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ON THE DIFFERENTIAL PRIME RADICAL OF A DIFFERENTIAL RING

D. Khadjiev & F. Çalhaly*

Abstract

In this paper we have obtained the following results for a differential ring (associative or nonassociative):

(1) For a differential ring (\mathcal{D} -ring) we introduce definitions of a \mathcal{D} -prime \mathcal{D} -ideal, \mathcal{D} -semiprime \mathcal{D} -ideal and a strongly \mathcal{D} -nilpotent element. We define the \mathcal{D} -prime radical as the intersection of all \mathcal{D} -prime \mathcal{D} -ideals. For any \mathcal{D} -ring the \mathcal{D} -prime radical, the intersection of all \mathcal{D} -semiprime \mathcal{D} -ideals and the set of all strongly \mathcal{D} -nilpotent elements are equal.

(2) For a \mathcal{D} -ring we introduce a definition of an s-nilpotent \mathcal{D} -ideal. If a \mathcal{D} -ring satisfies the ascending chain condition for \mathcal{D} -ideals then its \mathcal{D} -prime radical is s-nilpotent.

(3) Let \mathcal{Q} be a field of rational numbers. If δ is a differentiation of a \mathcal{Q} -algebra R with 1 then $\delta(\text{Pr.rad}(R)) \subseteq \text{Pr.rad}(R)$.

(4) Let K be a differential ring. Then every radical \mathcal{D} -ideal of K is an intersection of \mathcal{D} -prime \mathcal{D} -ideals.

1. The differential prime radical

This paper is a continuation of our papers [1-3]. Further, we use notions and notations of books [4-6].

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Definition ([7], p.556). A differential ring (\mathcal{D} -ring) is a system (K, \mathcal{D}) , where K is a ring (associative or nonassociative) and \mathcal{D} is a set of differentiations of K . A \mathcal{D} -subgroup H is an additive subgroup of the ring K such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.

Let K be a \mathcal{D} -ring. Denote by $Ad(K)$ the set of all \mathcal{D} -subgroups of K . $Ad(K)$ is a complete lattice with respect to the inclusion relation. Introduce a multiplication operation on it by the following manner ([5], p.12): For $A, B \in Ad(K)$ $A \cdot B$ consists of all finite sums $\sum_{i=1}^n a_i b_i$, where $a_i \in A, b_i \in B$.

Definition ([7], p.556). A differential ideal (\mathcal{D} -ideal) of K is an ideal H of K such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.

Denote by $Id(K)$ the set of all \mathcal{D} -ideals of K . $Id(K)$ is a complete lattice with respect to the inclusion relation. If K is associative then $A \cdot B \in Id(K)$ for all $A, B \in Id(K)$.

But there is nonassociative \mathcal{D} -ring K and $A, B \in Id(K)$ such that $A \cdot B \notin Id(K)$.

Therefore for any \mathcal{D} -ring we define a multiplication operation of \mathcal{D} -ideals in the following manner. For $A, B \in Id(K)$ denote by $A \cdot B$ the intersection of all \mathcal{D} -ideals of K containing the set $\{x \in K : x = a \cdot b, a \in A, b \in B\}$.

Proposition 1.1. For any \mathcal{D} -ring K the lattice $Id(K)$ with above multiplication operation is a complete l -groupoid.

Proof. Let $A, B_t \in Id(K), t \in T$. The inequality

$$A \cdot (\bigvee_{t \in T} B_t) \geq \bigvee_{t \in T} (A \cdot B_t)$$

is obvious. We now prove the inverse inequality. The \mathcal{D} -ideal $A \cdot (\bigvee_{t \in T} B_t)$ is the smallest \mathcal{D} -ideal containing all elements $a \cdot (b_1 + b_2 + \dots + b_k)$, where $a \in A, b_i \in B_{t_i}$. From the equality $a \cdot (b_1 + b_2 + \dots + b_k) = ab_1 + \dots + ab_k$ we obtain that $a \cdot (b_1 + b_2 + \dots + b_k) \in \bigvee_{t \in T} (A \cdot B_t)$. Therefore

$$A \cdot (\bigvee_{t \in T} B_t) \leq \bigvee_{t \in T} (A \cdot B_t).$$

A proof of the equality

$$(\bigvee_{t \in T} B_t) \cdot A = \bigvee_{t \in T} (B_t \cdot A)$$

is similar. □

Definition A \mathcal{D} -ideal P of K is \mathcal{D} -prime if $P \neq K$ and $A \cdot B \subseteq P$, $A, B \in \text{Id}(K)$, implies that $A \subseteq P$ or $B \subseteq P$.

For $A \in \text{Id}(K)$, $A \neq K$, denote by $R(A)$ the intersection of all \mathcal{D}_s -semiprime \mathcal{D} -ideals of K containing A . Put $r^s(A) = K$ if there are none.

For $A \in \text{Id}(K)$, denote by $\langle A \rangle$ the groupoid generated by A . An element of the groupoid $\langle A \rangle$ will be denoted by $f(A)$.

Definition A \mathcal{D} -ideal H of K is \mathcal{D}_w -semiprime if $H \neq K$ and $f(A) \subseteq H$, $A \in \text{Id}(K)$, $f(A) \in \langle A \rangle$, implies that $A \subseteq H$.

For $A \in \text{Id}(K)$, $A \neq K$, denote by $r^w(A)$ the intersection of all \mathcal{D}_w -semiprime \mathcal{D} -ideals of K containing A . Put $r^w(A) = K$ if there are none. It is clear $r^s(A) \subseteq r^w(A) \subseteq R(A)$ for all $A \in \text{Id}(K)$.

For $A \in \text{Id}(K)$ the \mathcal{D} -ideal $R(A)$ will be called \mathcal{D} -radical of A .

Definition A \mathcal{D} -ideal M of K is \mathcal{D} -maximal if $M \neq K$ and $M \subseteq B \subseteq K$, $B \in \text{Id}(K)$, implies that $M = B$ or $B = K$.

Proposition 1.2 Let K be a \mathcal{D} -ring satisfying the ascending chain condition for \mathcal{D} -ideal. Then every \mathcal{D} -ideal of K is contained in some \mathcal{D} -maximal \mathcal{D} -ideal. In particular, there is a \mathcal{D} -maximal \mathcal{D} -ideal of K .

A proof is standard.

Proposition 1.3. Let K be a \mathcal{D} -ring such that $K^2 = K$. Then any \mathcal{D} -maximal \mathcal{D} -ideal K is \mathcal{D} -prime.

Proof. Let M be \mathcal{D} -maximal \mathcal{D} -ideal of K . Suppose that $A \cdot B \subseteq M$, $A, B \in \text{Id}(K)$. If $A \not\subseteq M$, then $A \vee M = K$. Therefore

$$K \cdot K = (A \vee M) \cdot (B \vee M) = A \cdot B \vee A \cdot M \vee M \cdot B \vee M \cdot M \subseteq M \subseteq K.$$

We obtain $M = K$. This is a contradiction. □

Remark *The condition of Proposition 1.3 fulfils for \mathcal{D} -rings with 1.*

Proposition 1.4 *Let K be a \mathcal{D} -ring satisfying the ascending chain condition for \mathcal{D} -ideals. Then the following conditions are equivalent:*

- (1) $K^2 = K$;
- (2) *Every \mathcal{D} -maximal \mathcal{D} -ideal of K is \mathcal{D} -prime.*

Proof. (1) \Rightarrow (2) follows from proposition 1.3.

(1) \Rightarrow (2): Assume that $K^2 \neq K$. By proposition 1.2 there exists a \mathcal{D} -maximal M of K such that $K^2 \subseteq M$. It is contradiction since M is \mathcal{D} -prime. □

For an element $a \in K$ denote by $[a]$ the smallest \mathcal{D} -ideal containing a .

Every sequence $\{x_0, x_1, \dots, x_n, \dots\}$, where $x_0, x_{n+1} \in [x_n]^2$, will be called a \mathcal{D} -sequence of the element a .

Definition *An element $a \in K$ is strongly \mathcal{D} -nilpotent if every its \mathcal{D} -sequence is ultimately zero.*

Remark *This definition is a generalization of differential rings of the similar definition in ([5], p.55; [1], p.574).*

Denote by $n(0)$ the set of all strongly \mathcal{D} -nilpotent elements of K , where 0 is zero ideal of K .

Theorem 1.5 *For any \mathcal{D} -ring K the equalities $n(0) = r^s(0) = r^w(0) = R(0)$ hold.*

Proof. First we prove that $n(0) \subseteq r^s(0)$. If there are no \mathcal{D}_s -semiprime \mathcal{D} -ideals then $r^s(0) = K$. Hence $n(0) \subseteq r^s(0)$. Assume that there is a \mathcal{D}_s -semiprime \mathcal{D} -ideal. Let

$a \in n(0)$ and S be and \mathcal{D}_s -semiprime \mathcal{D} -ideal. Prove that $a \in S$. Assume that $a \notin S$. Then $[x_0] \not\subseteq S$, where $x_0 = a$. There exists $x_1 \in [x_0]^2$ such that $x_1 \notin S$. Continuing in this manner we obtain a \mathcal{D} -sequence $\{x_0, x_1, \dots, x_n, \dots\}$ of the element a such that $x_n \notin S$ for all n . But it is a contradiction since every \mathcal{D} -sequence of the element a is ultimately zero.

Thus $a \in S$ and $a \in r^s(0)$ since S is any \mathcal{D}_s -semiprime \mathcal{D} -ideal. Hence $n(0) \subseteq r^s(0) \subseteq r^w(0) \subseteq R(0)$.

Prove that $R(0) \subseteq n(0)$. If $n(0) = K$ then $n(0) = r^s(0) = r^w(0) = R(0) = K$. Let $n(0) \neq K$. Let $b \in K$ such that $b \notin n(0)$. Then there exists a \mathcal{D} -sequence $X = \{x_0, x_1, \dots, x_n, \dots\}$ of the element b such that $X \cap 0 = \emptyset$, where 0 is the zero ideal of K .

Denote by \sum the set of \mathcal{D} -ideals M of K such that $X \cap M = \emptyset$. Then \sum is not empty since $0 \in \sum$. We can apply Zorn's lemma to the set \sum ; so there exists a maximal element P of \sum . Show that P is \mathcal{D} -prime.

First, P is proper since $b \in P$. Let $B, C \in Id(K)$, $B \not\subseteq P$, $C \not\subseteq P$. Then $P \vee B \neq P$ and $P \vee C \neq P$. By the maximality of P in \sum we have $P \vee B \notin \sum$ and $P \vee C \notin \sum$. Hence there exist $x_m \in X$, $x_q \in X$ such that $x_m \in P \vee B$, $x_q \in P \vee C$. Then

$$[x_m] \subseteq P \vee B, \quad [x_q] \subseteq P \vee C.$$

Hence

$$x_{m+1} \in [x_m]^2 \subseteq P \vee B, \quad x_{q+1} \in [x_q]^2 \subseteq P \vee C.$$

Continuing in this manner we find that

$$x_{m+t} \in P \vee B, \quad x_{q+t} \in P \vee C$$

for all t . Put $n = \max(m, q)$. Then

$$x_n \in P \vee B, \quad x_n \in P \vee C.$$

Hence

$$x_{n+1} \in [x_n]^2 \subseteq (P \vee B)(P \vee C) \subseteq P \vee B \cdot C,$$

by Proposition 1.1. But $x_{n+1} \notin P$. Hence $B \cdot C \not\subseteq P$. Therefore P is \mathcal{D} -prime. Thus there exists a \mathcal{D} -prime \mathcal{D} -ideal P such that $b \notin P$. Then $n(0) = r^s(0) = r^w(0) = R(0)$. □

The \mathcal{D} -ideal $R(0)$ will be called differential prime radical of K and will be denoted by $\mathcal{DPr.rad}(K)$.

Definition \mathcal{D} -ideal H of K is \mathcal{D} -radical if $H = R(H)$.

For $A \in Id(K)$ denote by $n(A)$ the set of all elements $x \in K$ that every \mathcal{D} -sequence of x meets A .

Corollary 1 For any $A \in Id(K)$ the following equalities hold:

$$n(A) = r^s(A) = r^w(A) = R(A).$$

In particular, every \mathcal{D}_s -semiprime \mathcal{D} -ideal is \mathcal{D} -radical.

Proof. Applying theorem 1.5 to the quotient \mathcal{D} -ring K/A , we obtain $n(A) = r^s(A) = r^w(A) = R(A) = \mathcal{DPr.rad}(K/A)$. □

Corollary 2. For a \mathcal{D} -ring K the following conditions are equivalent:

1. Every \mathcal{D} -ideal of K is \mathcal{D} -radical;
2. $A \cdot B = A \cap B$ for all $A, B \in Id(K)$;
3. $[a]^2 = [a]$ for all $a \in K$.

Proof. We use the following lemma: □

Lemma *Let K be a \mathcal{D} -ring. Then*

$$R(A \cdot B) = R(A \cap B) = R(A) \cap R(B)$$

for any $A, B \in Id(K)$. The proof of this lemma follows from proposition 1.6 in [8],

(1) \Rightarrow (2) : If every \mathcal{D} -ideal of K is \mathcal{D} -radical then using the lemma we obtain

$$A \cdot B = R(A \cdot B) = R(A) \cap R(B) = A \cap B.$$

(2) \Rightarrow (3) : Let $A \cdot B = A \cap B$ for any $A, B \in Id(K)$. Then $A^2 = A$ for any $A \in Id(K)$.

(3) \Rightarrow (1) : Prove that every \mathcal{D} -ideal of K is \mathcal{D}_s -semiprime. Let A be a \mathcal{D} -ideal of K . Then $A = \vee_{a \in A} [a]$. Using proposition 1.1 we have

$$\begin{aligned} A^2 &= (\vee_{a \in A} [a])^2 = (\vee_{a \in A} [a]^2) \vee (\vee_{a, b \in A} [a] \cdot [b]) = \\ &(\vee_{a \in A} [a]) \vee (\vee_{a, b \in A} [a] \cdot [b]) = \vee_{a \in A} [a] = A, \end{aligned}$$

since $[a] \cdot [b] \subseteq [a] \cap [b]$ for any $a, b \in A$. Thus $A^2 = A$ for any $A \in Id(K)$. Let $B^2 \subseteq A$, $B \in Id(K)$. Then $B = B^2 \subseteq A$. Therefore every \mathcal{D} -ideal A is \mathcal{D}_s -semiprime. By Corollary 1 A is \mathcal{D} -radical.

Remark *This corollary is a generalization of the similar theorem in ([7], ch.4, §5).*

Let $A \in Id(K)$. Put $A^{(0)} = A$, $A^{(n+1)} = (A^{(n)})^2$.

Corollary 3 *For a \mathcal{D} -ring K the following conditions are equivalent:*

- (1) $\mathcal{D}Pr.rad(K) = 0$
- (2) If $A^{(n)} = 0$, $A \in Id(K)$, for some n then $A = 0$;
- (3) If $A^2 = 0$, $A \in Id(K)$, then $A = 0$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) is obvious.

Prove that (3) \Rightarrow (1). Condition (3) implies that $r^s(0) = 0$. By theorem 1.5 we see $Pr.rad(K) = r^s(0) = 0$. □

Definition A \mathcal{D} -ideal A of K is s -nilpotent if $A^{(n)} = 0$ for some n .

Proposition 1.6 Let K be a \mathcal{D} -rign and $A, B \in Id(K)$, $A \subseteq B$. If A is s -nilpotent and B/A is s -nilpotent in K/A then B is s -nilpotent in K .

Proof. Since B/A is s -nilpotent the $B^{(n)} \subseteq A$ for some n . Then $B^{(n+m)} = (B^{(n)})^{(m)} = 0$ since A is s -nilpotent. □

Theorem 1.7 Let K be a \mathcal{D} -rign satisfying the ascending chain condition for \mathcal{D} -ideals. The $\mathcal{DPr.rad}(K)$ is s -nilpotent.

Proof. Denote by Σ the set of s -nilpotent \mathcal{D} -ideals of K . Σ is not empty since $0 \in \Sigma$. There exists a maximal element P in Σ . By proposition 1.6 the \mathcal{D} -ring K/P have the following property: if $(A/P)^2 = 0$, $A \in Id(K)$, $P \subseteq A$, then $A/P = 0$. By corollary 3 of theorem 1.5 we have $\mathcal{DPr.rad}(K/P) = 0$. This means that $\mathcal{DPr.rad}(K) \subseteq P$. But $P \subseteq \mathcal{DPr.rad}(K)$ since P is s -nilpotent. Therefore $\mathcal{DPr.rad}(K) = P$. □

Corollary Let K be a \mathcal{D} -ring satisfying the ascending chain condition for \mathcal{D} -ideals. Then the followings are equivalent:

- (1) $K^{(n)} = 0$ for some n .
- (2) K has not a \mathcal{D} -prime \mathcal{D} -ideal;
- (3) K has not a \mathcal{D}_s -semiprime \mathcal{D} -ideal.

Denote by $Id_r(K)$ the set all \mathcal{D} -radical \mathcal{D} -ideals of K . $Id_r(K)$ is a complete lattice with respect to the inclusion relation. Denote by \vee and \wedge the lattice operations in $Id_r(K)$.

Theorem 1.8 Let K be a \mathcal{D} -ring. Then the lattice $Id_r(K)$ satisfies the infinte \wedge -distributive condition:

$$A \wedge (\vee_{\tau \in T} B_\tau) = \vee_{\tau \in T} (A \wedge B_\tau)$$

for any $A, B \in Id_r(K)$. In particular, $Id_r(K)$ is distributive.

A proof follows from Theorem 1.3 in [8].

Theorem 1.9 *Let K be a \mathcal{D} -ring satisfying the ascending chain condition for \mathcal{D} -ideals. Then any \mathcal{D} -radical \mathcal{D} -ideal A is an intersection of finite \mathcal{D} -prime \mathcal{D} -ideals and a such representation of A is unique.*

Proof. First prove the following. □

Lemma *$A \in Id_r(K)$ is a \mathcal{D} -prime \mathcal{D} -ideal iff A is an \wedge -indecomposable element of the lattice $Id_r(K)$.*

Proof. Let A be a \mathcal{D} -prime \mathcal{D} -ideal of K and $A = A_1 \wedge A_2$, $A_1, A_2 \in Id_r(K)$. Then

$$A_1 \cdot A_2 \subseteq A_1 \cap A_2 \subseteq R(A_1 \cap A_2) = A_1 \wedge A_2 = A.$$

Hence $A_1 \subseteq A$ or $A_2 \subseteq A$ since A is \mathcal{D} -prime. Then $A = A_1$ or $A = A_2$.

Let A be an \wedge -indecomposable element of the lattice $Id_r(K)$ and $B \cdot C \subseteq A$, $B, C \in Id_r(K)$. Then $R(B \cdot C) \subseteq A$. By lemma 1.6 in [8] we have $R(B) \wedge R(C) = R(B \cdot C) \subseteq A$. We obtain

$$A = A \vee (R(B) \wedge R(C)) = (A \vee R(B)) \wedge (A \vee R(C))$$

since $Id_r(K)$ is distributive. Hence $A = A \vee R(B)$ or $A = A \wedge R(C)$ since A is an \wedge -indecomposable. This means that $B \subseteq R(B) \subseteq A$ or $C \subseteq R(C) \subseteq A$. Thus A is \mathcal{D} -prime. The lemma is proved.

By the lemma and the corollary in ([4], p.183), we obtain the every \mathcal{D} -radical \mathcal{D} -ideal of K is an intersection of finite \mathcal{D} -prime \mathcal{D} -ideals of K and such a representation is unique. □

Remark *This theorem is a generalization of the similar statement from the theory of associative rings.*

Let $A \in Id(K)$. Put $N_0(A) = A$. Denote by $N_1(A)$ the supremum of all $B \in Id(K)$ such that $B^{(n)} \subseteq A$ for some n (n depends from B). Put $N_\alpha(A) = N_1(N_\beta(A))$ for $\alpha = \beta + 1$ and $N_\alpha(A) = \vee_{\beta < \alpha} N_\beta(A)$ for α a limit ordinal

Put $L(K, \mathcal{D}) = N_\alpha(0)$ for any ordinal α of cardinality $\geq |K|$.

Theorem 1.10 $L(K, \mathcal{D}) = \mathcal{D}Pr.rad(K)$ for \mathcal{D} -ring K .

Proof. By theorem 1.5 it is enough to prove that $L(K, \mathcal{D}) \subseteq r^s(0)$.

It is clear that $N_1(0) \subseteq r^s(0)$. By transfinite induction we obtain $N_\alpha(0) \subseteq r^s(0)$ for any ordinal α . Hence $L(K, \mathcal{D}) \subseteq r^s(0)$.

Prove that $L(K, \mathcal{D})$ is \mathcal{D}_s -semiprime. Assume that $B^2 \subseteq L(K, \mathcal{D})$. Then there exists an ordinal α such that $B^2 \subseteq N_\alpha(0)$. Hence $B \subseteq N_{\alpha+1}(0)$ by definition of $N_{\alpha+1}(0)$. Therefore $B \subseteq L(K, \mathcal{D})$ and $L(K, \mathcal{D})$ is \mathcal{D}_s -semiprime.

By the definition of $r^s(0)$ we have $L(K, \mathcal{D}) = r^s(0)$. □

2. \mathcal{D} -algebras over the field of rational numbers

Let \mathcal{Q} be the field of rational numbers. Put $\mathcal{D}Pr.rad(K) = Pr.rad(K)$ and $L(K, \mathcal{D}) = L(K)$ if $\mathcal{D} = \emptyset$.

\mathcal{D} -sequence of $a \in K$ will be called n -sequence of a if $\mathcal{D} = \emptyset$.

For $a \in K$ we denote its n -sequence $\{x_0(a) = a, x_1, \dots, x_m, \dots\}$ in the form:

$$\{x_0(a) = a, x_1(a), \dots, x_m(a), \dots\}.$$

If R is associative then every element $x_m(a) \in [x_{m+1}(a)]^2 \subseteq [a]^{2^m}$ is a finite sum:

$$x_m(a) = \sum f_1 f_2 \dots f_s,$$

where $s = 2^m$ and every f_i has the form $f_i = r_{1i} a r_{2i}$. Thus $x_m(a)$ is a homogeneous polynomial of a a degree 2^m with coefficients from R .

If R is nonassociative then every element $x_m(a)$ is a homogeneous nonassociative polynomial of degree 2^m with coefficients from R .

Theorem 2.1 If R is \mathcal{Q} -algebra with 1 and δ is a differentiation of R , then

$$\delta(Pr.rad(R)) \subseteq Pr.rad(R).$$

Proof. Let $a \in Pr.rad(R)$. We consider any n -sequence of the element δa :

$$\{x_0(\delta a) = \delta a, x_1(\delta a), \dots, x_m(\delta a), \dots, \}.$$

We have

$$\delta^{2m}(x_m(a)) \in q \cdot x_m(\delta a) + [a],$$

where $q \neq 0$ is an integer. Then $x_m(\delta a) \in \frac{1}{q}\delta^{2m}(x_m(a)) + [a]$. Since $a \in Pr.rad(R)$ then there exist m_0 such that $x_m(a) = 0$ for all $m \geq m_0$. Then the following sequence

$$\{y_0(a) = a, y_1(a), \dots, y_k(a), \dots, \},$$

where $y_k(a) = x_{m_0+k}(\delta a)$, is an n -sequence of the element a . Therefore there exists k_0 such that $y_k(a) = 0$ for all $k \geq k_0$. This means that any n -sequence of the form

$$\{x_0(\delta a) = \delta a, x_1(\delta a), \dots, x_m(\delta a), \dots, \}$$

of element δa is ultimately zero. Thus $\delta a \in Pr.rad(R)$. □

Remark *This theorem is the Ritt's theorem if R is a commutative associative \mathcal{Q} -algebra ([6], p.12). If R is an associative \mathcal{Q} -algebra then there is only the formulation of this theorem and putline of its prof in ([10], p.207).*

Bu this instruction is not correct (the member n in proposition 2.6.28 in ([10], p.207) depends for r_{1i}, r_{2i}). By this instruction the theorem may be proved for only Noetherian rings.

Further we investigate connections between radical \mathcal{D} -ideals and \mathcal{D} -radical \mathcal{D} -ideals in \mathcal{D} -rings.

If $\mathcal{D} = \emptyset$ then in §1 we obtain results for usual rings.

Corollary *Let K be a differential \mathcal{Q} -algebra with 1. Then radical of any \mathcal{D} -ideal of K is a \mathcal{D} -ideal of K .*

Proof. Let H be a \mathcal{D} -ideal of K . Then by Theorem 2.1 $Pr.rad(K/H)$ is a \mathcal{D} -ideal. This means that $rad(H)$ is a \mathcal{D} -ideal of K . □

Let K be a ring. For $a \in K$ denote by (a) the intersection of all ideals of K containing a .

For ideals A, B of K denote by $A \star B$ the intersection of all ideals of K containing the set $\{x \in K : x = a \cdot b, a \in A, b \in B\}$.

Proposition 2.2 *Let K be a ring and B be an ideal of K . Then B is prime iff for every $t_1, t_2 \in K \setminus B$ such that $t_1 \star t_2 \in B$.*

A proof is obvious.

Theorem 2.3 *Let H be a radical \mathcal{D} -ideal of a \mathcal{D} -rign K . Then H is an intersection of \mathcal{D} -prime \mathcal{D} -ideals.*

Proof. Let $x \notin H$. Then there exists a prime ideal B of K such that $H \subseteq B$ and $x \notin B$.

Denote by Σ the set of \mathcal{D} -ideals A of K such that $H \subseteq A$ and $A \cap (K \setminus B) = \emptyset$. $\Sigma \neq \emptyset$ since $H \in \Sigma$. By Zorn's lemma there exists a maximal element P in Σ . Prove that P is a \mathcal{D} -prime \mathcal{D} -ideal. P is proper since $x \notin P$.

Let $A_1, A_2 \in Id(K)$ and $A_1 \not\subseteq P, A_2 \not\subseteq P$. Then $P \vee A_1 \neq P, P \vee A_2 \neq P$. There exist $t_1 \in K \setminus B, t_2 \in K \setminus B$ such that $t_1 \in P \vee A_1, t_2 \in P \vee A_2$. Then

$$(t_1) \subseteq P \vee A_1, \quad (t_2) \subseteq P \vee A_2.$$

By proposition 1.1 we have

$$(t_1) \star (t_2) \subseteq (P \vee A_1) \cdot (P \vee A_2) \subseteq P \vee A_1 \cdot A_2.$$

By proposition 2.2 for $t_1, t_2 \in K \setminus B$ there exists $t \in K \setminus B$ such that $t \in (t_1) \star (t_2)$. Then $t \in P \vee A_1 \cdot A_2$. This means that $A_1 \cdot A_2 \not\subseteq P$. Therefore P is a \mathcal{D} -prime \mathcal{D} -ideal and $H \subseteq P$. Thus for any $x \notin H$ there exists a \mathcal{D} -prime \mathcal{D} -ideal P such that $H \subseteq P$ and $x \notin P$. This means that H is an intersection of all \mathcal{D} -prime \mathcal{D} -ideals containing H . \square

Theorem 2.4 *Let K be a differential \mathcal{Q} -algebra with 1. Then $Pr.rad(K)$ is an intersection of all \mathcal{D} -prime \mathcal{D} -ideals of K containing $Pr.rad(K)$ and $\mathcal{D}Pr.rad(K) \subseteq Pr.rad(K)$.*

Proof. By theorem 2.1 $Pr.rad(K)$ is a \mathcal{D} -ideal of K . By theorem 2.3 $Pr.rad(K)$ is an intersection of all \mathcal{D} -prime \mathcal{D} -ideals of K containing $Pr.rad(K)$. Hence $\mathcal{D}Pr.rad(K) \subseteq Pr.rad(K)$. □

Theorem 2.5 *Let K be a differential \mathcal{Q} -algebra with 1. Assume that K satisfies the ascending chain condition for ideals. Then $\mathcal{D}Pr.rad(K) = Pr.rad(K)$.*

Proof. In this case $Pr.rad(K)$ is s -nilpotent by theorem 1.7 $\mathcal{D} = \emptyset$. Therefore $Pr.rad(K)^{(n)} = 0$ for some n . Since $Pr.rad(K)$ is a \mathcal{D} -ideal $[Pr.rad(K)]^{(r)} = (Pr.rad(K))^{(n)} = 0$. Then by theorem 1.10 $Pr.rad(K) \subseteq \mathcal{D}Pr.rad(K)$. Thus $\mathcal{D}Pr.rad(K) = Pr.rad(K)$. □

Corollary *Let K be a differential \mathcal{Q} -algebra with 1. Assume that K satisfies the ascending chain condition for ideals. Then every \mathcal{D} -radical \mathcal{D} -ideal of K is radical.*

Proof. The statement follows from theorem 2.5. □

Remark *Theorems 2.3-2.5 are known for commutative differential rings [1].*

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