrow{t} & R \end{array}$$

of crossed modules, where $\epsilon(c) = (-c, \partial(c))$, $\mu(c) = (c, 0)$ and $t(c, r) = \partial(c) + r$, so is the "target map" of the category structure of the cat^1-algebra of $\mathcal{M}$. Applying $\mathbf{H}$ to each term gives a short exact sequence of simplicial algebras. If we write $\Gamma_{<1>}\mathcal{M} = (0, C, 0)$, and $\Gamma_{\emptyset}\mathcal{M} = (C, C \rtimes R, \partial)$ then the corresponding exact sequence is

$$\mathbf{H}\Gamma_{<1>}\mathcal{M} \xrightarrow{\epsilon_*} \mathbf{H}\Gamma_{\emptyset}\mathcal{M} \longrightarrow \mathbf{H}\mathcal{M}$$

Our previous calculations shows that $\Gamma_{<1>}\mathcal{M} \simeq K(C, 0)$, $\Gamma_{\emptyset}\mathcal{M} \simeq K(R, 0)$ and that

$$(\pi_0 \mathbf{H}\Gamma_{<1>}\mathcal{M} \xrightarrow{\pi_0\epsilon_*} \pi_0 \mathbf{H}\Gamma_{\emptyset}\mathcal{M}) \cong (C \xrightarrow{\partial} R).$$

We note that $(\mathbf{H}\Gamma_{<1>}\mathcal{M},\ \mathbf{H}\Gamma_{\emptyset}\mathcal{M},\ \epsilon_*)$ is a simplicial inclusion crossed module, i.e. is derived from a simplicial ideal 2-ad and we have incidently proved.

Proposition 3.1 (dimension 1).

Any crossed module is isomorphic to π_0 of a simplicial inclusion crossed module all of whose simplicial algebras have trivial homotopy modules in dimension ≥ 1.

The idea behind the proof of the corresponding result for crossed n-cubes of algebras is to use the Γ-operation in each direction in turn. For this purpose we relabel them Γ_1 and Γ_0 respectively, (i.e. Γ_1 is the old $\Gamma_{<1>}$, etc.). These are two functors from crossed modules to themselves and $\epsilon : \Gamma_1 \longrightarrow \Gamma_0$ is a natural transformation.

If we apply these Γ's to a crossed square $\mathcal{M} = \begin{pmatrix} L & M \\ N & R \end{pmatrix}$, then we can do so in two directions as there are two crossed module structures. This gives four different crossed squares:

$$\Gamma_1^1(\mathcal{M}) = \begin{pmatrix} 0 & L \\ 0 & N \end{pmatrix} \qquad \Gamma_1^2(\mathcal{M}) = \begin{pmatrix} 0 & 0 \\ L & M \end{pmatrix}$$

$$\Gamma_0^1(\mathcal{M}) = \begin{pmatrix} L & L \rtimes M \\ N & N \rtimes R \end{pmatrix} \qquad \Gamma_1^2(\mathcal{M}) = \begin{pmatrix} L & M \\ L \rtimes N & M \rtimes R \end{pmatrix}$$

and with the ϵ's yields an inclusion crossed 4-cube.

Similarly, of course, Γ_0 and Γ_1 can be applied independently to the n-different directions of a crossed n-cube giving $2n$ different operations. Γ_m^i, $i \in < n >$, $m = 0$ or 1. If $i \neq j$, for any l, m, $\Gamma_l^i \Gamma_m^j = \Gamma_m^j \Gamma_l^i$, hence if $\mathcal{M} = (M_A)$ is any crossed n-cube , we can define a crossed n-cube $\Gamma_* \mathcal{M}$ of crossed n-cubes by

$$\text{if } A \subseteq < n >, \ \Gamma_A \mathcal{M} = \Gamma_{\alpha(n)}^n \cdots \Gamma_{\alpha(1)}^1 \mathcal{M}$$

where

$$\Gamma_{\alpha(i)}^i = \begin{cases} \Gamma_1^i, & \text{if } \ \mathrm{i} \in A \\ \Gamma_{0,}^i & \text{if } \ i \notin A. \end{cases}$$

If $i \in A$, $\mu_i : \Gamma_A \mathcal{M} \longrightarrow \Gamma_{A-\{i\}} \mathcal{M}$ is given by

$$\mu_i = \Gamma_{\alpha(n)}^n \cdots \Gamma_{\alpha(i+1)}^{i+1} \epsilon^i \Gamma_{\alpha(i-1)}^{i-1} \cdots \Gamma_{\alpha(1)}^1 \mathcal{M}$$

where $\epsilon^i : \Gamma_1^i \longrightarrow \Gamma_0^i$ in the ϵ-maps, sending m to $(m, \mu_i(m))$, in the i^{th} direction if $i \notin A, \mu_i$ is the identity.

As all the μ_i's are inclusions, we can take h-maps to be multiplications.

Since each of the original Γ-operations gave $K(\pi, 0)$'s, it is clear that for any $A \subseteq < n >$, $H^{(n)} \Gamma_A M \simeq K(M_A, 0)$. In fact, a simple inductive proof gives

Proposition 3.2 (General case)

For any crossed n-cube $\mathcal{M}$, there is a simplicial inclusion crossed n-cube, $\mathcal{E}(\mathcal{M}) = (H^{(n)}\Gamma_A\mathcal{M})$, all of whose simplicial algebras have the homotopy types of $K(\pi,0)$'s, such that $\pi_0\mathcal{E}(\mathcal{M}) \cong \mathcal{M}$.

Thus $\pi_0\mathcal{E}$ is naturally isomorphic to the identity ($\mathcal{E}$ is an embedding of $\mathbf{Crs}^n$ as a retract of **Simp(IncCrs**^{n}**)**.) The next result gives information on the other composite $\mathcal{E}\pi_0$.

Proposition 3.3 *There is a functor*

$$\mathcal{H} : \mathbf{Simp}(\mathbf{IncCrs}^n) \longrightarrow \mathbf{Simp}(\mathbf{IncCrs}^n)$$

with natural transformations

$$Id \stackrel{\delta}{\longrightarrow} \mathcal{H} \stackrel{\delta'}{\longrightarrow} \mathcal{E}\pi_0$$

such that for each $B \subseteq < n >$ and simplicial inclusion crossed n-cube, $\mathcal{M}$, $\delta(\mathcal{M})_B$ and $\delta'(\mathcal{M})_B$ induce isomorphisms on π_0.

Proof: Let M be in **Simp(IncCrs**^{n}**)**. Applying $H^{(n)}\Gamma$ to each level of M gives a bisimplicial inclusion crossed n-cube. There are two different directions in which we can take π_0. These give $H^{(n)}\Gamma\pi_0\mathcal{M}$ and $\mathcal{M}$ itself.

This implies that the induced maps

$$\delta : \mathrm{diag} H^{(n)}\Gamma M \longrightarrow M$$
$$\delta : H^{(n)}\Gamma M \longrightarrow H^{(n)}\Gamma\pi_0 M$$

are as required by statement of the proposition. We therefore can take $\mathcal{H}(M) = \mathrm{diag} H^{(n)}\Gamma M$. □

In [6], the functor $\mathcal{Q} : \mathbf{Simp}(\mathbf{Inc}, \mathbf{Crs}^n) \longrightarrow \mathbf{Simp\,Alg}$ was introduced. Restricting initially to the case $n = 1$, we have

$$\mathcal{Q}(\mathbf{F} \longrightarrow \mathbf{E}) = \mathbf{E}/\mathbf{F}$$

In particular, we can start with $\mathcal{M} = (C, R, \partial)$ forming up to exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & C & \xrightarrow{=} & C \\ \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \partial} \\ C & \xrightarrow{\epsilon} & C \rtimes R & \xrightarrow{t} & R \end{array}$$

then taking $\mathbf{H}$, we obtain the short exact sequence

$$\mathbf{H}(0, C, 0) \longrightarrow \mathbf{H}(C, C \rtimes R, \mu) \longrightarrow \mathbf{H}(\mathcal{M})$$

i.e. $\mathcal{Q}\mathbf{H}\Gamma\mathcal{M} = \mathbf{H}(\mathcal{M})$. This example extends easily to higher dimensions since both $\mathcal{Q}$ and Γ can be defined iteratively. We thus have the following lemma

Lemma 3.4 *The functors $\mathcal{E}$ and Q satisfy $\mathcal{Q}\mathcal{E} \cong \mathbf{H}$.*

4. The composite $\mathcal{M}\mathcal{Q}$

So far we have the diagram

$$\begin{array}{ccc} \mathbf{Simp}(\mathbf{IncCrs^n}) & \underset{\mathcal{E}}{\overset{\pi_0}{\rightleftarrows}} & \mathbf{Crs^n} \\ \mathcal{M}\uparrow \; \downarrow\mathcal{Q} & & \downarrow = \\ \mathbf{SimpAlg} & \underset{\mathbf{M}}{\overset{\mathbf{H}}{\leftrightarrows}} & \mathbf{Crs^n} \end{array}$$

and a reflexive subcategory $T_{n]}$ of **SimpAlg**. These functors satisfy: $\mathcal{Q}\mathcal{M} \cong \mathrm{Id}$, $\pi_0\mathcal{E} \cong \mathrm{Id}$, $\mathcal{Q}\mathcal{E} = \mathbf{H}$ and $M = \pi_0\mathcal{M}$ is an embedding when restricted to $T_{n]}$. There is also a functor

$$\mathcal{H} : \mathbf{Simp}(\mathbf{IncCrs}^n) \longrightarrow \mathbf{Simp}(\mathbf{IncCrs}^n)$$

together with a natural transformations

$$\mathrm{Id} \longleftarrow \mathcal{H} \longrightarrow \mathcal{E}\pi_0$$

so that if $\mathbf{E}$ is in $T_{n]}$

$$\mathcal{M}(\mathbf{E}) \overset{\simeq}{\longleftarrow} \mathcal{H}\mathcal{M}(\mathbf{E}) \overset{\simeq}{\longrightarrow} \mathcal{E}\pi_0\mathcal{M}(\mathbf{E})$$

and thus on applying $\mathcal{Q}$

$$\mathbf{E} \overset{\simeq}{\longleftarrow} \mathcal{QHM}(\mathbf{E}) \overset{\simeq}{\longrightarrow} \mathcal{EM}(\mathbf{E}, n).$$

To complete the list of information it is necessary to gain some insight into the composite **MH**. Experiments in low dimensions show it to be closely connected with $\mathbf{H}'\Gamma$, i.e. with the result of applying the diagonal of the multinerve $H^{(n)}$ in the "other" n-directions of Γ ($\mathcal{E} = \mathbf{H}\Gamma$ was formed using half of the $2n$ directions). In this connection, although "there", is clearly difficult to pin down and as there is an elementary alternative approach that attacks not **MH** directly but $\mathbf{M}\mathcal{Q}$, it seems preferable to attack **MH** by this second route. We know $\mathcal{QM} \cong$Id and the identity on **Simp(Inc.Crs**^{n}**)**, then we can hope to use the equations: $M\mathbf{H} = \pi_0\mathcal{MQE}$ and $\pi_0\mathcal{E} \cong$Id to extend that comparison to one between $M\mathbf{H}$ and the identity on $\mathbf{Crs}^n$.

We start with a lemma that was suggested by the work of Conduché. In [12], he introduced a notion of a 2-crossed module (module crosié généralisé de longuer 2). His unpublished work determines that there exists an equivalence (up to homotopy) between the category of crossed squares and that of 2-crossed modules of groups. The first author proved in [3] that this result is true for the algebra case. The outline of the proof is as follows: for any crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\vartheta]{} & R \end{array}$$

the chain complex

$$L \overset{(-\lambda,\lambda)}{\to} M \rtimes N \overset{\mu+\vartheta}{\to} R$$

is a 2-crossed module. In a 2-crossed module

$$C_2 \underset{\beta}{\to} C_1 \underset{\alpha}{\to} C_0$$

the crossed multiplication, $cd - \alpha(c) \cdot d$ for $c, d \in C_1$, for which if α was a crossed module would be zero, is required to be the image of an element $\{c, d\} \in C_2$. The idea of Conduché's proof is that the h-map from $M \times N$ to L is used to construct such a lifting of crossed multiplications. This suggests (and in fact implies) that in the case of a crossed square

$$\frac{M \rtimes N}{\mathrm{Im}(-\lambda, \lambda')} \longrightarrow R$$

should be a crossed module. We need a particular case of this result. As Conduché's verification is, for the general case, quite long, it is convenient that for inclusion crossed squares the result is more or less trivial as it is a consequence of the "Isomorphism Theorems" of elementary algebra theory.

Lemma 4.1 *Let*

$$\begin{array}{ccc} M \cap N & \xrightarrow{\lambda} & M \\ \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\vartheta]{} & R \end{array}$$

be an inclusion crossed square. Let $\triangle : M \cap N \to M \rtimes N$ *be the twisted diagonal* $\triangle(m) = (-m, m)$. *Then* $\mathrm{Im}\triangle$ *is a normal subalgebra of* $M \rtimes N$ *and the morphisms* μ, ν *induce an inclusion crossed module*

$$\frac{M \rtimes N}{\mathrm{Im}\triangle} \xrightarrow{\partial} R$$

where $\partial(m, n) + Im\triangle = \mu(m) + \nu(n)$.

Proof: The map $\partial' : M \rtimes N \to R$ giving by $\partial(m, n) = \mu(m) + \nu(n)$ is an algebra homomorphism as is easily checked. Its kernel is subalgebra consisting of those (m, n) such that $\mu(m) = -\nu(n)$ but μ and ν are inclusions so this is precisely $\mathrm{Im}\triangle$, hence ∂ is a monomorphism. Its image is clearly $M + N$ which is ideal in R. $\square$

We next look at the homotopy type of this crossed module.

Lemma 4.2 (Noether Isomorphism Theorem)

If

$$\begin{array}{ccc} M\cap N & \xrightarrow{\lambda} & M \\ {\scriptstyle\lambda'}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ N & \xrightarrow[\vartheta]{} & R \end{array}$$

is as before an inclusion crossed square and

$$\partial : M+N \rightarrow R$$

is the inclusion crossed module formed from it, then there is a natural map of crossed modules

$$\begin{array}{ccc} M+N & \xrightarrow{\partial} & R \\ & \big\downarrow{\scriptstyle q} & \\ N/M\cap N & \xrightarrow[\overline{\vartheta}]{} & R/M \end{array}$$

given by the obvious quotient and **H** *of the kernel of* q *is contractible.*

As suggested above, this is a direct consequence of the natural isomorphisms: $N/M\cap N \cong (M+N)/M$. The kernel is thus isomorphic to the identity map crossed module (M, M, Id) and **H** applied to this gives a contractible simplicial algebra. (We say q is a crossed module equivalence.)

We note that in fact there are two crossed module equivalences

$$\begin{array}{ccccc} M/M\cap N & & M+N & & M/M\cap N \\ {\scriptstyle\overline{\vartheta}}\big\downarrow & \xleftarrow{q_1} & \big\downarrow & \xrightarrow{q_2} & \big\downarrow{\scriptstyle\overline{\mu}} \\ R/M & & R & & R/N \end{array}$$

Crossed module equivalences induce isomorphisms on π_1 and π_0 i.e. on kernels and cokernels, but here the fact that the quotient crossed modules $\overline{\nu}$ and $\overline{\mu}$ are monomorphisms means that π_1 in both cases is trivial, whilst the result on π_0's is merely that q_1 and q_2 both induce the identity homomorphisms on $P/M+N$. However even though the immediate consequence of these two results may appear trivial, they themselves can be applied at each level in a simplicial inclusion crossed square to get useful information.

Suppose

$$0 \to \mathbf{M} \xrightarrow{\partial} \mathbf{P} \to \mathbf{Q} \to 0$$

is an exact sequence of simplicial algebras. In the context of this section, we think of $(\mathbf{M}, \mathbf{P}, \partial)$ as a simplicial inclusion crossed module $\mathcal{N}$ with $\mathbf{Q}$ as $\mathcal{Q}(\mathcal{N})$ and we wish the compare $(\mathbf{M}, \mathbf{P}, \partial)$ and $\mathcal{M}_1(Q)$.

We use $\mathcal{M}_1 : \mathbf{SimpAlg} \to \mathbf{Simp}(\mathbf{IncCrs}^1)$ applying it to both $\mathbf{M}$ and $\mathbf{P}$. This gives a diagram of simplicial algebras

$$\begin{array}{ccccc}
\mathrm{Ker}\delta_0^M & \longrightarrow & \mathrm{Ker}\delta_0^P & \longrightarrow & \mathrm{Ker}\delta_0^Q \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Dec}^1\mathbf{M} & \longrightarrow & \mathrm{Dec}^1\mathbf{P} & \longrightarrow & \mathrm{Dec}^1\mathbf{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{M} & \xrightarrow{\partial} & \mathbf{P} & \longrightarrow & \mathbf{Q}
\end{array}$$

in which all rows and columns are exact. The top left hand square is a simplicial inclusion crossed square and so we can apply the last two previous lemmas at each level to construct a simplicial inclusion crossed module, which for simplicity we denote

$$\mathbf{R} \to \mathrm{Dec}^1\mathbf{P}$$

(so $\mathbf{R} = (\mathrm{Dec}^1\mathbf{M}) + (\mathrm{Ker}\delta_0^P)$). There are quotient maps

$$\begin{pmatrix} \mathbf{M} \\ \downarrow \\ \mathbf{P} \end{pmatrix} \xleftarrow{q_1} \begin{pmatrix} \mathbf{R} \\ \downarrow \\ \mathrm{Dec}^1\mathbf{P} \end{pmatrix} \xrightarrow{q_2} \begin{pmatrix} \mathrm{Ker}\delta_0^Q \\ \downarrow \\ \mathrm{Dec}^1\mathbf{P} \end{pmatrix} = \mathcal{M}_1(\mathcal{Q})$$

whose kernels are both contractible at each level. Thus if we apply $\mathbf{H}$ in each level and then take diagonal of the resulting bisimplicial algebras we find q_1 and q_2 induce trivial fibriations with explicit natural constractions on the fibers. In particular if $(\mathbf{M}, \mathbf{P}, \partial)$ is a resolution of crossed module (M, R, ∂), then applying $\mathcal{M}$ to the nerve of (M, R, ∂) produce an equivalent simplicial inclusion crossed module.

We suppose that $\mathcal{M}_{.} = (M_A)$ is a simplicial inclusion crossed n-cube. As usual, we write $\mathcal{Q}(\mathcal{M})$ for the quotient simplicial algebra. As the process we will use will take n-steps, it will help to denote by $Q_1, Q_2, \cdots$ the effect of using the 1-dimensional quotienting operation in direction 1,2 etc. Thus, for example, in an inclusion crossed square we have

$$\begin{array}{ccccc}
M_{<2>} & \xrightarrow{\mu_1} & M_{\{1\}} & \xrightarrow{q_1} & M_{\{1\}}/M_{<2>} \\
\downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \overline{\mu_2}} \\
M_{\{2\}} & \longrightarrow & M_{\emptyset} & \longrightarrow & M_{\emptyset}/M_{\{2\}} \\
\downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_2} \\
M_{\{2\}}/M_{<2>} & \longrightarrow & M_{\emptyset}/M_{\{1\}} & \xrightarrow{q_1} & M_{\emptyset}/M_{\{1\}}M_{\{2\}}
\end{array}$$

$Q_1(\mathcal{M})$ is the crossed module $(M_{\{1\}}/M_{<2>}, M_{\emptyset}/M_{\{2\}}, \overline{\mu}_2)$ whilst $Q_2Q_1(\mathcal{M})$ is the algebra $M_{\{1\}}/M_{\{1\}}M_{\{2\}}$ which is $\mathcal{Q}(\mathcal{M})$.

We will consider an inclusion crossed n-cube $\mathcal{M}$ to be both an n-dimentional resolution of $\mathcal{Q}(\mathcal{M})$ and a crossed resolution of any of the $Q_i(\mathcal{M})$ which are its quotient crossed $(n-1)$-cubes.

We first apply our previous result to $(n-1)$-cube of exact sequences of simplicial inclusion crossed modules

$$(M_A \xrightarrow{\mu_n} M_{A-\{n\}} \xrightarrow{q_n} Q_n(\mathcal{M})_{A-\{n\}})$$

where $A \subseteq < n >$ is such that $n \in A$.

We find that there is a simplicial inclusion crossed n-cube $R_n = (r_n : R_{n,A} \rightarrow \mathrm{Dec}^1 M_{A-\{n\}})$

and maps

$$\begin{array}{ccccc} M_A & & R_{n,A-\{n\}} & & \mathrm{Ker}\delta_0 \\ \downarrow & \longleftarrow & \downarrow & \longrightarrow & \downarrow \\ M_{A-\{n\}} & & \mathrm{Dec}^1 M_{A-\{n\}} & & \mathrm{Dec}^1 Q_n(\mathcal{M})_{A-\{n\}} \end{array} \qquad = \mathcal{M}_1(Q_n(\mathcal{M}))$$

which have contractible kernels, i.e. the $H^{(n)}$-functor applied to the kernels gives contractible simplicial algebras. We repeat this using direction $n-1$ starting with the right-hand simplicial crossed n-cube. This second stage produces a new simplicial inclusion crossed n-cube

$$R_{n-1} = (r_{n-1} : R_{n-1,B-\{n-1\}} \to \mathrm{Dec}^2 Q_n(\mathcal{M})_{B-\{n-1\}})$$

with maps

$$\mathcal{M}_1(Q_n(M)) \leftarrow R_{n-1} \to \mathcal{M}_1(Q_{n-1}(\mathcal{M}_1(Q_n(M))))$$

which induce trivial fibrations (as before) on applying $H^{(n)}$. The simplicial inclusion crossed n-cube $\mathcal{M}_1(Q_n(M))$ has one direction in the form of an image of a (one dimensional) $\mathcal{M}$ functor; the result of the repeated process has this structure in two directions and is isomorphic to $\mathcal{M}_1^2(Q_{n-1}Q_n(\mathcal{M}))$ (by the third isomorphism theorem for algebras!). However, for any simplicial algebra, applying, $\mathcal{M}_1$, i.e. the functor from **SimpAlg** to **Simp(IncCrs**^{1}**)** twice yields $\mathcal{M}_2$:**SimpAlg**$\to$**Simp(IncCrs**^{2}**)** so we can string the two diagrams together to connect $\mathcal{M}$ by a chain with $\mathcal{M}_2(Q_{n-1}Q_n(\mathcal{M}))$. Continuing like this in direction $(n-2)$ and so on, gives

$$\mathcal{M} \leftarrow R_n \to \mathcal{M}_1 Q_n \mathcal{M} \leftarrow R_{n-1} \to \mathcal{M}_2 Q_{n-1} Q_n \mathcal{M} \leftarrow \cdots \to \mathcal{M}_{n-1} Q_2 \cdots Q_n \mathcal{M} \leftarrow R_1 \mathcal{M}_n Q \mathcal{M}.$$

Each of the maps in this “zigzag” has the property that in each simplicial dimension its kernel yields a naturally contractible simplicial algebra on application of $H^{(n)}$. Now this latter property will be preserved at connected component level; more precisely if we look at the image of the zigzag under π_0, we find that each kernel again has a trivial crossed module structure (M, M, Id) in at least direction and $H^{(n)}$ of that kernel will be contractible.

It is now the moment to introduce the notion of quasi-isomorphism for crossed n-cubes. We start with basic observation that has been used many times already. Any map $f : \mathbf{M} \to \mathbf{N}$ of crossed n-cubes that is an epimorphism at each corner of the n-cube induces an epimorphism (and hence a fibration) of the simplicial algebras, $H^{(n)}f : H^{(n)}\mathbf{M} \to H^{(n)}\mathbf{N}$. We will say f is a trivial epimorphism if $H^{(n)}f$ is a trivial fibriation, i.e. if $\mathrm{Ker}\,H^{(n)}f$ is contractible. We write $\sum$ for the class of trivial epimorphisms in $\mathbf{Crs}^n$ and $\mathrm{Ho}(\mathbf{Crs}^n)$ or $\mathbf{Crs}^n(\sum^{-1})$ for corresponding category of fractions. A map f in $\mathbf{Crs}^n$ will be called a quasi-isomorphism if the corresponding map $[f]$ in $\mathrm{Ho}(\mathbf{Crs}^n)$ is an isomorphism. We can thus summarise the discussion above as follows.

Proposition 4.3 *Given any simplicial inclusion crossed n-cube χ there is a zigzag of quasi-isomorphism between $\pi_0(\chi)$ and $\pi_0\mathcal{M}_n(\mathcal{Q}(\chi))$, i.e. $\pi_0(\chi)$ and $\pi_0\mathcal{M}_n(\mathcal{Q}(\chi))$ are isomorphic in Ho$(\mathbf{Crs}^n)$. In particular for any crossed n-cube M, M is isomorphic to $\mathbf{M}(\mathbf{H}(M), n)$ in Ho$(\mathbf{Crs}^n)$.*

One can equally well use the above discussion to prove a result purely within $\mathbf{Simp(IncCrs}^n)$. If we say an epimorphism f in $\mathbf{Simp(IncCrs}^n)$ is a trivial fibration if $\mathrm{Ker}\,f$ is contractible (by which we mean $\mathrm{D}^n\mathrm{Ker}\,f = \mathrm{diag}\,H^{(n)}\ \mathrm{Ker}\,f$ is contractible), then again one has a class $\sum$ of trivial fibrations and can formally invert them to form $\mathrm{Ho}(\mathbf{Simp(IncCrs}^n))$. We have proved above that any χ in $\mathbf{Simp(IncCrs}^n)$ is quasi isomorphic (i.e. isomorphic within $\mathrm{Ho}(\mathbf{Simp(IncCrs}^n)))$ to $\mathcal{M}_n\mathcal{Q}(\chi))$.

To complete the analysis of the functor $\mathbf{H}$ and $\mathbf{M}(-,n)$ it is necessary to check what they do to quasi-isomorphisms and homotopy equivalences. Firstly by definition $\mathbf{H}$ sends trivial fibrations to trivial fibrations and hence sends quasi-isomorphisms to homotopy equivalences. Now assume $f : \mathbf{E} \to \mathbf{F}$ is homotopy equivalence of simplicial algebras and that $\mathbf{E}$ and $\mathbf{F}$ are both in $T_{n]}$. Then f factors within $\mathbf{SimpAlg}$ as $j^f i^f$ where i^f is a trivial fibration and j^f is a splitting of a trivial fibration, p^f. The usual construction is via the mapping cocylinder, M^f. This is homotopically equivalent to X via the projection $p^f : M^f \to X$. This simplicial algebra M^f is not necessarily in $T_{n]}$, but $t_{n]}M^f$ is and p^f and the other map, $i^f : M^f \to Y$, both factor via $M^f \to t_{n]}M^f$. Thus the factorisation of f as a composite of trivial fibration and a splitting of a trivial fibration can be made within $T_{n]}$.

If f is already a trivial fibration, the diagram

$$\begin{array}{ccc} \mathbf{H}(\mathbf{M}(\mathbf{E},n)) & \xrightarrow{\simeq} & \mathbf{E} \\ {\scriptstyle \mathbf{H}(M(f))}\downarrow & & \simeq \downarrow {\scriptstyle f} \\ \mathbf{H}(\mathbf{M}(\mathbf{F},n)) & \xrightarrow{\simeq} & \mathbf{F} \end{array}$$

shows that $M(f)$ is a trivial fibration in $\mathbf{Crs}^n$. As in general $M(f) = M(j^f)+M(i^f)$, we get that if f is a homotopy equivalence, $[M(f)] = [M(i^f)] - [M(p^f)]$ is an isomorphism, hence that $M(f)$ is a quasi-isomorphism.

This argument relied on $\mathbf{E}$ and $\mathbf{F}$ being n-truncated, if they are not, then we can try for a result using $\mathcal{M}_n$ rather than $\mathbf{M}(\ ,n)$. We can again reduce to the case that f is a trivial fibration.

We want to show that $\mathcal{M}_n(f)$ is a quasi-isomorphism in **Simp(Inc.Crs**^{n}**)** i.e. that $D^n\mathcal{M}_n(f)$ is a homotopy equivalence in **SimpAlg**. Using Proposition 7 (strong version), this is simple. We have the diagram

$$\begin{array}{ccc} D^{(n)}\mathcal{M}_n(\mathbf{E}) & \xrightarrow{\simeq} & \mathbf{E} \\ {\scriptstyle D^{(n)}\mathcal{M}_n(f)}\downarrow & & \simeq \downarrow {\scriptstyle f} \\ D^{(n)}\mathcal{M}_n(\mathbf{F}) & \xrightarrow{\simeq} & \mathbf{F} \end{array}$$

and by construction $D^{(n)}\mathcal{M}_n(f)$ is an epimorphism. It follows that it is a trivial fibration. We thus have the following theorem

Theorem 4.4 *The functors* $\mathcal{M}_n$ *and* $D^{(n)}$ *induce an equivalence between* $\mathbf{Ho_n}(\mathbf{SimpAlg}$ $)$ *and* $\mathbf{Ho}(\mathbf{Simp}(\mathbf{IncCrs}^n))$.

Thus "entire connected homotopy types can be encoded" in this manner, but one has information about the m-types for $m > n$ still in mixed simplicial algebraic gadget. In the second part of this paper we will consider how one may obtain information in all dimensions with an algebraic model, namely the homotopy systems/crossed complexes of Whitehead [23] more recently studied in depth by Brown and Higgins, (cf. [8]).

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