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SEGAL ALGEBRA AS AN IDEAL IN ITS SECOND DUAL SPACE

H. S. Mustafayev

Abstract

For a locally compact group G , let $S(G)$ be a symmetric Segal algebra. We prove that $S(G)$ is an ideal in its second dual space if and only if G is compact, where the second dual is equipped with an Arens multiplication.

1. Introduction

Let A be an arbitrary Banach algebra. On the second dual A^{**} of A may be equipped two Banach algebra multiplication, known as first and second Arens multiplication [1,2,5] each of which is an extension of the original multiplication in A as canonically embedded in A^{**} . From now on, we shall denote by A^{**} , the algebra A^{**} equipped with the first Arens multiplication and consider A as subalgebra of A^{**} .

Let G be a locally compact group, $L^1(G)$ the group algebra of G . K. P. Wong in [19] has proved that if G is a compact group, then $L^1(G)$ is an ideal in its second dual. For the converse of Wong's result S. Watanabe gave two different proofs [17,18] (the case G abelian had earlier been proved by P. Civin [4]). Other proofs were also provided by M. Grosser [9], D. L. Johnson [12], A. Ülger ([16], Prop. 4.8) and J. Duncan and A. Ülger ([6], Prop. 2.5). In [7] F. Ghahramani has extended this results to weighted group algebras.

In this note we find a necessary and sufficient condition for a symmetric Segal algebra to be an ideal in its second dual space. This generalizes the above-mentioned

result for group algebras.

2. Preliminaries

Throughout, G will be a locally compact group and dg a fixed left Haar measure on G . In the Banach algebra $L^1(G)$ we have the left translation operator L_g and right translation operator R_g defined by

$$L_g f(s) = f(g^{-1}s), R_g f(s) = \Delta(g^{-1})f(sg^{-1}),$$

where $\Delta(g)$ is the modular function of G . We recall also that, for any $f \in L^1(G)$, f^v and \tilde{f} are defined by $f^v(g)=f(g^{-1})$ and $\tilde{f}(g)=\Delta(g^{-1})f(g^{-1})$.

A linear subspace of $L^1(G)$ is said to be a Segal algebra, and denoted by $S(G)$, if it satisfies the following conditions (1)-(4), [14,15].

- (1) $S(G)$ is dense in $L^1(G)$
- (2) $S(G)$ is a Banach space under some norm $\| \cdot \|_S$ and

$$\| f \|_1 \leq C \| f \|_S$$

for all $f \in S(G)$ and for some constant $C > 0$.

- (3) $S(G)$ is left norm-invariant: $f \in S(G) \Rightarrow L_g f \in S(G)$ and $\| L_g f \|_S = \| f \|_S$ for all $f \in S(G)$ and all $g \in G$.

- (4) The mapping $g \rightarrow L_g f$ of G into $S(G)$ is continuous.

“Right-hand” versions of (3) and (4) are the following conditions.

- (3') $S(G)$ is right norm-invariant: $f \in S(G) \rightarrow R_g f \in S(G)$ and $\| R_g f \|_S = \| f \|_S$ for all $f \in S(G)$ and all $g \in G$.

- (4') The mapping $g \rightarrow R_g f$ of G into $S(G)$ is continuous.

A Segal algebra is said to be symmetric if it satisfies (3') and (4'). About Segal algebras, ample information can be found in H. Reiter's books [14, 15]. We now give some concrete examples of Segal algebras [14, 15].

- (i) The continuous functions in $L^1(G)$ that vanish at infinity form a Segal algebra, the norm being defined by

$$\|f\|_S = \|f\|_1 + \|f\|_\infty$$

(ii) The algebra $L^1(G) \cap L^p(G)$ ($1 < p < \infty$), equipped with the norm

$$\|f\|_S = \|f\|_1 + \|f\|_1 + \|f\|_p$$

is a Segal algebra.

The examples (i) and (ii) are symmetric Segal algebras if and only if G is unimodular ([15], p. 24).

(iii) Let G be an abelian group with character group \hat{G} , For $1 < p < \infty$, $A_p(G)$ denotes the set of all $f \in L^1(G)$ whose Fourier transforms \hat{f} are in $L^p(\hat{G})$. $A_p(G)$ is a (symmetric) Segal algebra with the norm

$$\|f\|_{A_p} = \|f\|_1 + \|\hat{f}\|_p$$

Any Segal algebra $S(G)$ is a left Banach $L^1(G)$ - convolution module, that is, if $h \in L^1(G)$ and $f \in S(G)$, then $h \star f \in S(G)$ and

$$\|h \star f\|_S \leq \|h\|_1 + \|f\|_S \quad f \in S(G), h \in L^1(G).$$

In particular, $S(G)$ is a Banach subalgebra of $L^1(G)$ under $\|\cdot\|_S$. If $S(G)$ is symmetric, then $S(G)$ is also a right Banach $L^1(G)$ - convolution module. Since $L^1(G)$ has a bounded (two-sided) approximate identity, it follows from the Cohen-Hewitt factorization theorem [11,32.22] that, if $S(G)$ is a symmetric Segal algebra, then

$$S(G) = L^1(G) \star S(G) = S(G) \star L^1(G).$$

On the other hand, we see that if $S(G)$ is a symmetric Segal algebra, then $L^1(G)$ is a Banach $S(G)$ - convolution bimodule. Using Cohen-Hewitt factorization theorem again one can see that $S(G)$ cannot have bounded (in the Segal norm) approximate identity (left or right) unless $S(G) = L^1(G)$. However, a symmetric Segal algebra has approximate (two-sided) identity that have L^1 -norm one ([15], p.34). Later on, we shall consider symmetric Segal algebras only.

Next, we recall definitions of some function spaces which we shall use in this note.

Let $C(G)$ be the space of bounded continuous complex-valued functions on G with sup-norm and $C_0(G)$ the subspace of $C(G)$ consisting of functions vanishing at infinity. By $C_{lu}(G)$, $C_{ru}(G)$ and $C_u(G)$ we denote in order, the subspaces of $C(G)$ consisting of the left, right and both left and right uniformly continuous functions on G . It is well known ([11], 32. 45) that

$$C_{lu}(G) = L^1(G) \star L^\infty(G), C_{ru}(G) = L^\infty(G) \star L^1(G)^v,$$

where $L^1(G)^v = \{f^v | f \in L^1(G)\}$. By $WAP(G)$ we denote the subspace of $C(G)$ consisting of the weakly almost periodic functions on G . It is well known ([3], p. 42, Theorem 3.11) that $WAP(G)$ is a (norm) closed linear subspace of $C_u(G)$. Burckel ([3], p. 68, Theorem 4.10) proved that $C(G) = WAP(G)$ if and only if G is compact. In [8] Granirer provided the following improvement of this result: $C_u(G) = WAP(G)$ if and only if G is compact.

3. The main result

The main result of this note is the following theorem.

Theorem. *A symmetric Segal algebra $S(G)$ is a right (resp. left) ideal in its second dual algebra if and only if G is compact.*

For the proof of the theorem we need some preliminary results. If X is a Banach space, we denote by X^* its dual and by $X_{(1)}$ its closed unit ball. For x in X and φ in X^* , we denote by $\langle \varphi, x \rangle$ the natural duality between X and X^* . Now, let $S(G)$ be a Segal algebra. It follows from (1) and (2) that, $L^\infty(G)$ can in natural way be embedded in $S(G)^*$, that is if $\varphi \in L^\infty(G)$, then $\varphi \in S(G)^*$ and

$$\|\varphi\|_{S^*} \leq C \|\varphi\|_\infty.$$

Moreover, if $f \in S(G)$ then we have

$$\langle \varphi, f \rangle = \int_G \varphi(g)f(g)dg, \quad \varphi \in L^\infty(G).$$

As we have seen before if $S(G)$ is a symmetric Segal algebra, then $S(G)$ is a Banach $L^1(G)$ - convolution bimodule. It follows that $S(G)^*$ is a Banach $L^1(G)$ -bimodule under the adjoint action defined by

$$\begin{aligned} \langle h \circ \varphi, f \rangle &= \langle \varphi, f \star h \rangle, \\ \langle \varphi \circ h, f \rangle &= \langle \varphi, h \star f \rangle, \end{aligned}$$

where $h \in L^1(G)$, $f \in S(G)$, $\varphi \in S(G)^*$. It is easily verified that

$$\begin{aligned} (h \star f) \circ \varphi &= h \circ (f \circ \varphi) \\ \varphi \circ (f \star h) &= (\varphi \circ f) \circ h, \end{aligned}$$

$h \in L^1(G)$, $f \in S(G)$, $\varphi \in S(G)^*$.

An easy calculation will show that, for φ in $L^\infty(G)$ and f in $L^1(G)$, $f \circ \varphi = \varphi \star f^v$ and $\varphi \circ f = \tilde{f} \star \varphi$.

By $S(G) \circ S(G)^*$, $S(G)^* \circ S(G)$ and $S(G)^* \circ S(G)$ we denote, respectively, the sets $\{f \circ \varphi | f \in S(G), \varphi \in S(G)^*\}$, $\{\varphi \circ f | \varphi \in S(G)^*, f \in S(G)\}$ and $\{f \star h | f, h \in S(G)\}$. Put $S(G)^v = \{f^v | f \in S(G)\}$.

Lemma 1.a) $S(G) \circ S(G)^*$ is in $C_{ru}(G)$ and moreover

$$\overline{S(G) \circ S(G)^*} \|\cdot\|_\infty = C_{ru}(G)$$

$S(G)^* \circ S(G)$ is in $C_{lu}(G)$ and moreover

$$\overline{S(G)^* \circ S(G)} \|\cdot\|_\infty = C_{lu}(G)$$

Here “ $\overline{\|\cdot\|_\infty}$ ” denotes the sup-norm closure.

Proof. a). Assume that $f \in S(G)$ and $\varphi \in S(G)^*$. First we observe that $f \circ \varphi$ is in $L^\infty(G)$. From the equality $\langle f \circ \varphi, h \rangle = \langle \varphi, h \star f \rangle$, where $h \in S(G)$, we have \square

$$| \langle f \circ \varphi, h \rangle | \leq \| \varphi \|_{S^*} \| h \star f \|_S \leq \| \varphi \|_{S^*} \| f \|_S \| h \|_1 .$$

This inequality shows that $f \circ \varphi$ is bounded on $S(G)$ for the norm of $L^1(G)$. Hence, $S(G)$ being dense in $L^1(G)$, $f \circ \varphi$ can be extended in a unique way to $L^\infty(G)$. Thus $f \circ \varphi$ can be considered as an element of $L^\infty(G)$. On the other hand, since $S(G) = L^1(G) \star S(G)$, f can be represented as $f = h \star k$, where $h \in L^1(G)$, $k \in S(G)$. Now we have

$$f \circ \varphi = (h \star k) \circ \varphi = h \circ (k \circ \varphi).$$

Since $k \circ \varphi \in L^\infty(G)$, this gives

$$f \circ \varphi = (k \circ \varphi) \star h^v.$$

From this we deduce that $f \circ \varphi \in C_{ru}(G)$ for all $f \in S(G)$ and $\varphi \in S(G)^*$. Thus we have the following:

$$\overline{S(G) \circ S(G)^*}^{\|\cdot\|_\infty} \subset C_{lu}(G).$$

To prove the opposite inclusion it is enough to show that

$$\subset C_{lu}(G)L^\infty(G) \star L^1(G)^v \subset \overline{S(G) \circ L^\infty(G)}^{\|\cdot\|_\infty} = \overline{L^\infty(G) \star S(G)^v}^{\|\cdot\|_\infty}$$

Since $S(G)$ is dense in $L^1(G)$, it remains to observe that if a sequence (f_n) in $S(G)$ converges (in the L^1 -norm) to some $f \in L^1(G)$, then $\varphi \star f_n^v \rightarrow \varphi \star f^v$ uniformly for all $\varphi \in L^\infty(G)$.

The proff of b) is similar. ■

Now, let A be an arbitrary Banach algebra and a is an element of A . By $R_a : A \rightarrow A$ (resp. $L_a : A \rightarrow A$) we denote the right multiplication operator (left multiplication operator) defined by $R_a(b) = ba(L_a(b) = ab)$. The following lemma was proved in [5,7].

Lemma 2. *A is a right (resp. left) ideal of A^{**} if and only if, for each $a \in A$, the right multiplication operator R_a (the left multiplication operator L_a) is weakly compact.*

Now we can prove the main result of this note.

Proof of Theorem Assume that $S(G)$ is a right ideal in its second dual algebra. Then by Lemma 2 the set $\ell \star f \mid \ell \parallel_S \leq 1$ is relatively weakly compact in $S(G)$ for all $f \in S(G)$. Assume first that f is of the form: $f = h \star k$ where, $h, k \in S(G)$. Without loss of generality we can suppose that $\|h\|_S \leq 1$. Since $L_g f = L_g(h \star k) = L_g h \star k$ and since $\|L_g h\|_S = \|h\|_S \leq 1$ we have

$$\{L_g f \mid g \in G\} \subset \{\ell \star k \mid \ell \parallel_S \leq 1\}$$

It follows that, the set $\{L_g f \mid g \in G\}$ is relatively weakly compact for all f in $S(G) \star S(G)$. Since $g \rightarrow L_g$ is a (continuous) representation of G on $S(G)$, by Eberlein theorem ([3], p.36, Theorem 3.1), the function $g \rightarrow \langle \varphi, L_g f \rangle$ is in $WAP(G)$ for all $\varphi \in S(G)^*$ and $f \in S(G) \star S(G)$. Moreover, since $S(G)$ has an approximate identity, for any $f \in S(G)$, there is a sequence (f_n) in $S(G) \star S(G)$ such that $f_n \rightarrow f$ in the Segal norm. This implies that $\langle \varphi, L_g f_n \rangle \rightarrow \langle \varphi, L_g f \rangle$ uniformly. From this it follows that the function $g \rightarrow \langle \varphi, L_g f \rangle$ is in $WAP(G)$ for all $\varphi \in S(G)^*$ and $f \in S(G)$. Now, we claim that $\langle \varphi, L_g f \rangle = (f \circ \varphi)(g)$. In fact, for given any $h \in S(G)$ we can write

$$\begin{aligned} \int_G h(g) \langle \varphi, L_g f \rangle dg &= \langle \varphi, \int_G h(g) L_g f dg \rangle \\ &= \langle \varphi, h \star f \rangle = \langle f \circ \varphi, h \rangle = \int_G h(g) (f \circ \varphi)(g) dg. \end{aligned}$$

Since $S(G)$ is dense in $L^1(G)$ and since the functions $g \rightarrow \langle \varphi, L_g f \rangle$ and $g \rightarrow (f \circ \varphi)(g)$ are both continuous, the last equality clearly implies that

$$\langle \varphi, L_g f \rangle = (f \circ \varphi)(g).$$

Further, by Lemma 1 we have

$$\overline{C_{ru}(G) = S(G) \circ S(G)^*} \parallel \cdot \parallel_\infty \subset WAP(G),$$

and consequently $C_u(G) = WAP(G)$. However, this equality is possible only if G is compact [8].

Now assume that G is a compact group. We shall prove that the right multiplication operator $R_f : h \rightarrow h \star f$ is compact on $S(G)$ for all $f \in S(G)$. Before beginning the proof, we recall the following fact which is an immediate consequence of the Peter-Weyl theory: If T is a (continuous) representation of the compact group G on some Banach space X , then X is a closed linear span of finite dimensional invariant subspaces of T ([13], p.91, Corollary 1).

Now, since $g \rightarrow L_g$ is a (continuous) representation of G on $S(G)$, by virtue of the above-mentioned fact, is a closed linear span of finite dimensional invariant subspaces of $L_g (g \in G)$. Let J be an invariant (closed) subspace of L_g and let $f \in J$. By the very definition of vector-valued integral we have

$$h \star f = \int_G h(g)L_g f dg \in J, \quad h \in S(G)$$

Hence J is a (closed) left ideal of $S(G)$. Thus we see that, actually, $S(G)$ is a closed linear span of finite dimensional left ideals. This means that, for given any $f \in S(G)$ and $\epsilon > 0$, there exist finite dimensional left ideals J_1, \dots, J_n and $f_1 \in J_1, \dots, f_n \in J_n$ such that

$$\| f - \sum_{i=1}^n f_i \| < \epsilon$$

This implies that

$$\| R_f - \sum_{i=1}^n R_{f_i} \| < \epsilon$$

Since $J_i (i = 1, \dots, n)$ are finite dimensional left ideals, $R_{f_i} (i = 1, \dots, n)$ are finite rank operators. On the other hand, the preceding inequality show that R_f can be approximate (in the operator norm) by finite rank operators. From this we conclude that R_f is a compact operator.

“Right-version” of this arguments shows that $S(G)$ is a left ideal in its second dual if and only if G is compact. The proof is complete.

As an immediate corollary of the Theorem we have the following results.

Corollary 3. a) The algebra $L^1(G) \cap C_0(G)$ equipped with the norm $\|f\| = \|f\|_1 + \|f\|_\infty$ is a right (resp. left) ideal in its second dual space if and only if G is compact.

b) The algebra $L^1(G) \cap L^p(G)$ ($1 < p < \infty$) equipped with the norm

$\|f\| = \|f\|_1 + \|f\|_p$ is a right (resp. left) ideal in its second dual space if and only if G is compact.

Notice that, if G is compact, then $L^1(G) \cap L^p(G)$ ($1 < p < \infty$) is equal to $L^p(G)$ which is a reflexive.

Corollary 4. The algebra $A_p(G)$ is an ideal in its second dual if and only if G is compact.

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