Turkish Journal of Mathematics

Volume 23 | Number 2 Article 11

1-1-1999

Some Results on Space-Like Line Congruences and Their Space-Like Parameter Ruled Surface

E. ÖZYILMAZ

Follow this and additional works at: https://journals.tubitak.gov.tr/math

Part of the Mathematics Commons

Recommended Citation

ÖZYILMAZ, E. (1999) "Some Results on Space-Like Line Congruences and Their Space-Like Parameter Ruled Surface," *Turkish Journal of Mathematics*: Vol. 23: No. 2, Article 11. Available at: https://journals.tubitak.gov.tr/math/vol23/iss2/11

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Tr. J. of Mathematics
23 (1999) , 333 – 344.
© TÜBİTAK

SOME RESULTS ON SPACE-LIKE LINE CONGRUENCES AND THEIR SPACE-LIKE PARAMETER RULED SURFACES

E. Özyılmaz

Abstract

Using dual vectors, space-like ruled surfaces was introduced in [2]. In this paper, we define space-like line congruence and their space-like parameter ruled surfaces $\mathbf{R_{11}(t)}$ and $\mathbf{R_{21}(t)}$ of R^3_1. Then, by choosing space-like parameter ruled surfaces as space-like principle ruled surfaces, we obtain some relations among the magnitudes of space-like ruled surfaces $\mathbf{R_1(t)}$, $\mathbf{R_{11}(t)}$.

1. Introduction

Let $A = a + \epsilon a_0$ be a dual number, $A \in ID = \{(a, a_0) \mid a, a_0 \in R\}$ and ID be a commutative ring with a unit element. We call the dual number $\epsilon = (0,1) \in ID$ dual unit and $\epsilon^2 = (0,0).(ID^3, +)$ is a module on the dual number ring. We call it ID-module, and dual vectors are the elements of this modul. We denote dual unit vector $\mathbf{A}$ as

$$\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon \mathbf{a_0} \quad , \quad \mathbf{aa} = \mathbf{1}, \mathbf{aa_0} = \mathbf{0}, \tag{1}$$

where $\mathbf{a}, \mathbf{a_0} \in \mathbf{IR^3}$. The dual vectors with unit length correspond to oriented lines of E^3 [1].

Theorem 1.1. *The oriented lines in IR^3 are in one-to-one correspondence with the points of the dual unit sphere in ID^3.*

The scalar product of two dual vectors $\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon\mathbf{a_0}$ *and* $\mathbf{B} = (\mathbf{b}, \mathbf{b_0}) = \mathbf{b} + \epsilon\mathbf{b_0}$ *is*

$$\mathbf{AB} = \mathbf{a}.\mathbf{b} + \epsilon(\mathbf{ba_0} + \mathbf{ab_0}) = \cos\varphi - \epsilon\varphi^\star \sin\varphi, \tag{2}$$

where, φ *is the real angle between dual unit vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *and* $\varphi^\star$ *is the shortest distance between the lines.*

The Blaschke trihedron (A_1, A_2, A_3) *depends on the ruled surface striction of* $\mathbf{A_1(t)}$ *in dual space* D^3 *[1]. According to this, the first axis* $\mathbf{A_1}$ *of the trihedron is the generator which passes from the striction point of the ruled surface, the second axis* $\mathbf{A_2}$ *is the surface normal at this point and finally the third axis* $\mathbf{A_3}$ *is the tangent of the striction line at this point.*

2. Dual Lorentzian Space D_1^3

Let we consider vector space R_1^3 of R^3 provided with Lorentzian inner product of signature (+,+,-). For any vector $\mathbf{a}=(a_1, a_2, a_3)$ of R_1^3 ;

i) if $\langle a, a\rangle > 0$, **a** is said to be space-like,

ii) if $\langle a, a\rangle < 0$, **a** is said to be time-like,

iii) if $\langle a, a\rangle = 0$, **a** is said to be light-like (null)

The Lorentzian and hyperbolic sphere of radius 1 in R_1^3 are defined by

$$S_1^2 = \{a = (a_1, a_2, a_3) \in R_1^3, \langle a, a\rangle = 1\} \tag{3}$$

$$H_0^2 = \{a = (a_1, a_2, a_3) \in R_1^3, \langle a, a\rangle = -1\} \tag{4}$$

respectively.

By considering the Lorentzian inner product, we may write inner product of **A** and **B** as follows:

$$\mathbf{AB} = \mathbf{a}.\mathbf{b} + \epsilon(\mathbf{ba_0} + \mathbf{ab_0}) \tag{5}$$

We call it dual Lorentzian space which is defined and denote by ID_1^3.

Definition 2.1. *Let* $\mathbf{A} = (\mathbf{a}, \mathbf{a_0}) = \mathbf{a} + \epsilon\mathbf{a_0} \in ID_1^3$. *The dual vector* $\mathbf{A}$ *is said to be space-like if the vector* $\mathbf{a}$ *is space-like, time-like if the vector* $\mathbf{a}$ *is time-like, and light-like (or null) if the vector* $\mathbf{a}$ *is light-like.*

We also defined the time orientation as follows:

A time-like vector $\mathbf{A} = \mathbf{a} + \epsilon\mathbf{a_0}$ *is future pointing if the vector* $\mathbf{a}$ *is future pointing.*

Definition 2.2. *Let* $\mathbf{A}$, $\mathbf{B} \in ID_1^3$. *We define the Lorentzian cross product of* $\mathbf{A}$ *and* $\mathbf{B}$ *by*

$$A \wedge B = \begin{vmatrix} E_1 & E_2 & -E_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{6}$$

where, $\mathbf{A} = (A_1, A_2, A_3), \mathbf{B} = (B_1, B_2, B_3)$ *and* $E_1 \wedge E_2 = E_3, E_2 \wedge E_3 = -E_3, E_3, E_3 \wedge E_1 = -E_2$, *[2].*

3. Space-Like Congruence in Dual Lorentzian Space ID_1^3

A space-like line congruence in line space R_1^3 can be represented by a unit space like dual vector which is depending on two real parameters u and v as follows:

$$\mathbf{R}(u, v) = \mathbf{r}(u, v) + \epsilon\mathbf{r}^\star(u, v) \quad , \quad \mathbf{R^2} = 1 \tag{7}$$

The dual arc element of a space like ruled surface of space-like congruence can be given as

$$\begin{aligned} dS^2 &= d\mathbf{R}^2 = (\mathbf{R}_u du + \mathbf{R_v} dv)^2 \\ &= -Edu^2 + 2Fdudv + Gdv^2 \end{aligned} \tag{8}$$

Where

$$
\begin{aligned}
E = -R_u^2 \quad & F = R_u R_v \quad & G = R_v^2 \\
E = e + \epsilon e^\star \quad & F = f + \epsilon f^\star \quad & G = g + \epsilon g^\star \\
e = -r_u^2 \quad & f = r_u r_v \quad & g = r_v^2 \\
e^\star = 2r_u r_u^\star \quad & f^\star = r_u r_v^\star \quad & g^\star = 2r_v r_v^\star
\end{aligned}
\tag{9}
$$

The differential forms I and II of the space-like line congruence are

$$I = -edu^2 + 2fdudv + gdv^2$$

$$II = -e^\star du^2 + 2f^\star dudv + g^\star dv^2 \tag{10}$$

respectively. If we use the relations (8),(9) and (10), we have

$$dS^2 = I + \epsilon II \tag{11}$$

and the dral of a space-like ruled surface of space like congruence can write as

$$\frac{1}{d} = \frac{II}{2I} \tag{12}$$

Definition 3.3. *Let $\frac{1}{d_1}$ and $\frac{1}{d_2}$ be extremum values of the dral. These values are called principle drals.*
The principle drals can calculated following relation:

$$\begin{vmatrix} -edu + fdv & -e^\star du + f^\star dv \\ fdu + gdv & f^\star du + g^\star dv \end{vmatrix} \tag{13}$$

Thus, we may write mean dral and Gaussian dral as follows, respectively:

$$h = \frac{1}{2}(\frac{1}{d_1} + \frac{1}{d_2}) \tag{14}$$

$$k = \frac{1}{d_1 d_2} \tag{15}$$

Definition 3.4. *The space-like ruled surfaces which are obtained by the relation (13) of the space like congruence are called the space-like principle ruled surfaces.*

Definition 3.5. *The space-like ruled surfaces u=constant and v=constant of a space-like line congruence are called the space-like parameter ruled surfaces.*

4. The Relations Among The Magnitudes of the Space Like Ruled Surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$

Let we consider a space-like ruled surface **R=R**(t) of the space-like congruence **R=R**(u,v), Where, u and v are functions of t.

Let we write the space-like parameter ruled surfaces as

$$\mathbf{R_{11}} = \mathbf{R_{11}}(u, v_0) \text{and} \mathbf{R_{21}} = \mathbf{R_{21}}(u_0, v) \tag{16}$$

The space-like ruled surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$ have common space-like line which is defined by the following relation:

$$\mathbf{R_0} = \mathbf{R}(u_0, v_0) = \mathbf{R_1}(u_0, v_0) = \mathbf{R_{21}}(u_0, v_0) \tag{17}$$

Blaschke trihedrons of these space-like ruled surfaces are in the following form:

$$(\mathbf{R_0} = \mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}) \quad , (\mathbf{R_0} = \mathbf{R_{11}}, \mathbf{R_{12}}, \mathbf{R_{13}}) \quad , (\mathbf{R_0} = \mathbf{R_{21}}, \mathbf{R_{22}}, \mathbf{R_{23}}) \tag{18}$$

Where $\mathbf{R_2}, \mathbf{R_{12}}$ and $\mathbf{R_{23}}$ are time-like, $\mathbf{R_{13}}, \mathbf{R_3}, \mathbf{R_{22}}$ and $\mathbf{R_0}$ are space-like. Thus we may write

$$\begin{aligned}
&\mathbf{R_1^2} = \mathbf{R_3^2} = \mathbf{1}, \mathbf{R_2^2} = -\mathbf{1} \mathbf{R_3} \wedge \mathbf{R_1} = \mathbf{R_2}, \mathbf{R_2} \wedge \mathbf{R_3} = -\mathbf{R_1}, \mathbf{R_1} \wedge \mathbf{R_2} = -\mathbf{R_3} \\
&\mathbf{R_{11}^2} = \mathbf{R_{13}^2} = \mathbf{1}, \mathbf{R_{12}^2} = -\mathbf{1} \mathbf{R_{13}} \wedge \mathbf{R_{11}} = +\mathbf{R_{12}}, \mathbf{R_{12}} \wedge \mathbf{R_{13}} = -\mathbf{R_{11}}, \mathbf{R_{11}} \wedge \mathbf{R_{12}} = -\mathbf{R_{13}} \\
&\mathbf{R_{21}^2} = \mathbf{R_{22}^2} = \mathbf{1}, \mathbf{R_{23}^2} = -\mathbf{1} \mathbf{R_{23}} \wedge \mathbf{R_{21}} = -\mathbf{R_{22}}, \mathbf{R_{22}} \wedge \mathbf{R_{23}} = -\mathbf{R_{21}}, \mathbf{R_{21}} \wedge \mathbf{R_{22}} = \mathbf{R_{23}}
\end{aligned} \tag{19}$$

On the other hand, if we choose the space-like paremeter ruled surfaces as space like principle ruled surfaces, we may write f=0 and $f^{\star}=0$. Thus,

$$F=0, \quad \mathbf{R_u}.\mathbf{R_v}=0 \tag{20}$$

can write.

The dual arc elements of the space-like ruled surfaces $\mathbf{R_1}, \mathbf{R_{11}}$ and $\mathbf{R_{21}}$ can be given respectively as

$$dS=Pdt \quad , dS_1=P_1du \quad , dS_2=P_2dv \tag{21}$$

where

$$P_1=\sqrt{\mid \mathbf{R_u^2} \mid}=\sqrt{-\mathbf{R_u^2}}=\sqrt{E}P_2=\sqrt{\mathbf{R_v^2}}=\sqrt{\mathbf{G}} \quad P=\sqrt{\mathbf{R_I'^2}} \tag{22}$$

$$\mathbf{R_1'}=\mathbf{R}_u\frac{du}{dt}+\mathbf{R_v}\frac{dv}{dt}$$

Using (21) and (22), we have

$$dS_1=\sqrt{E}du\text{and}dS_2=\sqrt{G}dv \tag{23}$$

The derivative formulas of these Blaschke trihedrons, defined by (18), are

$$\begin{aligned}
&\mathbf{R_1'}=\mathbf{PR_2}\mathbf{R_2'}=\mathbf{PR_1}+\mathbf{QR_3}\mathbf{R_3'}=\mathbf{QR_2}\\
&\mathbf{R_{11}'}=\mathbf{P_1R_{12}}\mathbf{R_{12}'}=\mathbf{P_1R_{12}}+\mathbf{Q_1R_{13}}\mathbf{R_{13}'}=\mathbf{Q_1R_{12}}\\
&\mathbf{R_{21}'}=\mathbf{P_2R_{22}}\mathbf{R_{22}'}=-\mathbf{P_2R_{21}}-\mathbf{Q_2R_{23}}\mathbf{R_{23}'}=-\mathbf{Q_2R_{22}}
\end{aligned} \tag{24}$$

Thus, the Blaschke vectors of the Blaschke trihedrons can be given by the following relations:

$$\mathbf{B}=-\mathbf{QR_0}+\mathbf{PR_3} \quad ,\mathbf{B_1}=-\mathbf{Q_1R_0}+\mathbf{P_1R_{13}} \quad ,\mathbf{B_2}=-\mathbf{Q_2R_0}-\mathbf{P_2R_{23}} \tag{25}$$

respectively.

The dual unit vectors $\mathbf{R_2}, \mathbf{R_{12}}$ and $\mathbf{R_{22}}$ which are the second edges of the trihedrons (18), can be written as

$$\mathbf{R_2} = \frac{\mathbf{R_1'}}{\mathbf{P}} = \frac{\mathbf{1}}{\mathbf{P}}(\mathbf{R_u}\frac{\mathbf{du}}{\mathbf{dt}} + \mathbf{R_v}\frac{\mathbf{dv}}{\mathbf{dt}}) \tag{26}$$

$$\mathbf{R_{12}} = \frac{\mathbf{R_{11}'}}{\mathbf{P_1}} = \frac{\mathbf{R_u}}{\mathbf{P_1}} = \frac{\mathbf{R_u}}{\sqrt{\mathbf{E}}} \tag{27}$$

$$\mathbf{R_{22}} = \frac{\mathbf{R_{21}'}}{\mathbf{P_2}} = \frac{\mathbf{R_v}}{\mathbf{P_2}} = \frac{\mathbf{R_v}}{\sqrt{\mathbf{G}}} \tag{28}$$

Using the relation

$$\mathbf{R} = \frac{\mathbf{R_u x R_v}}{\| \mathbf{R_u} \times \mathbf{R_v} \|} \tag{29}$$

, we can express the relation (29) for the common space-like line $\mathbf{R_0}$ as

$$\mathbf{R_0} = \frac{\mathbf{R}_u(u_0, v_0) x \mathbf{R}_v(u_0, v_0)}{\| \mathbf{R}_u(u_0, v_0) x \mathbf{R}_v(u_0, v_0) \|} = \mathbf{R}(u_0, v_0) \tag{30}$$

In our study, we will take the space-like parameter ruled surfaces as the space-like principle ruled surfaces. Thus, if we consider (20), (27) and (28), we have

$$\mathbf{R_{12}}.\mathbf{R_{22}} = 0 \tag{31}$$

From the relation below

$$\mathbf{R_{12}} \times \mathbf{R_{22}} = \frac{\mathbf{R_u} \times \mathbf{R_v}}{\sqrt{\mathbf{EG}}} = \mathbf{R_0} \tag{32}$$

and from (26), (27) and (28), we obtain

$$\mathbf{R_2} = \frac{\mathbf{P_1}}{\mathbf{P}}\frac{\mathbf{du}}{\mathbf{dt}}\mathbf{R_{12}} + \frac{\mathbf{P_2}}{\mathbf{P}}\frac{\mathbf{dv}}{\mathbf{dt}}\mathbf{R_{22}} \tag{33}$$

Thus, using the dual angle between the unit dual vectors $\mathbf{R_2}$ and $\mathbf{R_{12}}$, we may write

$$\mathbf{R_2} = ch\Phi \mathrm{R}_{12} + sh\Phi \mathrm{R}_{22} \tag{34}$$

where

$$ch\Phi = \frac{dS_1}{dS} = \sqrt{E}\frac{du}{dS} \quad , sh\Phi = \frac{dS_2}{dS} = \sqrt{G}\frac{dv}{dS} \tag{35}$$

$$tg\Phi = \frac{dS_2}{dS_1} = \sqrt{\frac{G}{E}}\frac{dv}{du} \tag{36}$$

and from (19) and (34)

$$-\mathbf{R_3} = \mathbf{R_0} \times \mathbf{R_2} = \mathbf{R_0} \times (ch\Phi\mathbf{R_{12}} + sh\Phi\mathbf{R_{22}}) \tag{37}$$

$$-\mathbf{R_3} = ch\Phi\mathbf{R_{13}} - sh\Psi\mathbf{R_{23}} \tag{38}$$

are obtained. Then, by the relations (19) and (32), we have

$$\mathbf{R_{12}} \times \mathbf{R_{13}} = -\mathbf{R_0}, \quad \mathbf{R_{22}} \times \mathbf{R_{23}} = -\mathbf{R_0} \quad , \mathbf{R_{12}} \times \mathbf{R_{22}} = \mathbf{R_0}$$

$$\mathbf{R_{12}} \times (\mathbf{R_{13}} + \mathbf{R_{22}}) = \mathbf{0} \rightarrow \mathbf{R_{13}} + \mathbf{R_{22}} = \mathbf{MR_{12}} \tag{39}$$

$$\mathbf{R_{22}} \times (\mathbf{R_{23}} - \mathbf{R_{12}}) = \mathbf{0} \rightarrow \mathbf{R_{23}} - \mathbf{R_{12}} = \mathbf{NR_{22}} \tag{40}$$

where M and N are dual scalars. Taking dot product of the both sides of (39) and (40) by the unit dual vectors $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$, respectively, and considering the relations (18) and (31), we have M=0 and N=0. Then, if we insert the values M and N into (39) and (40), we have

$$\mathbf{R_{12}} = -\mathbf{R_{22}} \tag{41}$$

$$\mathbf{R_{23}} = \mathbf{R_{12}}$$

Finally, we obtain following teorem:

Theorem 4.2. *The edges of Blascke trihedrons of the space-like parameter ruled surfaces coincide with each other under the condition that their directions and orders are not the same.*

Result 1. The Blaschke vectors $\mathbf{B}$, $\mathbf{B_1}$ and $\mathbf{B_2}$ can be expressed following form by the vectors $\mathbf{R_{12}}$, $\mathbf{R_{22}}$ and $\mathbf{R_0}$.

It is clear that

$$\begin{aligned} \mathbf{B} &= -\mathbf{Q}\mathbf{R_0} + \mathbf{P}(-\mathbf{ch\Phi R_{22}} - \mathbf{sh\Phi R_{12}}) \\ \mathbf{B_1} &= -\mathbf{Q_1}\mathbf{R_0} - \mathbf{P_1}\mathbf{R_{22}} \\ \mathbf{B_2} &= -\mathbf{Q_2}\mathbf{R_0} - \mathbf{P_2}\mathbf{R_{12}} \end{aligned} \tag{42}$$

can be written easily from (38) and (41)

Result 2. If the trihedron $(\mathbf{R_0}, \mathbf{R_{12}} = \mathbf{R_{23}}, \mathbf{R_{13}} = -\mathbf{R_{22}})$ moves on the striction curves of the space-like parameter ruled surface $\mathbf{R_{11}}$, it changes as a function of dual arc S_1 of v=constant ruled surface. If the $(\mathbf{R_0}, \mathbf{R_{12}} = \mathbf{R_{23}}, \mathbf{R_{13}} = -\mathbf{R_{22}})$ moves on the striction curves of the space-likeparameter ruled surface $\mathbf{R_{11}}$, it changes as a function of dual arc S_2 of u=constant ruled surface. Thus, the edges of this trihedron are depend on two parameters

Theorem 4.3. *If we consider Blaschke trihedrons and their derivative formulaes of the space-like ruled surface which are determined by (18) and (24), we have*

$$\begin{aligned} &\frac{\partial \mathbf{R_0}}{\partial u} = \mathbf{B_1} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{23}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{23}}, \frac{\partial \mathbf{R_{22}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{22}}, \\ &\frac{\partial \mathbf{R_0}}{\partial v} = \mathbf{B_2} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{13}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{13}}, \frac{\partial \mathbf{R_{12}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{12}}, \end{aligned} \tag{43}$$

Proof. If we write derivative formulas of the edges of Blaschke trihedrons by the Blaschke vectors, we have

$$\begin{aligned} &\frac{\partial \mathbf{R_0}}{\partial u} = \mathbf{B_1} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{12}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{12}}, \frac{\partial \mathbf{R_{13}}}{\partial u} = \mathbf{B_1} \times \mathbf{R_{13}}, \\ &\frac{\partial \mathbf{R_0}}{\partial v} = \mathbf{B_2} \times \mathbf{R_0}, \frac{\partial \mathbf{R_{22}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{22}}, \frac{\partial \mathbf{R_{23}}}{\partial v} = \mathbf{B_2} \times \mathbf{R_{23}}, \end{aligned} \tag{44}$$

If we insert (41) into (44), we get (43). □

Theorem 4.4. *If we use space-like paremeter ruled surfaces, we can write below form:*

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= -\mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_1} = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_2} &= -\mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2} = \frac{(\sqrt{G})_u}{\sqrt{EG}} \end{aligned} \tag{45}$$

Proof. From (22) and by taking derivatives of $(\sqrt{E})^2 = -\mathbf{R}_u^2, (\sqrt{G})^2 = \mathbf{R}_v^2$, we have

$$\sqrt{E}(\sqrt{E})_v = -\mathbf{R}_u\mathbf{R}_{uv}, \sqrt{G}(\sqrt{G})_u = \mathbf{R}_v\mathbf{R}_{vu} \tag{46}$$

and taking derivatives of (27) and (28)

$$\frac{\partial \mathbf{R}_{12}}{\partial v} = \frac{\mathbf{R}_{uv}\sqrt{E} - (\sqrt{E})_v\mathbf{R}_u}{E}, \quad \frac{\partial \mathbf{R}_{22}}{\partial u} = \frac{\mathbf{R}_{uv}\sqrt{G} - (\sqrt{G})_u\mathbf{R}_v}{G} \tag{47}$$

are obtained respectively. Then, from the relations above (27), (28), (46) and (20).

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial u} &= \frac{\mathbf{R}_u}{\sqrt{E}}\frac{\partial \mathbf{R}_{22}}{\partial u} = \frac{\mathbf{R}_u\mathbf{R}_{uv}}{\sqrt{EG}} = -\frac{(\sqrt{E})_v}{\sqrt{G}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial v} &= \frac{\mathbf{R}_v}{\sqrt{G}}\frac{\partial \mathbf{R}_{12}}{\partial v} = \frac{\mathbf{R}_v\mathbf{R}_{uv}}{\sqrt{EG}} = \frac{(\sqrt{G})_u}{\sqrt{E}} \end{aligned} \tag{48}$$

are faund. If we use (23) in (48), we have

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= R_{12}\frac{\partial \mathbf{R}_{22}}{\partial u}\frac{1}{\sqrt{E}} = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_2} &= R_{22}\frac{\partial \mathbf{R}_{22}}{\partial v}\frac{1}{\sqrt{G}} = \frac{(\sqrt{G})_u}{\sqrt{EG}} \end{aligned} \tag{49}$$

$$\begin{aligned} \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1} &= -R_{22}\frac{\partial \mathbf{R}_{12}}{\partial S_1} \\ \mathbf{R}_{22}\frac{\partial \mathbf{R}_{22}}{\partial S_2} &= -R_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2} \end{aligned} \tag{50}$$

□

Result 3. There is following relation between $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$:

$$\mathbf{R}_{12}.d\mathbf{R}_{22} = -d\mathbf{R}_{12}.\mathbf{R}_{22} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du - \frac{(\sqrt{G})_u}{\sqrt{E}}dv \tag{51}$$

Proof. If we differentiate (31) and consider (23) and (45), we obtain

$$\begin{aligned} &-\mathbf{R}_{12}.d\mathbf{R}_{22} = \mathbf{R}_{12}d\mathbf{R}_{22} = \mathbf{R}_{12}(\frac{\partial \mathbf{R}_{22}}{\partial S_1}dS_1 + \frac{\partial \mathbf{R}_{22}}{\partial S_2}dS_2) \\ &= \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_1}\sqrt{E}du + \mathbf{R}_{12}\frac{\partial \mathbf{R}_{22}}{\partial S_2}\sqrt{G}dv \\ &= -\frac{(\sqrt{E})_v}{\sqrt{EG}}\sqrt{E}du - \frac{(\sqrt{G})_u}{\sqrt{EG}}\sqrt{G}dv \end{aligned} \tag{52}$$

Thus, we have the result as below. □

Result 4. There is following relation between $\mathbf{R_{12}}$ and $\mathbf{R_{22}}$:

$$\frac{\partial}{\partial v}(\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial u}) - \frac{\partial}{\partial u}(\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial v}) = -\frac{\partial}{\partial v}(\frac{(\sqrt{E})_v}{\sqrt{G}}) + \frac{\partial}{\partial u}(\frac{(\sqrt{G})_u}{\sqrt{E}}) \tag{53}$$

Proof. Taking derivative of (31) with respect to the parameters v and u

$$\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial v} = -\mathbf{R}_{22}.\frac{\partial \mathbf{R}_{12}}{\partial v} = -\frac{(\sqrt{G})_u}{\sqrt{E}}, R_{12}\frac{\partial R_{22}}{\partial u} = -\frac{(\sqrt{E})_v}{\sqrt{G}} \tag{54}$$

is written. Then, if we consider (48), (53) is obtain. □

Theorem 4.5. *There are following relations for the magnitudes Q_1, Q_1 and Q of the space-like ruled surfaces $\mathbf{R_{11}}$, $\mathbf{R_{21}}$ and $\mathbf{R_1}$, respectively.*

$$Q_1 = -\frac{(\sqrt{E})_v}{\sqrt{G}} \quad , \quad Q_2 = -\frac{(\sqrt{G})_u}{\sqrt{E}} \tag{55}$$

$$Q^2 = -sh^2\Phi(\Phi' - Q_2)^2 + ch^2\Phi(\Phi' - Q_1)^2 \tag{56}$$

Proof. If we consider relations (43) and (23), we have

$$\frac{\partial \mathbf{R}_{22}}{\partial S_1} = \frac{\partial \mathbf{R}_{22}}{\sqrt{E}} = \frac{1}{\sqrt{E}}\mathbf{B}_1 \times \mathbf{R}_{22}$$

Then, using the relation above and (45), (32) and (25), we may write

$$-\frac{(\sqrt{E})_v}{\sqrt{EG}} = \mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial S_1} + \mathbf{R}_{12}\frac{\mathbf{B}_1 \times \mathbf{R_{12}}}{\sqrt{E}} = -\frac{\mathbf{R}_{12} \times \mathbf{R}_{22}}{\sqrt{E}}.\mathbf{B}_1 = -\frac{\mathbf{R}_0.\mathbf{B}_1}{\sqrt{E}} = \frac{Q_1}{\sqrt{E}}$$

By the same way and from the relations (45), (23), (44), (32) and (25), we have

$$\frac{(\sqrt{G})_u}{\sqrt{EG}} = -\mathbf{R}_{12}.\frac{\partial \mathbf{R}_{22}}{\partial S_2} = -\frac{1}{\sqrt{G}}\mathbf{R}_{12}.\frac{\mathbf{R}_{22}}{\partial v} = -\frac{\mathbf{R}_{12}.(\mathbf{B}_2 \times \mathbf{R}_{22})}{\sqrt{G}} = \frac{\mathbf{R}_0.\mathbf{B}_2}{\sqrt{G}} = -\frac{Q_2}{\sqrt{G}}$$

Finally, if we take derivative of the relation (38) by using the derivative formulas (24) and consider (41) and (31), it can be reached (55) and (56). □

References

[1] Blaschke, W. "Vorlesunger Über Differential Geometrie Und Geometriscke Grundlagen Von Einsteins Relativitatsthearie" Verlag Vonjulius Springer. Berlin, 1924.

[2] Uğurlu, H. H; Çalışkan, A. "Ruled Surface in Dual Lorentzian Space ID_1^3", (preprint).

[3] Birman, G. S.; Nomizu, K. "Trigonometry in Lorentzian Geometry" Ann. Math. Mont. 91(9), 543-549, 1984.

[4] O'nell, B. "Semi-Riemannnian Geometry with uplications to relativity" Academic Press, London, 1983."

[5] Çalışkan, A. "On the Studying of A Line Congruence By Choosing Parameter Ruled Surface As Principle Ruled Surfuce, Journal of Faculty of Ege University, Vol. 10, 1987.

E. ÖZYILMAZ
Ege Üniversitesi, Fen Fakültesi,
Matematik Bölümü 35100,
Bornova, İzmir-TURKEY

Received 28.01.1999