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A new approach to immersion theory

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A new approach to immersion theory

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Dedicated to Rob Kirby on the occasion of his 60th birthday

1. Introduction

The classification of smooth immersions by Smale and Hirsch [3, 6] in the late 1950's was one of the early spectacular successes of differential topology. They classified immersions of one smooth manifold M in another Q in terms of *tangential maps* of M in Q. A tangential map of M in Q is a map with a given extension to a map of the tangent bundle TM in TQ which is a *bundle monomorphism*, ie. which embeds each fibre of TM linearly in a fibre of TQ.

An immersion of M in Q gives a tangential map by differentiating and the classification theorem states that differentiation induces a 1–1 correspondence between regular homotopy classes of immersions of M in Q and homotopy classes of tangential maps of M in Q. To see the power of this result observe that, given a map of M in Q, the problem of extending it to a tangential map is purely homotopy theoretic: one just has to construct a cross-section of an appropriate associated Stiefel manifold bundle. A famous corollary is that there is only one regular homotopy class of maps of the 2–sphere S^2 in $\mathbb{R}^3$ since the obstruction to deforming one tangential map into another lies in $\pi_2(SO(3)) = 0$. In other words a sphere "can be turned inside out". (This result is due to Smale, who classified immersions of spheres in Euclidean spaces; Hirsch extended Smale's result to the classification for general M and Q.)

In the proof a new machine was invented—the *immersion theory machine*. The key to the machine is to prove far more than is needed. Consider the space $\mathrm{Imm}(M, Q)$ of all immersions of M in Q and the space $\mathrm{Mon}(TM, TQ)$ of bundle monomorphisms of TM in TQ, then differentiation gives a map $d\colon \mathrm{Imm}(M, Q) \to \mathrm{Mon}(TM, TQ)$. The machine proves that d is a (weak) homotopy equivalence and the classification theorem follows by just considering path-components. The proof is by induction on a handle decomposition of M. For a 0–handle the result is easily seen by shrinking to a neighbourhood of a point. Now let M be obtained from M_0 by attaching a handle. It is an easy exercise to prove that restriction $\mathrm{Mon}(TM, TQ) \to \mathrm{Mon}(TM_0, TQ)$ is a fibration. Suppose that we can prove the same result for $\mathrm{Imm}(M, Q) \to \mathrm{Imm}(M_0, Q)$, then comparing the two fibrations and using the five lemma, we can deduce the result for $\mathrm{Imm}(M, Q) \to \mathrm{Mon}(TM, TQ)$. Thus

0

0

m

3

C^∞

W W

$x \in W$ $_x W$

Compression Theorem. *Let* ^{m} *be a compact manifold embedded in* $^q \times$ *and equipped with a normal vector field. Assume* $q - m \geq$ *then the vector field can be straightened (ie. made parallel to the positive direction) by an isotopy of and normal field in* $\times$.

Proof of the Compression Theorem.

$\times$

α

α ε ε

$\varepsilon >$

Lemma 2.1. *Under the hypotheses of the Compression Theorem the normal field may be assumed to be perpendicular and grounded.*

Proof. α

α α

α α

$\times$ $|$

$q - m >$

□

α α ε

$\varepsilon >$

α

μ $< \mu < \varepsilon$ $p \in$ α p

P p $_p$ $\times$ α p

β p P p

α p $\frac{\pi}{2} - \mu$

α p β p *be* α p

β p α

β

α

continuous *smooth*

β

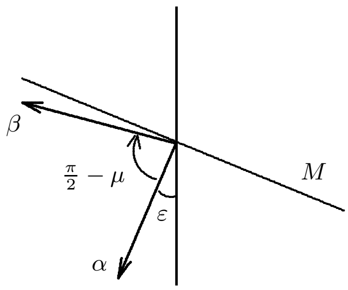

Figure 1.

$\varepsilon - \mu$ β
$\varepsilon - \mu$ μ
p P p
β γ $\times$ β
N $\times$
β fl
γ fl $\times$
γ $\varepsilon{-}\mu$
fl fl
fl fl γ
$\gamma|$ β fl $\gamma|$
fl

$\square$

C^0

downset
hori-
zontal set

$\times$ ^{r} $r \geq$

The horizontal set

m $q \times$ r $m \leq q$ *horizontal*
r *vertical* H H $\{ x\ y \in$
$| _{(x,y)} \{x\} \times$ r $\subset$ $_{(x,y)}$ $\}$ $\times$ r
H *horizontal*
H $\{ x\ y \in$ $| \nu_{(x,y)} \subset$ $_x$ $\}$ $\nu_{(x,y)}$
$x\ y$

Proposition 3.1. *After a small isotopy of in* $\times$ ^{r} *the horizontal set* H *can be assumed to be a submanifold of .* ☐

The downset

r H
c $q - m$ ψ
$- H$ ψ
α
D $\{x \in$ $| \alpha\ x$ $-\psi\ x \}$ D α *downmost*

Proposition 3.2. *After a small isotopy of* α *we can assume that* α *is transverse to* $-\psi$*, which implies that* D *is a manifold, and further we can assume that the closure of* D *is a manifold with boundary* H*.* ☐

Localisation

α
ψ
$\overline{D}$ $\overline{D}$
H $\overline{D}$
α ψ $\frac{W}{-W}$ W
∂W W
W

fl

C^0

Straightening multiple vector fields

$\times$ ^{n} n

n

n ^{n}

Multi-Compression Theorem. *Suppose that* ^{m} *in embedded in* $^q \times$ ^{n} *with* n *independent normal vector fields and that* $q - m \geq$. *Then* *is isotopic (by a* C^0*–small isotopy) to a parallel embedding.*

Proof.

$\times$ $^{n-1}$ $n -$

$n -$

$\times D^{n-1}$ $\times$ ^{n} $\times$ $^{n-1}$

$\times$ $^{n-1}$ $\times$ $^{n-1}$

$\times$ $^{n-1}$

$\times$ ^{n}

^{n} □

n *rotating the fibres*

n

$\oplus$

$v \in$

v_v v $\oplus$

v v_v

$$v \mapsto cv \quad sv_v \quad v_v \mapsto cv_v - sv$$

$$c \quad \tfrac{\pi}{2}t \quad s \quad \tfrac{\pi}{2}t \quad \leq t \leq \quad .$$

Basic Immersion Theorem. *Suppose that $q - m \geq$ and that we are given a bundle monomorphism f . Then the restriction $f|$ is homotopic to an immersion.*

Proof. f g
q ^{n} n
q ^{n}
q $g \times q$
$\times$ ^{n} n
^{n} n
$| \sim \oplus \times |$
$\times |$ n
n

$\square$

^{n}

$\times$

4 3

1 in 3

3

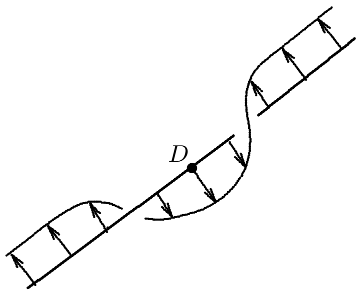

Figure 2.

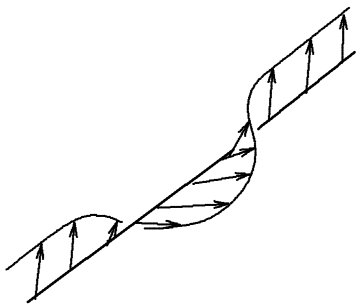

Figure 3.

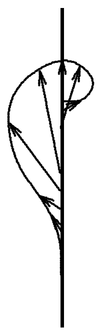

Figure 4.

ﬂ

ﬂ ﬂ

FIGURE 5.

fl

2 in 4

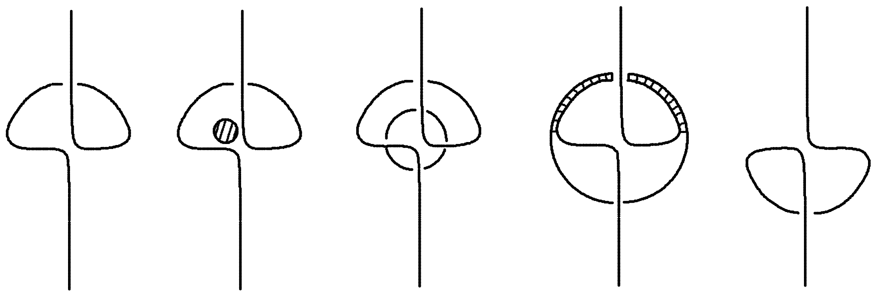

FIGURE 6.

fl

fl

3 in 5

fl

n n n

Removal of a Whitney umbrella

3

4

3

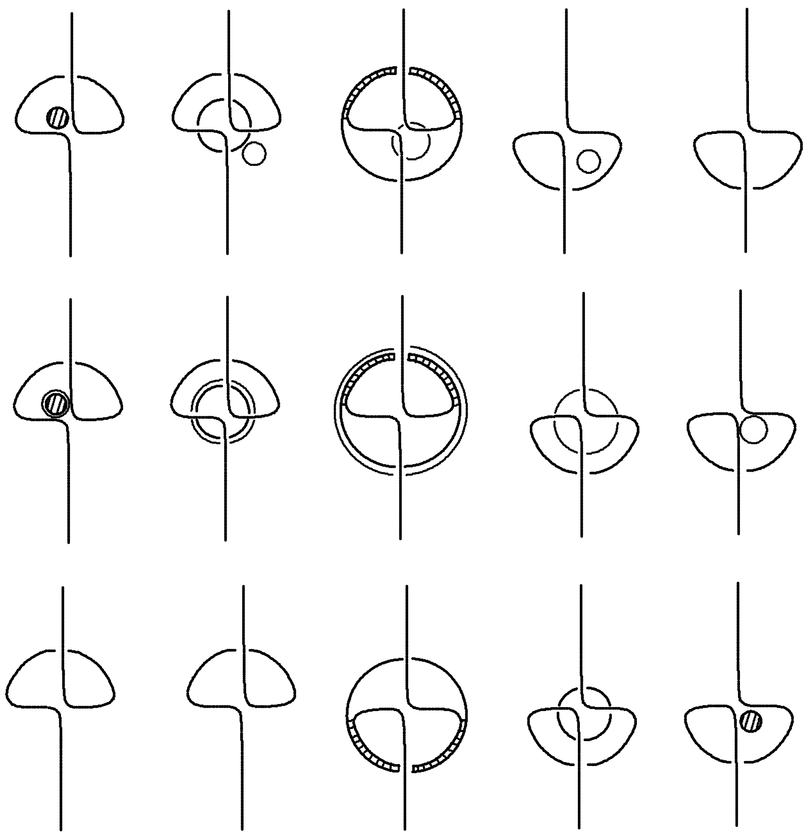

FIGURE 7.

downhill

2 4

2 3

H

ﬂ

$\alpha \times I$

H

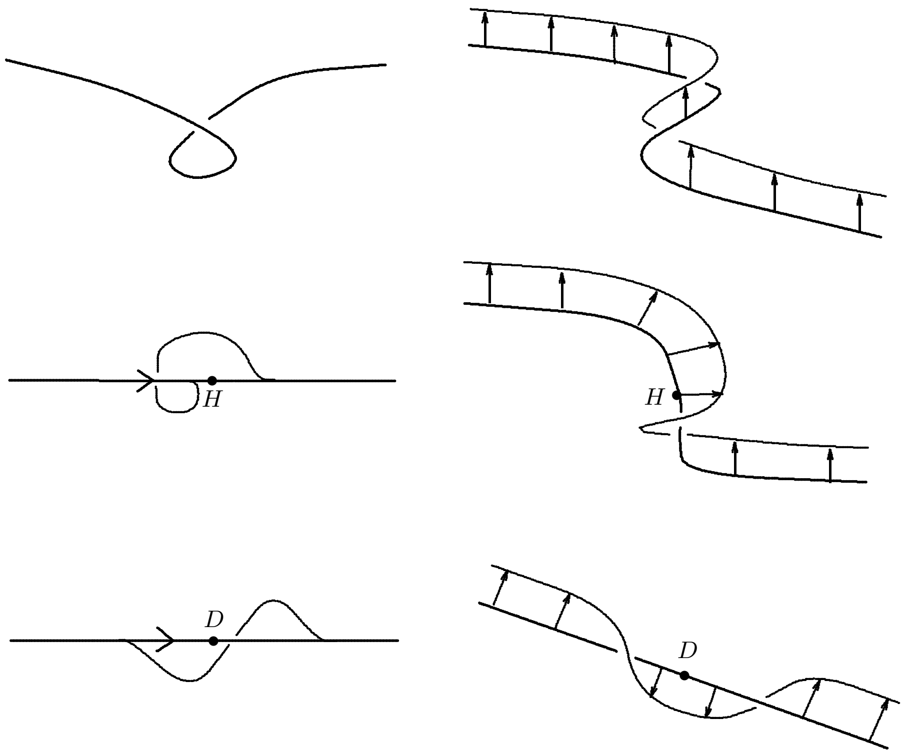

Figure 8.

Cross-capto Boy's surface

P^2 ^{3}

P^2 ^{3}

P^2 ^{2} $^3 \times$ ^{5}

P^2

$^3 \times$ ^{1} ^{4}

P^2 ^{3}

^{4}

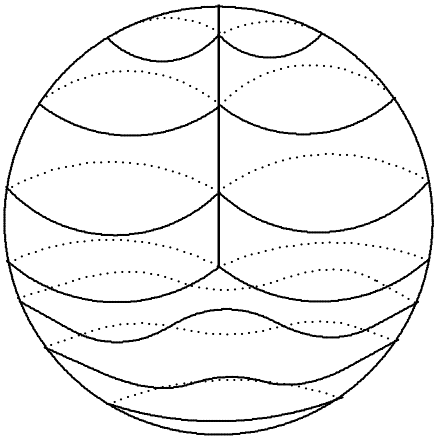

FIGURE 9. P^2 3

H_1

B H_2

H_1

X X
fl

4 3

outside
inside X
3

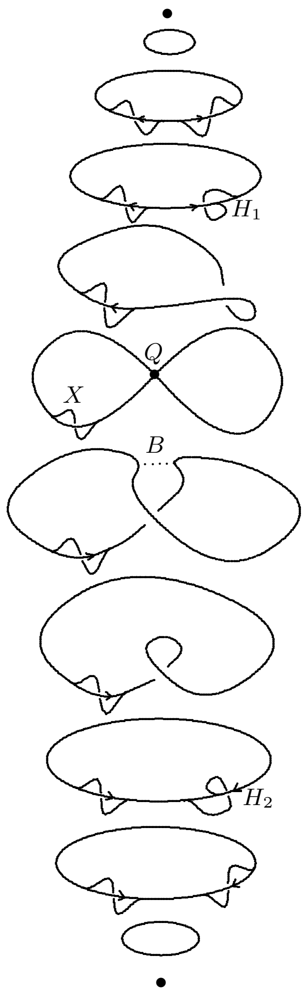

FIGURE 10.

4

X

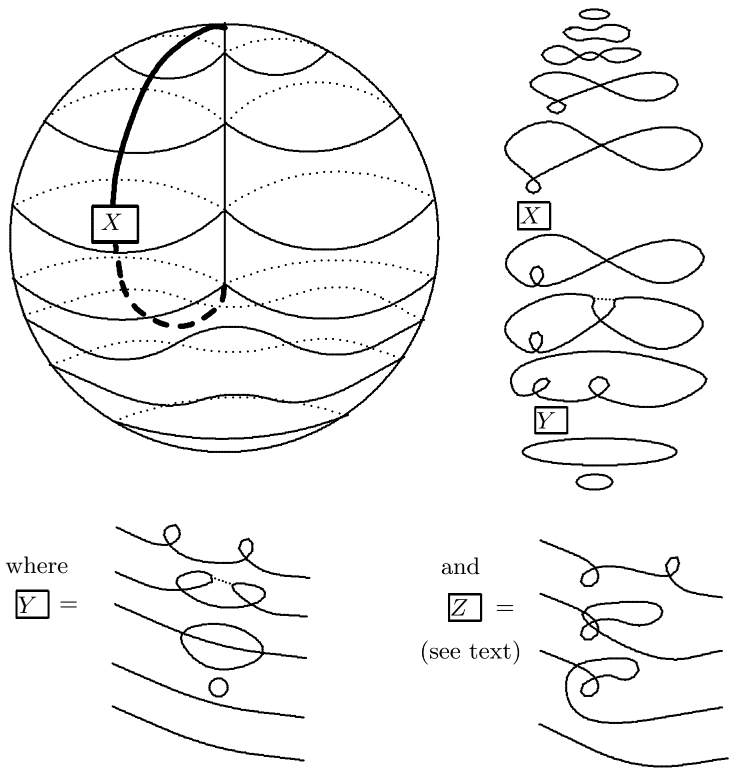

FIGURE 11. P^2 3

Y Y

X

Y

Z

D

D

D H

H_1 H_2

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