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Seiberg-Witten invariants of mapping tori,symplectic fixed points, and Lefschetz numbers

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Seiberg-Witten invariants of mapping tori, symplectic fixed points, and Lefschetz numbers

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1. Introduction

Let Y be a compact oriented smooth 3-manifold with nonzero first Betti number. Two nonzero vector fields on Y are called **homologous** if they are homotopic over the complement of a ball in Y. An **Euler structure** on Y is an equivalence class of homologous vector fields (see Turaev [33]). Let $\mathcal{E}(Y)$ denote the space of Euler structures on Y. If Y carries a Riemannian metric then an Euler structure can also be defined as a cohomology class $e \in H^2(SY;\mathbb{Z})$ on the unit sphere bundle SY in TY which restricts to a positive generator on each fiber (with the orientation given by the complex structure $\eta \mapsto v \times \eta$). The correspondence assigns to each unit vector field $v : Y \to SY$ the Euler structure

$$e_v = \mathrm{PD}(v_*[Y]) \in H^2(SY;\mathbb{Z}).$$

With the second description it follows that there is a free and transitive action of $H^2(Y;\mathbb{Z})$ on the space of Euler structures, given by

$$H^2(Y;\mathbb{Z}) \times \mathcal{E}(Y) \to \mathcal{E}(Y) : (h,e) \mapsto h \cdot e = e + \pi^* h.$$

Moreover there is a natural map

$$\mathcal{E}(Y) \to H^2(Y;\mathbb{Z}) : e \mapsto c(e)$$

which assigns to $e = \mathrm{PD}([v])$ the Euler class of the normal bundle $v^\perp$. These maps are related by $c(h \cdot e) = c(e) + 2h$. Turaev introduces a torsion invariant

$$\mathcal{T} : \mathcal{E}(Y) \to \mathbb{Z}$$

which is a kind of refinement of the Reidemeister-Milnor torsion. In the case $b_1(Y) = 1$ this function depends on a choice of orientation of $H_1(Y)$.

A unit vector field $v : Y \to SY$ also determines a spin^c structure γ_v on Y (see Example 3.1 below). Turaev [33] observes that two such spin^c structures γ_{v_0} and γ_{v_1} are isomorphic if and only if the vector fields v_0 and v_1 are homologous, and hence there is a natural bijection between $\mathcal{E}(Y)$ and the set $\mathcal{S}^c(Y)$ of isomorphism classes of spin^c structures on Y (see also [26]). Now the Seiberg-Witten invariants of Y take the form of a function

$$\mathrm{SW} : \mathcal{S}^c(Y) \to \mathbb{Z}$$

H_1 Y b_1 Y

Y Y $_1$ Y

T *Let be a compact oriented Riemann surface and f be an orientation preserving diffeomorphism. Denote by Y_f the mapping torus of f. Then*

$$Y_f, \gamma \qquad Y_f, e$$

for every nonzero vector field v on Y_f.

$\partial/\partial t$ e_f Y_f γ_f Y_f e_f Y_f $\sim$ Y_f Y_f Y_f H^2 Y_f H^2 Y_f Y_f f E E f

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & E \\ \downarrow & & \downarrow \\ & \xrightarrow{f} & \end{array} .$$

d E $\langle c_1$ E $,$ $\rangle$ $e_{d,\tilde f}$ Y_f $\gamma_{d,\tilde f}$ Y_f f

$$Y_f, \gamma_{d,\tilde f} \qquad Y_f, e_{d,\tilde f}$$

E f E E f

2

M ϕ M M Ω_ϕ x $\mathbb{R}$ M x t ϕ x t . x Ω_ϕ x π_0 Ω_ϕ $\phi_0, \mathcal{P}_0$ $\phi_1, \mathcal{P}_1$ $\mathcal{P}_i$ π_0 Ω_{ϕ_i} **nj** ψ M M ϕ_1 $\psi^{-1} \circ \phi_0 \circ \psi$ $\mathcal{P}_1$ ψ $\mathcal{P}_0$ **pi** s ϕ_s ϕ_0 ϕ_1 , $\mathbb{R}$ M s, t x_s t x_s Ω_{ϕ_s} s x_0 $\mathcal{P}_0$ x_1 $\mathcal{P}_1$ x ϕ x Ω_ϕ $\mathcal{P}$ π_0 Ω_ϕ $\phi, \mathcal{P}$

---

[1] While this paper was written the author received a message that Turaev had proved the conjecture for general 3-manifolds [34]. Turaev's proof is based on the work by Meng-Taubes [20].

x ϕ x $\mathcal{P}$ ϕ x ϕ
n nd n $- d\phi\ x\ /$

$$x, \phi \qquad - d\phi\ x$$

fix d p in ind x x
L f z inv i n $\phi, \mathcal{P}$ ϕ M M
$\mathcal{P}$ $\pi_0\ \Omega_\phi$

(Fix d p in ind x): ϕ $\phi, \mathcal{P}$

$$L\ \phi, \mathcal{P} \qquad \sum_{x \in \mathrm{Fix}(\phi, \mathcal{P})} \qquad x, \phi\ .$$

(H py):
(N i y):
(T f): L f z n b f ϕ

$$L\ \phi \qquad \sum_{\mathcal{P} \in \pi_0(\Omega_\phi)} L\ \phi, \mathcal{P} \qquad \sum_i - \quad {}^i \qquad \phi \quad H_i\ M \qquad H_i\ M\ \ .$$

(Z f n i n): z f n i n ϕ

$$\zeta_\phi\ t \qquad \left(\sum_{k=1}^{\infty} \frac{t^k}{k} L\ \phi^k \ \right)$$

$$\prod_{i=0}^{\dim M} \qquad - tH_i\ \phi\ \ {}^{(-1)^{i+1}}$$

$$\sum_{d=0}^{\infty} t^d L\ S^d \phi\ .$$

ϕ^k k ϕ $S^d\phi$ S^dM S^dM
d S^dM ϕ

(P d f): ϕ

$$\zeta_\phi\ t \quad \prod_{k=1}^{\infty} \prod_{\bar{x} \in \mathcal{P}(\phi, k)/\mathbb{Z}_k} \qquad - \varepsilon\ x, \phi^k\ t^k\ {}^{-\varepsilon(x,\phi^k)\mathrm{ind}(x,\phi^k)}.$$

$\varepsilon\ x, \phi^k$ $d\phi^k\ x$ $\mathcal{P}\ \phi, k$
k

"homotopy" *"fixed point index"*
"trace formula" *"product formula"*

Proof of (1) and the product formula.

$$- A \; ^{-1} \qquad \left(\sum_{k\geq 1} \frac{A^k}{k} \right).$$

$$L \; S^d \phi \quad \sum_{j=0}^{d} - \quad ^{j} \qquad ^{j} H_{\mathrm{odd}} \; \phi \qquad S^{d-j} H_{\mathrm{ev}} \; \phi \quad ,$$

$$- A \quad \sum_{j\geq 0} - \quad ^{j} \qquad ^{j} A \;, \qquad - A \; ^{-1} \quad \sum_{k\geq 0} - \quad ^{k} \qquad S^k A \;.$$

$$x, \phi^{k\ell} \qquad x, \phi^k \; \varepsilon \; x, \phi^k \; ^{\ell-1}.$$

$p^{\pm} \; \phi, k$ $\mathcal{P} \; \phi, k \; / \; _k$

$\varepsilon \; x, \phi^k$ $\pm$

$$L \; \phi^k \quad \sum_{n|k} \frac{k}{n} \left(p^{+} \; \phi, k/n \qquad - \quad ^{n-1} p^{-} \; \phi, k/n \; \right).$$

$$\sum_{k=1}^{\infty} t^k L \; \phi^k \quad \sum_{k=1}^{\infty} \left(p^{+} \; \phi, k \; \frac{kt^k}{- t^k} \quad p^{-} \; \phi, k \; \frac{kt^k}{t^k} \right).$$

t □

f

$f \quad E \quad E$ d d

$\mathcal{P}_{d,\tilde{f}} \quad \pi_0 \; \Omega_{S^d f}$ d $\mathcal{P}_{1,\tilde{f}}$

$\Omega_{S^d f}$

T 2 *Let be a compact oriented Riemann surface, E be a Hermitian line bundle of degree d, f be an orientation preserving diffeomorphism and $f \quad E \quad E$ be an automorphism that descends to f. Then*

$$Y_f, \gamma_{d,\tilde{f}} \qquad L \; S^d f, \mathcal{P}_{d,\tilde{f}} \;.$$

Theorem 2.1 implies Theorem 1.1.

$$Y_f, e_{d,\tilde{f}} \qquad L \; S^d f, \mathcal{P}_{d,\tilde{f}} \;.$$

α

fl

α

α

$\square$

C y 2 2 *Let be a compact oriented Riemann surface of genus g and f be an orientation preserving diffeomorphism. Then*

$$\sum_{\gamma \in \mathcal{S}^c(Y_f)} \quad Y_f, \gamma \; t^{\;(\gamma)\cdot\Sigma/2} \quad t^{1-g}\zeta_f \; t \; ,$$

Proof. $\gamma_{d,\tilde{f}}$ $c\;\gamma_{d,\tilde{f}}$ d $-\;g$ $\square$

ζ_f

f H^1 H^1 $b_1 \; Y_f$

3

Y Y W, γ W Y

γ TY W

$$\gamma \; v \; \gamma \; w \quad \gamma \; v \quad w \; - \langle v, w \rangle$$

v, w T_yY **i i** γ $c\;\gamma$ $c_1 \; W$ $H^2 \; Y$

Example . v Y TY W $, \gamma$ W $\mathbb{C} \oplus v$

$$\gamma \quad \eta \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} \quad \begin{pmatrix} -i\langle \eta, v\rangle\theta_0 \quad \langle \eta, \theta_1\rangle \quad i\langle v \quad \eta, \theta_1\rangle \\ \langle \eta, v\rangle v \quad \theta_1 - \quad \theta_0 \quad \eta - \langle \eta, v\rangle v \; - \quad \theta_0 \; v \quad \eta \end{pmatrix}$$

θ_0 $\mathbb{C}$ θ_1 v η TY $c\;\gamma$ $c_1 \; v$

$\mathcal{A}\;\gamma$ $W^{\;1/2}$ W A $\mathcal{A}\;\gamma$ ∇_A W TY Y

$$\mathcal{D}_A \quad , \quad \gamma \; {*F_A} \quad {*\eta} \quad\quad {}_0,$$

A $\mathcal{A}\;\gamma$ $C^\infty \; Y, W$ $\mathcal{D}_A$ $C^\infty \; Y, W$ $C^\infty \; Y, W$ ∇_A F_A $\Omega^2 \; Y, i\mathbb{R}$ A ${}_0$ $C^\infty \; Y,$ W ${}_0\theta$ $\langle \; , \theta\rangle \; - | \; |^2\theta/$ θ $C^\infty \; Y, W$ TY $T \; Y$ γ $T \; Y \otimes \mathbb{C}$ ${}_0 \; W$ W W

η $\Omega^2 \; Y, i\mathbb{R}$ $d\;\gamma^{-1}$ ${}_0$ i $\langle \mathcal{D}_A \quad , \quad \rangle$ η

Remark . **(i)**

$\mathcal{C}$ $\mathcal{D}_\eta$ $\mathcal{A}$ γ C^∞ Y, W $\mathbb{R}$

$$\mathcal{C} \quad \mathcal{D}_\eta \ A, \qquad -- \int_Y A - A_0 \ \wedge \ F_A \quad F_{A_0} \qquad \eta \ -- \int_Y \quad \langle \mathcal{D}_A \quad , \quad \rangle \qquad .$$

(ii) $A,$ $\not\equiv$

$$_Y \ | \quad |^2 \leq \quad _Y \left(\ |\eta| - \frac{s}{} \right),$$

s Y $\mathbb{R}$

(iii)

$\mathcal{H}_{A,\Theta}$ $\Omega^0 \ Y, i\mathbb{R} \ \oplus \Omega^1 \ Y, i\mathbb{R} \ \oplus C^\infty \ Y, W$

$$\mathcal{H}_{A,\Theta} \begin{pmatrix} \psi \\ \alpha \\ \theta \end{pmatrix} \quad \begin{pmatrix} & d \ \alpha - i \quad \langle \ , \theta \rangle & \\ d\psi & *d\alpha - \gamma^{-1} \ \theta & \theta \ \ _0 \\ & -\mathcal{D}_A\theta - \gamma \ \alpha \quad - \psi & \end{pmatrix}.$$

$A,$ $/$

$$\mathcal{H}_{A,\Theta}\mathcal{H}_{A,\Theta} \begin{pmatrix} \psi \\ \alpha \\ \theta \end{pmatrix} \quad \begin{pmatrix} \psi \quad | \quad |^2\psi \\ \alpha \quad | \quad |^2\alpha - \ i \quad \langle \nabla_A \ , \theta \rangle \\ \mathcal{D}_A\mathcal{D}_A\theta \quad | \quad |^2\theta - \ \nabla_{A,\alpha} \end{pmatrix}$$

ψ, α, θ $\mathcal{H}_{A,\Theta}$ ψ $d^2\mathcal{C}$ $\mathcal{D}_\eta$ $A,$

$$\Omega^1 \ Y, i\mathbb{R} \quad C^\infty \ Y, W \ / \{ \ d\xi, -\xi \quad | \ \xi \quad \Omega^0 \ Y, i\mathbb{R} \ \}.$$

$A,$ $/$ **n nd n** $\mathcal{H}_{A,\Theta}$

η

$A,$

$\mu^{\mathrm{SW}} \ A,$ fl

$- \ ,$ $\ni s$ $\mathcal{H}_s$ $\mathcal{H}_s$ $\mathcal{H}_{A,s\Theta}$ $\leq s \leq$

$$\mathcal{H}_s \quad \begin{pmatrix} s\pi_0 & d & \\ d & *d \quad s\pi_1 & \\ & & \mathcal{D}_A \end{pmatrix}, \qquad - \ \leq s \leq \ .$$

$s <$ fl

$\mu^{\mathrm{SW}} \ A,$ $\mathcal{H}_{A,\Theta}$

$$\mu^{\mathrm{SW}} \ u \ A, u^{-1} \quad - \mu^{\mathrm{SW}} \ A, \qquad \left[\frac{u^{-1}du}{\pi i} \right] \ c_1 \ W$$

u Y S^1

Y, γ

$$Y, \gamma \qquad \sum_{[A,\Theta]\in \mathrm{Crit}(\mathcal{CSD}_\eta)} - \quad \mu^{\mathrm{SW}}(A,\Theta)$$

η

b_1 Y $>$ η

γ

Remark . b_1 Y

η η

$$\left[\frac{i\eta}{\pi}\right] \quad c_1 \; W$$

H^1 Y

g Y α_g Ω^1 Y

H^1 Y

$$\varepsilon_\gamma \; g, \eta \qquad -\int_Y \frac{i\eta}{\pi} \wedge \alpha_g - c_1 \; W \qquad \alpha_g \; <$$

V q

g ω Ω^2

$\mathcal{J}$

E

$$d \quad \langle c_1 \; E \; , \quad \rangle$$

$\mathcal{A}$ E E J $\mathcal{J}$

$\mathcal{A}$ E E

A $\mathcal{A}$ E J $\mathcal{J}$

$\partial_{J,A}$ C^∞ $, E$ $\Omega^{0,1}_J$ $, E$

J

$$\partial_{J,A} \; {}_0 \qquad , \qquad *iF_A \qquad \frac{| \; {}_0|^2}{} \qquad \tau$$

A $\mathcal{A}$ E ${}_0$ C^∞ $, E$ τ $\mathbb{R}$

$$\int_\Sigma \tau\omega > \; \pi d.$$

$$\mathcal{M}\ J,\tau \qquad \mathcal{M}_{\Sigma,d}\ J,\tau \qquad \frac{\{\ A, \quad _0 \qquad \mathcal{A}\ E \qquad C^\infty \quad ,E\ | \qquad \}}{,S^1}.$$

$\mathcal{A}\ E$ $C^\infty \ ,E$ Ω

$$\Omega \quad \alpha,\theta_0\ ,\ \alpha\ ,\theta_0 \qquad -\int_\Sigma \alpha\wedge\alpha \qquad \int_\Sigma \qquad \langle\theta_0,\theta_0\rangle\omega$$

$\alpha,\theta_0 \qquad *\alpha, i\theta_0\ .$ $\mathcal{G}$ $,S^1$

$$\mathcal{A}\ E \qquad C^\infty \quad ,E \qquad C^\infty \qquad A, \quad _0 \qquad *iF_B \quad |\quad _0|^2/\ .$$

$$\mathcal{X}_J \quad \{\ A, \quad _0\ |\ \partial_A \quad _0 \qquad , \quad _0 \not\equiv \quad \}$$

$\mathcal{A}\ E \qquad C^\infty \quad ,E$ $\mathcal{G}$

$\mathcal{M}\ J,\tau$

$\mathcal{X}_J /\!\!/ \mathcal{G}\ \tau$

Remark . $\mathcal{M}_{\Sigma,d}\ J,\tau$ $A, \quad _0$ θ_0,α_1 $C^\infty \quad ,E$ $\Omega^{0,1}$

$$\partial_{J,A}\theta_0 \quad \alpha_1 \quad _0 \qquad , \qquad \partial_J\ \alpha_1 - -\langle \quad _0,\theta_0\rangle \qquad .$$

α_1 , α $\Omega^1 \quad ,i\mathbb{R}$ $\partial\ \alpha^{0,1}$ $d\ \alpha - *id\alpha$ $*id\alpha$ $\langle \quad _0,\theta_0\rangle$ $d\ \alpha - i \quad \langle \quad _0,\theta_0\rangle$.

$\mathcal{D}_{A,\Theta_0}$ $\mathcal{D}_{A,\Theta_0}\ \mathcal{D}_{A,\Theta_0}$ $\bar\partial$ $|\quad _0|^2/$ $\mathcal{M}\ J,\tau$

Remark . E

$$_{\Sigma,d}\ J \qquad \frac{\mathcal{A}^\omega\ E}{\mathcal{G}} \sim \frac{\mathcal{A}\ E}{\mathcal{G}}, \qquad \mathcal{A}^\omega\ E \qquad \left\{A\,|\,*iF_A \qquad \frac{\pi d}{\quad}\right\}.$$

$\mathcal{G}$ $,\mathbb{C}$ $\mathcal{A}\ E$

$$u\ A \quad A \quad u^{-1}\partial u - u^{-1}\partial u.$$

u e^{-f} $\mathbb{R}$ $u\ A \quad A \quad *idf$ $*iF_{u^*A} - *iF_A$ $d\ df$ $u\ A$ $\mathcal{A}^\omega\ E$ $d\ df$ $\pi d/$ $- *iF_A$ f $\mathcal{A}\ E$ $\mathcal{A}^\omega\ E$

Remark . $\mathcal{M}_{\Sigma,d}\ J,\tau$ $\mathcal{X}_J/\mathcal{G}$

u e^{-f} $\mathbb{R}$

$*iF_{u^*A} - *iF_A$ $d\ df$ $u\ A, u^{-1}\ {}_0$

$$d\ df \quad e^{2f} \frac{|\ {}_0|^2}{} \quad \tau - *iF_A.$$

f $\mathbb{R}$

$$\mathcal{M}_{\Sigma,d}\ J,\tau \quad \mathcal{X}_J /\!\!/ \mathcal{G}\ \tau \ \sim \mathcal{X}_J/\mathcal{G}\ .$$

$\mathcal{M}_{\Sigma,d}\ J,\tau$ ${}_{\Sigma,d}\ J$

$A,\ {}_0$ A $\partial_A\ \leq$

$A\ \ \mathcal{A}\ E$

Remark . $\mathcal{M}_{\Sigma,d}\ J,\tau\ \sim \mathcal{X}_J/\mathcal{G}$

$$\mathcal{M}_{\Sigma,d}\ J,\tau\ \sim S^d \quad \frac{}{S_d}.$$

$\mathcal{X}_J$ S^d $A,\ {}_0$ ${}_0$

J $\mathcal{J}$ S^d

J

5 T

$\mathcal{M}_{\Sigma,d}\ J,\tau$

τ

T **5** *Let* , $\mathcal{J}$ C^∞ t $J\ ,\tau$ *be a smooth function such that* $\int_\Sigma \tau\,\omega$ *and choose* , Ω^1 t σ *such that* τ $*d\sigma$ *. Then there is a symplectomorphism*

$$\psi \quad \psi_{\{J_t,\tau_t,\sigma_t\}}\ \mathcal{M}\ J_0,\tau_0 \quad \mathcal{M}\ J_1,\tau_1$$

defined by A , ${}_0$ A , ${}_0$, *where*

$$iA \quad \langle\ {}_0,\ {}_1\rangle - \sigma, \quad i\ {}_0\ \ \partial_{J,A}\ {}_1,$$

and ${}_1$ ${}_1\ t$ $\Omega^{0,1}_{J_t}$ $,E$ *is the unique solution of the elliptic equation*

$$\partial_{J,A}\,\partial_{J,A}\ {}_1 \quad \frac{|\ {}_0|^2}{}\ {}_1 \quad - \partial_{J,A}\ {}_0 \circ J \quad \sigma^{0,1}\ {}_0.$$

If J_0 J_1, τ_0 τ_1, *and* $\int_0^1 \sigma_s\,ds$ *then* ψ *is Hamiltonian.*

σ $* df$ f $\mathbb{R}$

τ d ${}^t df$ $\psi_{\{J_t,\tau_t\}}$ $\mathcal{M}$ J_0, τ_0 $\mathcal{M}$ J_1, τ_1

niv H i ni n nn i n $\mathcal{J}$ C^∞_m

$\mathcal{M}$ J, τ C^∞_m

$m >$ πd

Remark . A t ${}_0$ t ${}_1$ t

i $A - d\Psi$ $\langle$ ${}_0,$ ${}_1\rangle - \sigma,$ i ${}_0$ Ψ ${}_0$ $\partial_{J,A}$ ${}_1,$

, $\mathcal{G}$ t u t

$u^{-1}u$ Ψ .

A A $u^{-1}du,$ ${}_0$ u^{-1} ${}_0,$ ${}_1$ u^{-1} ${}_1$

Exercise . J $\equiv J$ τ $\equiv \tau$ ψ $\mathcal{M}$ J, τ $\mathcal{M}$ J, τ

σ dh ψ

H $A,$ ${}_0$ $-\int_\Sigma ih$ $F_A.$ $\{\psi\}$ H^1 $\mathcal{M}$ J, τ

$$T_{[A,\Theta_0]}\mathcal{M}\ J,\tau \quad \mathbb{R} \quad \alpha,\theta_0 \quad \int_\Sigma i\sigma\wedge\alpha, \quad \sigma \quad \int_0^1 \sigma_s\, ds.$$

ﬂ σ

$$\mathcal{X}_{J,\sigma} \quad \{ \ A, \ {}_0 \quad \mathcal{A}\ E \quad C^\infty\ X,E\ |\ \partial_{J,A+i\sigma}\ {}_0 \quad , \ {}_0 \not\equiv \ \}$$

J $\mathcal{J}$ σ Ω^1

L 5 2 *For every* J $\mathcal{J}$ *and every* α Ω^1 *the space* $\mathcal{X}_{J,\sigma}$ *is a complex submanifold of* $\mathcal{A}$ E C^∞ X, E *with respect to the complex structure* α, θ_0 $*_J\alpha, i\theta_0$.

Proof. $\mathcal{X}_{J,\sigma}$ $A,$ ${}_0$

$\mathcal{D}_{J,A+i\sigma,\Theta_0}$ Ω^1 $, i\mathbb{R}$ C^∞ $, E$ $\Omega^{0,1}$ $, E$

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}\ \alpha,\theta_0 \quad \partial_{J,A+i\sigma}\theta_0 \quad \alpha^{0,1} \quad {}_0.$$

$*_J\alpha$ ^{0,1} $i\alpha^{0,1}$ L^2

$\mathcal{D}_{J,A+i\sigma,\Theta_0}$ $\Omega^{0,1}$ $, E$ Ω^1 $, i\mathbb{R}$ C^∞ $, E$

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}\ \theta_1 \quad i \quad \langle \ {}_0, \theta_1\rangle,\ \partial_{J,A+i\sigma}\ \theta_1 \ .$$

i $\langle$ ${}_0, \theta_1\rangle$ ^{0,1} $\langle$ ${}_0, \theta_1\rangle/$

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}\mathcal{D}_{J,A+i\sigma,\Theta_0}\ \theta_1 \quad \partial_{J,A+i\sigma}\,\partial_{J,A+i\sigma}\ \theta_1 \quad -|\ \ {}_0|^2\,\theta_1.$$

$\mathcal{D}_{J,A+i\sigma,\Theta_0}$ $\mathcal{X}_{J,\sigma}$

∎

$\mathcal{M}$ J, τ

$$\bigcup_{J,\sigma} \{ J, \sigma \} \quad \mathcal{X}_{J,\sigma} \quad - \quad \mathcal{J} \qquad \Omega^1 \quad .$$

$\mathcal{J}$ Ω^1 $\mathcal{A}\, E$ C^∞ $, E$

$J, \sigma, A,$ $_0$ Ω $T_{(A,\Theta_0)}\mathcal{X}_J$

niv y p i nn i n n

P p i i n 5 3 *A smooth path* , t $J\ t\ , \sigma\ t\ , B\ t\ ,$ $_0\ t$ *is horizontal with respect to the universal connection on* *if and only if*

$$iA \quad \langle \ _0, \ _1 \rangle, \quad i \ _0 \quad \partial_{J,A+i\sigma} \ _1,$$

$$\partial_{J,A+i\sigma}\, \partial_{J,A+i\sigma} \ _1 \quad \frac{| \ _0|^2}{} \ _1 \quad - \ \partial_{J,A+i\sigma} \ _0 \circ J \quad \sigma^{0,1} \ _0.$$

Every horizontal path satisfies

$$\frac{d}{dt}\left(*iF_A \quad \frac{| \ _0|^2}{} \right) \quad .$$

Proof. t $J\ t\ , \sigma\ t\ , A\ t\ ,$ $_0\ t$

$$*_J A, i \ _0 \ \perp \quad \mathcal{D}_{J,A+i\sigma,\Theta_0}$$

t

$$*_J A, i \ _0 \quad \mathcal{D}_{J,A+i\sigma,\Theta_0} \ .$$

$$*_J A \quad i \quad \langle \ _0, \ _1 \rangle, \quad i \ _0 \quad \partial_{J,A+i\sigma} \ _1$$

$_1$ $\Omega^{0,1}$ $, E$ $*_J$ $\langle \ _0, \ _1 \rangle$ $\langle \ _0, i \ _1 \rangle$ $\langle \ _0, \ _1 \rangle,$ $A,$ $_0$ $\mathcal{X}_{J,\sigma}$ t

$$\frac{d}{dt}\partial_{J,A+i\sigma} \ _0$$

$$\partial_{J,A+i\sigma} \ _0 \quad A^{0,1} \ _0 \quad i\sigma^{0,1} \ _0 \quad \frac{i}{} \ d_{A+i\sigma} \ _0 \circ J$$

$$-i\partial_{J,A+i\sigma}\, \partial_{J,A+i\sigma} \ _1 - i\frac{| \ _0|^2}{} \ _1 \quad \frac{i}{} \ \partial_{J,A+i\sigma} \ _0 \circ J \quad i\sigma^{0,1} \ _0$$

$_1$

t $J\ t$ $, \sigma\ t$ $, A\ t$ $,$ ${}_0\ t$

A $,$ ${}_0$ $\mathcal{X}_{J(0),\sigma(0)}$

$$\frac{d}{dt}\partial_{J,A+i\sigma}\ {}_0 \quad \frac{i}{}\ \partial_{J,A+i\sigma}\ {}_0 \circ J$$

$\partial_{J,A+i\sigma}\ {}_0$ t

$\partial_{J,A+i\sigma}\theta_0$ $\alpha^{0,1}\ {}_0$ $*_J A$ i $\langle i\ {}_0, \ {}_1\rangle,$

$$\Omega\ A, \ {}_0\ , \ \alpha, \theta_0 \quad \int_\Sigma \Big(\ \langle *_J A, \alpha\rangle \quad \langle i\ {}_0, \theta_0\rangle\Big)\,\omega$$

$$\int_\Sigma \big(\ \langle i\ \ \langle i\ {}_0, \ {}_1\rangle, \alpha\rangle \quad \langle \partial_{J,A+i\sigma}\ {}_1, \theta_0\rangle\big)\,\omega$$

$$\int_\Sigma \ \langle\ {}_1, \partial_{J,A+i\sigma}\theta_0 \quad \alpha^{0,1}\ {}_0\rangle\,\omega$$

$.$

$d\ \langle\ {}_0, \ {}_1\rangle$ $\langle\ {}_0, \partial_{J,A+i\sigma}\ {}_1\rangle - \langle\partial_{J,A+i\sigma}\ {}_0, \ {}_1\rangle.$ $*idA$

$d\ *_J iA$ d $\langle\ {}_0, i\ {}_1\rangle$

$$\frac{d}{dt}\left(*iF_A \quad \frac{|\ {}_0|^2}{}\right) \quad *idA \quad \langle\ {}_0, \ {}_0\rangle$$

$$d \quad \langle\ {}_0, i\ {}_1\rangle - \quad \langle\ {}_0, i\,\partial_{J,A+i\sigma}\ {}_1\rangle$$

$$- \quad \langle\partial_{J,A+i\sigma}\ {}_0, i\ {}_1\rangle$$

$.$

□

Proof of Theorem 5.1. $A\ t$ $\mathcal{A}\ E$ $\sigma\ t$ Ω^1 $, E$

$$A\ t \quad A\ t\ - i\sigma\ t\ , \quad \sigma\ t \quad \int_0 \sigma_s\,ds.$$

$\mathcal{X}_{J(\ )}$ $\mathcal{X}_{J(\ ),\sigma'(\ )}$ $A\ t$ $,$ ${}_0\ t$ $A\ t$ $,$ ${}_0\ t$ ¨

iA iA σ $\langle\ {}_0, \ {}_1\rangle,$ $i\ {}_0$ $\partial_{J,A}\ {}_1$ $\partial_{J,A'+i\sigma'}\ {}_1$

1 A σ A σ

$\mathcal{X}_{J(0),\sigma'(0)}$ $\mathcal{X}_{J(1),\sigma'(1)}$ A $,$ ${}_0$ A $,$ ${}_0$

$\mathcal{X}_{J(\ ),\sigma'(\ )}$ $\mathcal{X}_{J(\ )}$

$$\mathcal{X}_{J(0)} \xrightarrow{\tilde\psi} \mathcal{X}_{J(1)} \quad A \quad , \ {}_0 \qquad A \quad , \ {}_0$$

J J σ σ

$$\frac{d}{dt}\left(*iF_{A'} \quad \frac{|\ \ _0|^2}{\ }\right)$$

$$\frac{d}{dt}\left(\tau \ - *iF_A - \frac{|\ \ _0|^2}{\ }\right) \quad \frac{d}{dt}\ \tau \quad *id\ \ A\ - A \quad \frac{d}{dt}\tau \quad *d\sigma \quad ,$$

ψ J, τ J_0, τ_0

J, τ J_1, τ_1 ψ $\mathcal{M}$ J_0, τ_0 $\mathcal{M}$ J_1, τ_1

ψ J J $\int_0^1 \sigma_s\, ds$ σ σ

ψ ψ □

6

d $j \leq d$

$$H^j\ \ S^d \quad \sim \quad ^j \oplus \quad ^{j-2} \oplus \quad ,$$

^{j} $^j H^1$

$$\chi\ \ S^d \qquad \sum_{j=0}^{d} - \quad ^j\ d \qquad -j \begin{pmatrix} g \\ j \end{pmatrix} \qquad - \quad ^d \begin{pmatrix} g - \\ d \end{pmatrix}.$$

$$L\ \ S^d f \qquad \sum_{j=0}^{d} - \quad ^j\ d \qquad -j \qquad\qquad ^j f$$

$S^d f$ S^d f

H^1

d $E\ > \ g-$

E d

S^d $H^0\ \ , E$ d $-g$ $\mathbb{P}H^0\ \ , E\ \sim \mathbb{C}P^{d-g}$

$$\mathbb{C}P^{d-g} \hookleftarrow \quad S^d \quad - \qquad _{\Sigma, d}.$$

c_1 $c_1\ TS^d$

A $\pi_2\ \ S^d$

$$c_1\ \ A \qquad d \qquad -g$$

d $g-$ d

P p i i n 6 *The space*

$$\mathcal{M}_{\Sigma,d} \quad \mathcal{M}_{\Sigma,d}\ J,\tau \quad \{\ A, \quad {}_0 \quad \mathcal{A}\ E \quad C^{\infty} \quad ,E\ | \quad \}$$

is connected. If d *then* $\mathcal{M}_{\Sigma,d}$ *is simply connected and*

$$\pi_1\ \mathcal{M}_{\Sigma,d} \quad \pi_0\ \mathcal{G} \qquad {}^{2g}.$$

If d *then* $\mathcal{M}_{\Sigma,1} \sim$ *and* $\pi_1\ \mathcal{M}_{\Sigma,1}/S^1$ *is the Torelli group.*

Proof. $\mathcal{M}_{\Sigma,d}$

$$\mathcal{G} \quad \mathcal{M}_{\Sigma,d} \quad \mathcal{M}_{\Sigma,d}.$$

$A, \quad {}_0 \quad \mathcal{M}_{\Sigma,d} \quad {}_0 \quad d \quad \mathcal{M}_{\Sigma,d}$
$u \quad \mathcal{G} \quad A, \quad {}_0 \quad u\ A, u^{-1} \quad {}_0$
$\mathcal{M}_{\Sigma,d}$
$g \quad \pi_0\ \mathcal{G} \quad C \subset$
${}_0 \quad u \quad S^1 \quad u$
C

$$\left[\frac{u^{-1}du}{\pi i}\right] \qquad C \quad .$$

$A, \quad {}_0 \quad u\ A, u^{-1} \quad {}_0$
${}_0 \quad C \quad \mathcal{M}_{\Sigma,d}$
d

$$\pi_1\ S^d \quad \sim H_1 \qquad \sim \quad {}^{2g}.$$

c
$S^d \quad \gamma_1, \ldots, \gamma_d \quad S^1 \quad S^d \quad d$
$\gamma_i \quad S^1$

$$\gamma_1, \ldots, \gamma_d \ \sim\ c, \ldots, c, \gamma_1 \quad \gamma_d\ .$$

$\gamma_i \quad \pi_1\ S^d \quad \gamma \quad S^1$
$\alpha \quad H^1$
$\langle \alpha,\ \gamma\ \rangle \quad S^d$
$c, \ldots, c, \gamma \quad \pi_1\ S^d \quad H_1$
$d \quad J, A \quad \mathcal{J} \quad \mathcal{A}\ E$

$$\partial_{J,A} \quad .$$

$\mathbb{C}P^1 \quad \mathbb{C}^2$
$H \quad \mathbb{C}P^1 \quad \ell \quad \mathbb{C}P^1$
$\ell \quad \ell, \mathbb{C} \quad H \quad s\ \ell \quad \phi|_\ell \quad \phi$
$\mathbb{C}^2, \mathbb{C}$
$u \quad \mathbb{C}P^1 \quad d \quad E \quad u\ H \quad d$
$A \quad \mathcal{A}\ E \quad H \quad J \quad \mathcal{J}$

$\mathbb{C}P^1$ $\partial_{J,A}$

d $\mathcal{M}_{J,d}$ J
τ J

$$\pi_1 \ \mathcal{G} \quad \pi_1 \ \mathcal{M}_{\Sigma,d} \quad \pi_1 \ \mathcal{M}_{\Sigma,d} \quad \pi_0 \ \mathcal{G} \quad .$$

$\pi_1 \ \mathcal{M}_{\Sigma,d} \ \sim \ ^{2g}$ d $\pi_0 \ \mathcal{G} \ \sim \ ^{2g}$
$\pi_1 \ \mathcal{M}_{\Sigma,d}$ $\pi_0 \ \mathcal{G}$
$\pi_1 \ \mathcal{M}_{\Sigma,d}$ $\pi_1 \ \mathcal{M}_{\Sigma,d}$ $\pi_1 \ \mathcal{G}$
$\pi_1 \ \mathcal{G}$ $\pi_1 \ \mathcal{M}_{\Sigma,d}$

$$S^1 \quad \mathcal{M}_{\Sigma,d} \ J, \tau \quad e^i \quad A, e^i \quad _0 \ .$$

d J $\mathcal{J}$
A $\mathcal{A}$ E $\partial_{J,A}$
$\pi_1 \ \mathcal{G}$ $\pi_1 \ \mathcal{M}_{\Sigma,d}$
J J
$\mathcal{M}_{\Sigma,d}$ □

7

$$, \omega \ / \qquad , \omega \ - \qquad \mathcal{M} \ J, \tau \ , \Omega \ / \qquad \mathcal{M} \ J, \tau \ , \Omega \ .$$

$, \omega$
$, \omega$
f $, \omega$ f f E
f, f E E

$$f \quad m \ u \ \circ f \quad f \circ m \ u \circ f$$

u $\mathcal{G}$ $m \ u$ E E u $\mathbb{R}$ $\mathcal{J}$
t J

$$J_{\ +1} \quad f \ J \ .$$

ψ $\mathcal{M} \ J_0, \tau$ $\mathcal{M} \ J \ , \tau$
τ τ σ

$$\phi_{d,f} \quad \phi_{d,f,\{J_t\}} \quad \psi_1{}^{-1} \circ f \quad \mathcal{M} \ J_0, \tau \quad \mathcal{M} \ J_0, \tau$$

f
$\{J \ \}$

$\Omega_{\phi_{d,f}}$ $\mathcal{P}_{d,\tilde f}$

$\mathbb{R}$ $\mathcal{A}\, E$ C^∞ $, E$ t $A\, t\,,\ {}_0\, t$ $A\, t\,,\ {}_0\, t$

$\mathcal{M}\ J\,,\tau$

$$A\ t \quad f\ A\ t\,, \quad {}_0\ t \quad f \quad {}_0\ t\,.$$

$\mathcal{G}_f$ $\mathbb{R}$ $\mathcal{G}\ t$ $u\ t$

$$u\ t \quad t \quad u\ t\ \circ f$$

$$\mathcal{P}_{d,\tilde f} \quad \mathcal{P}_{d,\tilde f}/\mathcal{G}_f.$$

$\Omega_{\phi_{d,f}}$

t $A\ t\,,\ {}_0\ t$ $\mathcal{P}_{d,\tilde f}$ γ $\mathbb{R}$ $\mathcal{M}\ J_0, \tau$ $\gamma\ t$

$\psi^{\ -1}$ $A\ t\,,\ {}_0\ t$ $\Omega_{\phi_{d,f}}$ $\mathcal{P}_{d,\tilde f}$

f $\mathcal{P}_{d,\tilde f}$ $\Omega_{\phi_{d,f}}$

$$\pi_0\ \Omega_{\phi_{d,f}} \sim \frac{H^1}{-\ f}.$$

L 7 *Suppose that d . Then, for every unitary lift f E E of f, the space $\mathcal{P}_{d,\tilde f}$ is a connected component of $\Omega_{\phi_{d,f}}$. Two such lifts f and f determine the same component if and only if there exists a u $\mathcal{G}$ such that f f ∘ m u and*

$$\left[\frac{u^{-1}du}{\pi i}\right] \qquad -\ f \quad \subset H^1 \qquad .$$

Proof.

$\mathcal{P}_{d,\tilde f}$ $\mathcal{P}_{d,\tilde f}$ f f

f

(i): $\mathcal{P}_{d,\tilde f} \quad \mathcal{P}_{d,\tilde f'}$

(ii): $\mathcal{P}_{d,\tilde f} \cap \mathcal{P}_{d,\tilde f'}\ /\ \emptyset$

(iii): u $\mathcal{G}$ f $f \circ m\ u$

u S^1

σ Ω^1 $u^{-1}du/\ \pi i - \sigma$ f σ

v S^1 $v^{-1}dv/\ \pi i$ σ $v \circ f\ u$ S^1 v

$\mathbb{R}$ $\mathcal{G}$ t $v\ t$ v v

$$v\ t \qquad v\ t\ \circ f\ u.$$

t $A\ t\,,\ {}_0\ t$ $\mathcal{P}_{d,\tilde f}$

$$A\ t \quad v\ t\ \ A\ t\,, \quad {}_0\ t \quad v\ t^{\ -1}\ {}_0\ t\,, \quad f \quad f \circ m\ u\,.$$

$A \quad t \qquad v \ t \qquad A \ t$

$v \ t \qquad f \ A \ t$

$u \quad v \ t \ \circ f \quad f \ A \ t$

$u \ f \ v \ t \quad A \ t$

$f \quad A \quad t \ .$

$A \ t \qquad {}_0 \ t \qquad t$

$A \quad t \ , \quad {}_0 \ t \qquad \mathcal{P}_{d,\tilde{f}'}$

$$\mathcal{P}_{d,\tilde{f}} \quad \mathcal{P}_{d,\tilde{f}'} \quad \{A \ t \ , \quad {}_0 \ t \ \} \qquad \{v \ t \quad A \ t \ , v \ t \ ^{-1} \quad {}_0 \ t \ \} \ .$$

$\mathcal{P}_{d,\tilde{f}} \ / \ \emptyset$

□

$\phi_{d,f} \qquad \mathcal{P}_{d,\tilde{f}}$

$\mathbb{R} \qquad \mathcal{A} \ E \quad C^{\infty} \quad , i\mathbb{R} \quad C^{\infty} \quad , E \quad \Omega^{0,1} \quad , E$

$t \qquad A \ t \ , \Psi \ t \ , \quad {}_0 \ t \ , \quad {}_1 \ t$

$$\partial_{J_t,A} \quad {}_0 \qquad , \qquad *iF_A \quad \frac{| \quad {}_0|^2}{} \quad \tau,$$

$$* \quad A - d\Psi \quad i \quad \langle \quad {}_0, \quad {}_1 \rangle, \qquad i \quad {}_0 \quad \Psi \quad {}_0 \qquad \partial_{J,A} \quad {}_1,$$

$$\partial_{J_t,A} \partial_{J_t,A} \quad {}_1 \quad \frac{| \quad {}_0|^2}{} \quad {}_1 \quad - \ \partial_{J_t,A} \quad {}_0 \ \circ J \ ,$$

$$A \ t \qquad f \ A \ t \ , \qquad \Psi \ t \qquad \Psi \ t \ \circ f,$$
$${}_0 \ t \qquad f \quad {}_0 \ t \ , \qquad {}_1 \ t \qquad f \quad {}_1 \ t \ .$$

$A \ t \ , \quad {}_0 \ t \qquad \mathcal{M} \ J \ , \tau \qquad t$

$t \qquad A \ t \ , \quad {}_0 \ t$

$t \qquad A \ t \ , \quad {}_0 \ t \qquad \mathcal{P}_{d,\tilde{f}}$

$$A, \Psi, \quad {}_0, \quad {}_1 \qquad B \quad u^{-1}du, \Psi \quad u^{-1}u, u^{-1} \quad {}_0, u^{-1} \quad {}_1$$

$u \quad \mathcal{G}_f$

8 M

g ω

f $,\omega$

$$Y_f \quad \mathbb{R} \quad / \sim$$

$$t \quad ,z \sim t, f\, z \;.$$

$\mathbb{R}$ $\mathcal{J}$ J_{+1} f J

$$\langle\, ,\, \rangle \quad \omega \;, J \quad i\omega \;,$$

T J ω

Y_f

T n ni pin^c

Y_f $\{J\}$

γ_f TY_f W_f .

W_f Y_f

$$W_f \quad \Big\{ t, z, \quad _0, \quad _1 \,|\, t \quad \mathbb{R}, z \quad , \quad _0 \quad \mathbb{C}, \quad _1 \quad {}^{0,1}_{J_t} T_z \Big\} \Big/ \sim .$$

$$t \quad ,z, \quad _0, \quad _1 \sim t, f\, z\,, \quad _0, \quad _1 \circ df\, z^{\;-1}$$

γ_f

$$\gamma_f\; t, z\;\; \tau, \zeta \begin{pmatrix} _0 \\ _1 \end{pmatrix} \quad \begin{pmatrix} -i\tau \quad _0 - \sqrt{\;} \quad _1\, \zeta \\ i\tau \quad _1 \quad \langle\, ,\zeta\rangle \quad _0/\sqrt{\;} \end{pmatrix}$$

t, τ $\mathbb{R}$ ζ T_z γ

v $\partial/\partial t$ T ${}^{0,1}T$ θ_1 $_1$

$-\langle\, ,\theta_1\rangle/\sqrt{\;}$.

L 8 *Let* η $\eta_2 - \eta_1 \wedge dt$ $\Omega^2\; Y_f, i\mathbb{R}$ *, i.e.* $\eta_2\; t$ Ω^2 $, i\mathbb{R}$ *and* $\eta_1\; t$ Ω^1 $, i\mathbb{R}$ *satisfy* $\eta_i\; t$ f $\eta_i\; t$ *. Then*

$$\gamma_f \;\, *_3\; \eta_2 - \eta_1 \wedge dt \qquad\qquad _0$$

if and only if

$$*i\eta_2 \quad \frac{|\quad _0|^2 - |\quad _1|^2}{} \quad , \qquad *\eta_1 - i\sqrt{\;} \quad \langle\quad _0, \quad _1\rangle \quad .$$

Proof. $*$ Y_f

$$*_3 \ \eta_2 - \eta_1 \wedge dt \quad *_2\eta_2 dt \quad *_2\eta_1,$$

$*_2$ $*$ v T

η_1 Jv $*$ η_1 $-$ $\eta_1 \circ J$

$$\theta_1 \ Jv \quad \langle \eta_1^{0,1}, \theta_1 \rangle, \quad \langle \ , Jv \rangle \quad \eta_1^{0,1}$$

$$\gamma_f \ *_3 \ \eta_2 - \eta_1 \wedge dt \ \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} \quad \gamma_f \ *_2\eta_2 \ dt \quad *_2\eta_1 \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix}$$
$$\begin{pmatrix} -i \ *_2\eta_2 \ \theta_0 - i\sqrt{\ } \theta_1 \ Jv \\ i \ *_2\eta_2 \ \theta_1 \quad i\langle \ , Jv \rangle \theta_0 / \sqrt{\ } \end{pmatrix}$$
$$\begin{pmatrix} - \ *_2 i\eta_2 \ \theta_0 - i\sqrt{\ } \langle \eta_1^{0,1}, \theta_1 \rangle \\ *_2 i\eta_2 \ \theta_1 \quad i\sqrt{\ } \eta_1^{0,1} \theta_0 \end{pmatrix}.$$

$$_0\theta \quad \begin{pmatrix} \lambda\theta_0 \quad \langle \ _1, \theta_1 \rangle \ _0 \\ -\lambda\theta_1 \quad \langle \ _0, \theta_0 \rangle \ _1 \end{pmatrix}, \qquad \lambda \quad \frac{| \ _0|^2 - | \ _1|^2}{}$$

$*i\eta_2$ λ

$$\langle \ _0, \ _1 \rangle \quad i\sqrt{\ } \eta_1^{0,1} \quad i\eta_1/\sqrt{\ } - \eta_1 \circ J/\sqrt{\ } \quad i\eta_1/\sqrt{\ } \quad *\eta_1/\sqrt{\ }$$

η_1 i $\langle \ _0, \ _1 \rangle$ $*\eta_1/\sqrt{\ }$ □

T n ni pin^c nn i n

Y_f S^1

$\mathbb{C}$ W_f ∇ ∇_f W_f

∇_f ω , J

$\{t\}$ $\partial/\partial t$

$$\nabla \ _1 \quad _1 \ - \ _1 \circ JJ.$$

^{1} , $\nabla \ _1$ A_f

W_f $^{1/2}$ ∇_f A_f

$$F_{A_f} \quad -\frac{iK}{}\omega - \frac{\alpha}{} \wedge dt,$$

K $\mathbb{R}$ ω , J α Ω^1 , $i\mathbb{R}$

$$\alpha \ J \quad \nabla \quad -J\nabla J.$$

T S ib -Wi n q i n

E f E E f
E

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & E \\ \downarrow & & \downarrow \\ & \xrightarrow{f} & \end{array} .$$

$E_{\tilde f}$ $\mathbb{R}$ $E_f/\sim$ Y_f

$$t \quad , z, \theta_0 \ \sim \ t, f \ z \ , f \ z \ \theta_0 \ .$$

$E_{\tilde f}$ $A\ t$ $\Psi\ t\ dt$ $A\ t$ $\mathcal{A}\ E$ $\Psi\ t$ Ω^0 $, i\mathbb{R}$

$$F_{A+\Psi\, d} \quad F_A - \ A - d\Psi \ \wedge dt.$$

$$\gamma_{d,\tilde f} \ \ TY_f \qquad W_{d,\tilde f} \ , \qquad W_{d,\tilde f} \quad W_f \otimes E_{\tilde f}.$$

J $\sqrt{\ }$

$$\nabla \ \ _0 \qquad _0 \ \ \Psi \ \ _0, \qquad \nabla \ \ _1 \qquad _1 \ \ \Psi \ \ _1 \ \ - \ \ _1 \circ JJ$$

$_0$ $_0\ t$ C^∞ $, E$ $_1$ $_1\ t$ $\Omega^{0,1}$ $, E$

$$-i\nabla \ \ _0 \quad \sqrt{\ }\,\partial_{J,A} \ \ _1 \qquad , \qquad i\nabla \ \ _1 \quad \sqrt{\ }\,\partial_{J,A} \ \ _0 \qquad .$$

$$*i\ F_A \quad \eta_2 \quad \frac{K}{\ } \quad \frac{| \ \ _0|^2 - | \ \ _1|^2}{\ } \qquad ,$$

$$*\ \left(A - d\Psi \quad \frac{\alpha}{\ } \quad \eta_1\right) \quad i\sqrt{\ } \quad \langle \ \ _0, \ \ _1\rangle.$$

η $\eta_2 - \eta_1 \wedge dt$ Ω^2 $Y_f, i\mathbb{R}$
Y_f $\gamma_{d,\tilde f}$

$\phi_{d,f}$ $\mathcal{P}_{d,\tilde f}$

$$\eta \quad \eta_2 - \eta_1 \wedge dt, \qquad \eta_2 \quad i\left(\frac{\tau}{\ } \quad \frac{K}{\ }\right)\omega, \qquad \eta_1 \quad -\frac{\alpha}{\ }.$$

τ t

$\sqrt{}$ $_0$

$$_0^{\text{new}} \quad \sqrt{} \quad _0^{\text{old}}, \qquad \omega^{\text{new}} \quad -\omega^{\text{old}}, \qquad K^{\text{new}} \quad K^{\text{old}}.$$

$*$ $*$

$*^{\text{new}}$ $*^{\text{old}}$

$\sqrt{}$ $K\,\omega$ α

$$i\nabla \quad _0 \quad \partial_{J,A} \quad _1, \qquad -i\nabla \quad _1 \quad \partial_{J,A} \quad _0,$$

$$*iF_A \quad \frac{| \quad _0|^2 - | \quad _1|^2}{} \quad \tau,$$

$$* \left(A - d\Psi\right) \quad i \quad \langle \quad _0, \quad _1\rangle.$$

T C n-Si n -Di f n i n

$A_0\ t$ $\mathcal{A}\ E$ $A_0\ t$ $f\ A_0\ t$ Y_f $\gamma_{d,\tilde{f}}$ A_f A_0 η $i\tau\omega/ \ - F_{A_f}$ ω $_0$

$$\begin{aligned} \mathcal{C}\ \mathcal{D}_\tau\ A, \Psi, \qquad & - \int_0^1 \int_\Sigma A - A_0 \ \wedge \ A \quad A_0 \ dt \\ & - \int_0^1 \int_\Sigma \Big(\Psi\ F_A \quad i\tau\omega \qquad \langle \quad _1, \partial_{J_t,A} \quad _0\rangle\omega \Big)\, dt \\ & - \int_0^1 \int_\Sigma \Big(\quad \langle i\nabla \quad _0, \quad _0\rangle - \quad \langle i\nabla \quad _1, \quad _1\rangle \Big)\omega\, dt \end{aligned}$$

$_1$ $A\ t\ ,\ \ _0\ t$ $\mathcal{M}\ J\ ,\tau$

$$\mathcal{C}\ \mathcal{D}_\tau\ A, \Psi, \qquad - \int_0^1 \int_\Sigma \Big(A - A_0 \ \wedge \ A \quad A_0 \qquad \langle i \quad _0, \quad _0\rangle\omega \Big)\, dt.$$

t $A\ t\ ,\ \ _0\ t$

9 A

ε^2 τ

$$\omega_\varepsilon \quad \varepsilon^2\omega, \qquad \tau_\varepsilon \quad \varepsilon^{-2}\tau.$$

$\tau\omega$ η

$$i\nabla \quad {}_0 \quad \varepsilon^{-2}\,\partial_{J,A} \quad {}_1, \qquad -i\nabla \quad {}_1 \quad \partial_{J,A} \quad {}_0,$$

$$\varepsilon^{-2} * iF_A \quad \frac{| \quad {}_0|^2 - \varepsilon^{-2}| \quad {}_1|^2}{} \quad \varepsilon^{-2}\tau,$$

$$* \left(A - d\Psi\right) \quad i \quad \langle \quad {}_0, \quad {}_1\rangle.$$

$*$

ε 0

1

$${}_0{}^{\text{new}} \quad \varepsilon \quad {}_0{}^{\text{old}}, \qquad {}_1{}^{\text{new}} \quad \varepsilon^{-1} \quad {}_1{}^{\text{old}}.$$

$$i\nabla \quad {}_0 \quad \partial_{J_t,A} \quad {}_1, \qquad -i\nabla \quad {}_1 \quad \varepsilon^{-2}\partial_{J_t,A} \quad {}_0,$$

$$\varepsilon^{-2}\left(*iF_A \quad \frac{| \quad {}_0|^2}{} - \tau\right) \quad \frac{| \quad {}_1|^2}{},$$

$$* \left(A - d\Psi\right) \quad i \quad \langle \quad {}_0, \quad {}_1\rangle.$$

ε 1

ε

1 1

L 9 *Every solution of (29), (30), and (31) satisfies*

$$\partial_{J_t,A}\,\partial_{J_t,A} \quad {}_1 \quad \frac{| \quad {}_0|^2}{} \quad {}_1 - - \ \partial_{J_t,A} \quad {}_0 \circ J \quad \varepsilon^2\nabla\nabla \quad {}_1.$$

Proof.

$$\nabla\,\partial_{J_t,A} \quad {}_0 \quad \frac{d}{dt}\,\partial_{J_t,A} \quad {}_0 \quad \Psi\partial_{J_t,A} \quad {}_0 \quad - \ \partial_{J_t,A} \quad {}_0 \circ JJ.$$

$$id_A \quad {}_0 \quad \partial_{J,A} \quad {}_0 \circ J - \ \partial_{J,A} \quad {}_0 \circ J$$

$$\nabla \partial_{J_t,A} \quad {}_0 - \partial_{J_t,A} \nabla \quad {}_0 \qquad A - d\Psi \;{}^{0,1} \quad {}_0 \quad - \; \partial_{J_t,A} \quad {}_0 \circ J \; J \; .$$

$$i \; A - d\Psi \;{}^{0,1} \quad -\langle \quad {}_0, \quad {}_1 \rangle .$$

$$\partial_{J_t,A} \, \partial_{J_t,A} \quad {}_1 \qquad i\partial_{J_t,A} \nabla \quad {}_0$$

$$i\nabla \, \partial_{J_t,A} \quad {}_0 - i \; A - d\Psi \;{}^{0,1} \quad {}_0 - \frac{i}{} \; \partial_{J_t,A} \quad {}_0 \circ J \; J$$

$$\varepsilon^2 \nabla \nabla \quad {}_1 - - \langle \quad {}_0, \quad {}_1 \rangle \quad {}_0 \quad - \; \partial_{J_t,A} \quad {}_0 \circ J \; .$$

□

Remark .

Y S^1

γ_d TY W_d ,

W_d S^1 $E \oplus$ ${}^{0,1}T$ $\otimes E$ E d

J t

ε

$$\partial_{J_t,A} \, \partial_{J_t,A} \quad {}_1 - \nabla \nabla \quad {}_1 \quad \frac{| \quad {}_0|^2}{} \quad {}_1 \qquad .$$

1

$$\int_0^1 \int_\Sigma \left(| \, \partial_A \quad {}_1|^2 \quad |\nabla \quad {}_1|^2 \quad - | \quad {}_0|^2 \, | \quad {}_1|^2 \right) \omega \, dt \qquad .$$

${}_0 \equiv$ ${}_1 \equiv$ $\tau - *iF_A$

${}_1 \equiv$

$\Psi \; t$ t $A \; t$ A

${}_0 \; t$ ${}_0$ t

S^1

$$S^1 \qquad , \gamma_d \qquad \chi \; S^d \qquad S^1 \qquad , e_d \; .$$

f

L^2

$\tau - *iF_A$ $_1$ t ^{0} $\Omega^{0,1}_{J_t}$ $, E$

$_1 -$ $_1$ ε

$_0$ $_1$ $_1$

$\phi_{d,f}$ $\mathcal{P}_{d,\tilde{f}}$

$\varepsilon >$

$Y_f, \gamma_{d,\tilde{f}}$

μ^{SW} fl

0 F

$\mathbb{R}$ Y_f

$$\nabla_s \; _0 \quad i\nabla \; _0 \quad \partial_{J_t,A} \; _1, \qquad \nabla_s \; _1 - i\nabla \; _1 \quad \varepsilon^{-2}\partial_{J_t,A} \; _0.$$

$$\varepsilon^{-2}\left(*iF_A \quad \frac{| \; _0|^2}{} - \tau\right) \quad \frac{| \; _1|^2}{} \quad i \; \partial \quad - \partial_s \Psi \; ,$$

$$\partial_s A - d \quad * \;\; \partial \; A - d\Psi \quad i \quad \langle \; _0, \; _1\rangle.$$

s A ds $\Psi \, dt$ $\mathbb{R}$ $E_{\tilde{f}}$

$\mathbb{R}$ Y_f ε

$\mathcal{M}_{\Sigma,d} \; J, \tau \;\; \sim S^d$

$$\partial_{J_t,A} \; _0 \quad , \qquad *iF_A \quad | \; _0|^2/ \quad \tau,$$

$$\partial_s A - d \quad * \;\; \partial \; A - d\Psi \quad i \quad \langle \; _0, \; _1\rangle,$$

$$\nabla_s \; _0 \quad i\nabla \; _0 \quad \partial_{J_t,A} \; _1,$$

$$\partial_{J_t,A} \partial_{J_t,A} \; _1 \quad \frac{| \; _0|^2}{} \; _1 \quad - \; \partial_{J_t,A} \; _0 \circ J \; .$$

ε

C nj 0 *For every* f $,\omega$ *and every lift* f *of* f *to a unitary automorphism of a line bundle* E *of degree* d *there is a natural isomorphism between Seiberg-Witten and symplectic Floer homologies*

$$^{\rm SW}\; Y_f, \gamma_{d,\tilde f} \qquad ^{\rm symp}\; \phi_{d,f}, \mathcal{P}_{d,\tilde f}\ .$$

These isomorphisms intertwine the natural product structures:

$$\begin{array}{ccccc} ^{\rm SW}\; Y_f, \gamma_{d,\tilde f} & \otimes & ^{\rm SW}\; Y_g, \gamma_{d,\tilde g} & & ^{\rm SW}\; Y_{fg}, \gamma_{d,\tilde f \tilde g} \\ \downarrow & & \downarrow & & \downarrow \\ ^{\rm symp}\; \phi_{d,f}, \mathcal{P}_{d,\tilde f} & \otimes & ^{\rm symp}\; \phi_{d,g}, \mathcal{P}_{d,\tilde g} & & ^{\rm symp}\; \phi_{d,fg}, \mathcal{P}_{d,\tilde f \tilde g} \end{array}\ .$$

fl

ε

$\mathcal{M}_{\Sigma,d}$ g

d

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